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Spector first constructed a function h whose degree of recursive unsolvability is minimal—that is to say that any function recursive in h is either recursive or of the same degree as h. Define a set Q of degrees to be an initial segment of the upper semi-lattice of degrees of unsolvability if

$$a \in Q \wedge b < a \,.\!\rightarrow\!.\, b \in Q \,.$$

Spector's result can then be interpreted as saying that a certain partially ordered set occurs as an initial segment of the degrees; it was conjectured that the same is true for every finite partially ordered set which has a least member. Sacks then constructed two minimal degrees a and b such that $a \cup b$ has $a, b, 0$ as its only predecessors.

In this paper their methods are extended to obtain the following result. Let T be the upper semi-lattice of all finite subsets of N. Then T can be embedded as an initial segment of the degrees. From this it follows that any finite partial ordering which can be embedded as an initial segment of $P(B)$ (the power set of B), with B finite, can also be embedded as an initial segment of the degrees.

We will define a function h containing a countable infinity of functions h_i, each of minimal degree, such that $\underline{h}_{i_1} \cup \underline{h}_{i_2} \cup \cdots \cup \underline{h}_{i_t}$ will represent the finite subset $\{i_1, \cdots, i_t\}$ of N.

We first define a recursive function ϕ as follows: Let $\psi(k) = (\mu n)((n+1)((n+1)+1)/2) > k$ so that

$$\frac{\psi(k)(\psi(k)+1)}{2} \leqq k < \frac{(\psi(k)+1)((\psi(k)+1)+1)}{2} \,,$$

and define $\phi(k) = k - (\psi(k)(\psi(k)+1))/2$. Then $\phi(k)$ takes on successively the values

$$0, 0, 1, 0, 1, 2, 0, 1, 2, 3, \cdots, n, 0, 1, 2, \cdots, n, n+1, 0, \cdots \,.$$

We will want to arrange things so that in the k'th interval of g, the $\phi(k)$'th function (carried on the powers of $p_{\phi(k)}$) will be the *only* one for which f_0 and f_1 have different values. (The reader who finds this sentence mysterious is encouraged to read the next few definitions and then return to this remark.)

Let p_i be the i'th prime, and define recursive predicates P_i and P as follows:

$$P_i(x) \equiv x = p_i^{(x)_i}$$
$$P(x) \equiv (Ei)P_i(x) \,.$$

Thus $P(x)$ if and only if x is a prime-power.

A triple (f_0, f_1, g) of functions is *special* if:

(i) $(i)_{i<2}(t)[f_i(t) = 0 \vee f_i(t) = 1]$

(ii) $g(0) = 0 \wedge (n)[g(n) < g(n+1)]$

(iii) $\sim P(x) \rightarrow f_0(x) = f_1(x) = 0$

(iv) $$(n)[(Ex)\{g(n+1) \leqq x < g(n+2) \wedge P_{\phi(n)}(x) \wedge f_0(x) \neq f_1(x)\} \wedge (z)\{g(n+1) \leqq z < g(n+2) \wedge \sim P_{\phi(n)}(z) \,.\!\rightarrow\!.\, f_0(z) = f_1(z)\}]\,.$$

If (f_0, f_1, g) satisfies (i), (ii), and (iii) then we define $F(f_0, f_1, g)$ as follows:

$$h \in F(f_0, f_1, g) \equiv (n)(Ei)_{i<2}(x)[g(n) \leqq x < g(n+1) \,.\!\rightarrow\!.\, h(x) = f_i(x)]\,.$$

If (f_0, f_1, g) and (u_0, u_1, v) are special triples, then we say that (u_0, u_1, v) is a *contraction* of (f_0, f_1, g) if:

(i) $u_0, u_1 \in F(f_0, f_1, g)$

(ii) $(\mu t)(u_0(t) \neq u_1(t)) > (\mu t)(f_0(t) \neq f_1(t))$

(iii) $(n)(Em)[v(n) = g(m)]$.

If $F(f_0, f_1, g)$ is defined, then we define $F^*(f_0, f_1, g)$ to be the set of all initial segments of members of $F(f_0, f_1, g)$—that is,

$$s \in F^*(f_0, f_1, g) \equiv (En)(g(n) = lhs) \wedge (m)_{g(m)<lhs}(Ei)_{i<2}(x)[g(m) \leqq x < g(m+1) \,.\!\rightarrow\!.\, (s)_x = f_i(x)]\,.$$

LEMMA 1. *Let* (f_0, f_1, g) *be a special triple and let* $s \in F^*(f_0, f_1, g)$. *Then there is a special triple* (u_0, u_1, v) *such that:*

(i) (u_0, u_1, v) *is a contraction of* (f_0, f_1, g)

(ii) u_0, u_1, v *are recursive in* f_0, f_1, g

(iii) $v(1) > lhs$

(iv) $(x)_{x<lhs}(u_0(x) = u_1(x) = (s)_x)$.

Proof. Let m be such that $g(m) = lhs$. Define $v(0) = 0$, $v(1) = g(m+2)$, $u_0(x) = u_1(x) = (s)_x$ for $x < lhs$, and $u_0(x) = u_1(x) = f_0(x)$ for $lhs \leqq x < g(m+2)$.

Suppose now that $v(n+1)$ is defined; we show how to define $v(n+2)$ and $u_i(x)$ for $v(n+1) \leqq x < v(n+2)$. Let m^* be such that $g(m^*) = v(n+1)$ and let r^* be such that

$$r^* = (\mu r)(\phi(r) = \phi(n) \wedge r + 1 \geqq m^*)\,.$$

Then define $v(n+2) = g(r^*+2)$. We also define $u_i(t)$ for $v(n+1) \leqq t < v(n+2)$ as follows:

$$u_i(t) = f_0(t) \quad \text{for } v(n+1) \leqq t < g(r^*+1)$$
$$u_i(t) = f_i(t) \quad \text{for } g(r^*+1) \leqq t < v(n+2)\,.$$

It is clear that the triple (u_0, u_1, v) is special and has the properties (i)-(iv).

Let $0 \leqq a_0 < a_1 < a_2 < \cdots < a_{k-1}$ be a finite sequence of integers and let h be any function whose range is $\{0, 1\}$. We define $h_{a_0\cdots a_{k-1}}$ as follows:

$$\begin{aligned} h_{a_0\cdots a_{k-1}}(x) &= 0 \qquad \text{if } \sim(P_{a_0}(x) \vee \cdots \vee P_{a_{k-1}}(x)) \\ &= h(x) \qquad \text{if } P_{a_0}(x) \vee \cdots \vee P_{a_{k-1}}(x) \,. \end{aligned}$$

Where there is no danger of ambiguity we will abbreviate $h_{a_0\cdots a_{k-1}}$ to h_-. We also use the convention that $x_0, \cdots, \hat{x}_j, \cdots, x_r$ means $x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_r$.

We construct a function h so that for each j, the function h_j is of minimal degree. The degree $\underline{h}_j$ will correspond to the finite set $\{j\}$. We see that $\underline{h}_{a_0} \cup \cdots \cup \underline{h}_{a_{k-1}} = \underline{h}_{a_0\cdots a_{k-1}}$, so we will assign $\underline{h}_{a_0\cdots a_{k-1}}$ to be the degree corresponding to the set $\{a_0, \cdots, a_{k-1}\}$. Let A be a finite subset of N, and let $A_1, \cdots, A_t$ be all sets obtained by deleting one element from A. The lemma below is concerned with guaranteeing the degree analog of the following statement: If $B \leqq A$, then either $B = A$ or for some j, $B \leqq A_j$.

Note that if $\{a_0, \cdots, a_{k-1}\} = \phi$, then $h_{a_0\cdots a_{k-1}} \equiv 0$.

LEMMA 2. *Let* (f_0, f_1, g) *be a special triple and let* e *and* $0 \leqq a_0 < a_1 < \cdots < a_{k-1}$ *be given. Then there is a special triple* (u_0, u_1, v) *such that*:

(i) (u_0, u_1, v) *is a contraction of* (f_0, f_1, g)

(ii) u_0, u_1, v *are recursive in* f_0, f_1, g

(iii) *if* $h \in F(u_0, u_1, v)$ *then either*

(A) $\{e\}^{h_-}(n)$ *is undefined for some* n

or

(B_j) $\{e\}^{h_-}(n)$ *is recursive in* $h_{a_0\cdots\hat{a}_j\cdots a_{k-1}}, f_0, f_1, g$ *for some* $j, 0 \leqq j \leqq k-1$

or

(C) h_- *is recursive in* $\{e\}^{h_-}, f_0, f_1, g$.

Proof. Consider $F(f_{0-}, f_{1-}, g)$, which is precisely

$$\{h_{a_0\cdots a_{k-1}} \mid h \in F(f_0, f_1, g)\} \,,$$

and denote $F^*(f_{0-}, f_{1-}, g)$ by F^*.

Case 1.

$$(Es)(En)(w)[s \in F^* \wedge (w \in F^* \wedge w \operatorname{ext} s \;.\!\rightarrow\!.\; \{e\}^w(n) \text{ is undefined})] \,.$$

Let s_0, n_0 be the least pair satisfying the case hypothesis. Let

$$\begin{aligned} s = (\mu w)(w \in F^*(f_0, f_1, g) \wedge lh\ w = lhs_0 \\ \wedge (x)_{x<lhs_0}(P_{a_0}(x) \vee \cdots \vee P_{a_{k-1}}(x) \;.\!\rightarrow\!.\; (w)_x = (s_0)_x)) \,. \end{aligned}$$

Apply Lemma 1 to get a special triple (u_0, u_1, v) satisfying (i) and (ii). We need only show that (u_0, u_1, v) also satisfies (iii) (A).

But if $h \in F(u_0, u_1, v)$ and $\{e\}^{h_-}(n_0)$ is defined, then there is an initial segment w of h_- such that $\{e\}^w(n_0)$ is defined. But s is an initial segment of h; and if $w \operatorname{ext} s_0$ then $\{e\}^w(n_0)$ is undefined and if $s_0 \operatorname{ext} w$ then $\{e\}^{s_0}(n_0)$ is defined. Both of these are impossible. Hence $\{e\}^{h_-}(n_0)$ is undefined.

Case 2_j. The hypothesis of Case 1 is false and in addition:

$$\begin{aligned} (Es)(n)(u)(v)[s \in F^* \wedge \{u \in F^* \wedge v \in F^* \wedge lhu = lhv \\ \wedge \{e\}^u(n) \text{ is defined} \wedge u \operatorname{ext} s \\ \wedge \{e\}^v(n) \text{ is defined} \wedge v \operatorname{ext} s \\ \wedge (x)_{lhs \leq x < lhu}(P_{a_0}(x) \vee \cdots \vee \hat{P}_{a_j}(x) \vee \cdots \vee P_{a_{k-1}}(x) \\ .\!\rightarrow\!.\; (u)_x = (v)_x)\} \rightarrow \{e\}^u(n) = \{e\}^v(n)] \,. \end{aligned}$$

This hypothesis will be referred to as statement (j).

Let s_0 be the least s satisfying the statement (j), where j is the smallest i such that statement (i) holds. Let

$$\begin{aligned} s = (\mu w)(w \in F^*(f_0, f_1, g) \wedge lh\ w = lhs_\sigma \\ \wedge (x)_{x<lhs_0}(P_{a_0}(x) \vee \cdots \vee P_{a_{k-1}}(x) \;.\!\rightarrow\!.\; (w)_x = (s_0)_x) \,. \end{aligned}$$

Apply Lemma 1 to get a special triple (u_0, u_1, v) satisfying (i) and (ii). We show that (u_0, u_1, v) also satisfies (iii) (B_j).

So let $h \in F(u_0, u_1, v)$ be such that $\{e\}^{h_-}(n)$ is defined for all n. Then s_0 must be an initial segment of h_-. We show how to compute $\{e\}^{h_-}(n)$ from $f_0, f_1, g, h_{a_0 \cdots \hat{a}_j \cdots a_{k-1}}$.

Now since $\{e\}^{h_-}(n)$ is defined, $h_- \in F(f_{0-}, f_{1-}, g)$, and the hypothesis of Case 1 is false, there is a $w \in F^*$ such that $w \operatorname{ext} s_0 \wedge \{e\}^w(n)$ is defined and

$$(x)_{lhs \leq x < lhw}(P_{a_0}(x) \vee \cdots \vee \hat{P}_{a_j}(x) \vee \cdots \vee P_{a_{k-1}}(x) \;.\!\rightarrow\!.\; (w)_x = h(x)) \,.$$

One can easily find such a w by examining sufficiently large segments of f_0, f_1, g, and $h_{a_0 \cdots \hat{a}_j \cdots a_{k-1}}$. We claim that $\{e\}^w(n) = \{e\}^{h_-}(n)$. Indeed, if v is an initial segment of h_- such that $v \in F^* \wedge v \operatorname{ext} s_0 \wedge \{e\}^v(n)$

is defined $\wedge \{e\}^v(n) = \{e\}^{h-}(n)$, then $\{e\}^v(n) = \{e\}^w(n)$ by statement (j). Hence $\{e\}^w(n) = \{e\}^{h-}(n)$ and so we have computed $\{e\}^{h-}(n)$ from $f_0, f_1, g, h_{a_0 \cdots \hat{a}_j \cdots a_{k-1}}$; thus $\{e\}^{h-}$ is recursive in these functions.

Case 3. Case 1 and Case 2_j are false for all j. We claim that for each j, $0 \leqq j \leqq k - 1$, following statement (j') holds:

$$\begin{aligned}&(s)(y)(En)(Eu)(Ew)[s \in F^* \wedge y \in F^* \wedge lhs = lhy\\ &\quad \to \{u \in F^* \wedge w \in F^* \wedge u \operatorname{ext} s \wedge w \operatorname{ext} y\\ &\wedge \{e\}^u(n) \text{ is defined } \wedge \{e\}^w(n) \text{ is defined } \wedge lhu = lh\, w\\ &\wedge \{e\}^u(n) \neq \{e\}^w(n) \wedge (x)_{lhs \leqq x < lhu}(P_{a_0}(x) \vee \cdots \vee\\ &\hat{P}_{a_j}(x) \vee \cdots \vee P_{a_{k-1}}(x) .\!\to\!. (u)_x = (w)_x)\}] .\end{aligned}$$

To prove (j') suppose $s, y \in F^*$. Since statement (j) is false there is an n, a u' and a u'' such that $u', u'' \in F^*$ and $lhu' = lhu''$ and $\{e\}^{u'}(n)$ is defined and $\{e\}^{u''}(n)$ is defined and u' extends s and u'' extends s and $\{e\}^{u'}(n) \neq \{e\}^{u''}(n)$ and

$$(x)_{lhs \leqq x < lhu'}(P_{a_0}(x) \vee \cdots \vee \hat{P}_{a_j}(x) \vee \cdots \vee P_{a_{k-1}}(x) .\!\to\!. (u')_x = (u'')_x) .$$

Let y' be defined by $(y')_x = (y)_x$ for $x < lhy$ and $(y')_x = (u')_x$ for $lhy \leqq x < lhu'$. Then since Case 1 is false there is a w such that $w \in F^*$ and w extends y' and $\{e\}^w(n)$ is defined. Now either $\{e\}^{u'}(n)$ or $\{e\}^{u''}(n)$ is different from $\{e\}^w(n)$; suppose without loss of generality that the first one is. Define u by $(u)_x = (u')_x$ for $x < lhu'$ and $(u)_x = (w)_x$ for $lhu' \leqq x < lhw$. Then n, u, and w have the properties described in statement (j').

We will use the truth of these statements (j') to define u_0, u_1, and v.

Let $m^* = (\mu n)(f_0(n) \neq f_1(n))$. Let $v(0) = 0$, $v(1) = g(m^* + 1)$, and $u_i(m) = f_0(m)$ when $m < v(1)$. Fix $t \geqq 0$ and suppose that $v(m)$ has been defined for $m \leqq t + 1$, that $v(0) < v(1) < \cdots < v(t + 1)$, and that $u_i(m)$ has been defined for all $m < v(t + 1)$. We shall define $v(t + 2)$ and $u_i(m)$ for $v(t + 1) \leqq m < v(t + 2)$.

We must first examine t to decide which of the infinite number of functions should be the one varying in this interval. If $\phi(t) = a_j$ for some j we shall use statement (j'); if $\phi(t) \neq a_j$ for all j we shall use statement $(0')$.

For each $i \leqq 2^{t+1}$ we define a pair (x_i, y_i) of partial functions with finite domains. Let x_0 and y_0 be the partial functions whose domain is empty. Fix i so that $0 < i \leqq 2^{t+1}$ and suppose that (x_{i-1}, y_{i-1}) has been defined. Let $i = c_0 2^0 + c_1 2^1 + \cdots + c_{t+1} 2^{t+1}$ where each c_j is either 0 or 1. We assume that domain of $x_{i-1} =$ domain of $y_{i-1} = \{m \mid v(t + 1) \leqq m < z\}$ where $z = g(r)$ for some r.

We define two initial segments, s and y:

$$
\begin{array}{ll}
s(m) = y(m) = u_{a-}(m) & \text{if } v(j) \leqq m < v(j+1),\, c_j = a, \text{ and } j \leqq t\,; \\
s(m) = x_{i-1}(m) & \text{if } v(t+1) \leqq m < z\,; \\
y(m) = y_{i-1}(m) & \text{if } v(t+1) \leqq m < z. \quad \text{Then } s, y \in F^*\,.
\end{array}
$$

Hence by the appropriate assumption (j') there is a natural number n and segments u, w such that s, y, n, u, w have the properties described in statement (j'). Choose these minimal. Define $x_i(m) = u(m)$ and $y_i(m) = w(m)$ for all m such that $v(t+1) \leqq m < lhu$. The assumptions we made concerning x_{i-1} and y_{i-1} remain true when $i-1$ is replaced by i. We thus proceed to get x_{t^*} and y_{t^*} where $t^* = 2^{t+1}$.

Let m_0 be such that $g(m_0) = v(t+1) +$ (cardinality of domain of x_{t^*}), and let $r_0 = (\mu r)(\phi(r) = \phi(t) \wedge g(r+1) \geqq g(m_0))$ and define $v(t+2) = g(r_0+2)$.

If $\phi(t) = a_j$, then we want variation in the a_j'th function in this interval, so define

$$
\begin{aligned}
u_0(m) = \{(\mu s)(s \in F^*(f_0, f_1, g) \wedge lhs = g(m_0) \wedge \\
(x)_{v(t+1) \leqq x < lhs}(P_{a_0}(x) \vee \cdots \vee P_{a_{k-1}}(x) \,.\!\!\longrightarrow\!.\, (s)_x = (x_{t^*})_x)\}_m\,,
\end{aligned}
$$

and $u_1(m)$ to be the same except that y_{t^*} replaces x_{t^*}, for all m such that $v(t+1) \leqq m < g(m_0)$. Furthermore, for $g(m_0) \leqq m < g(r_0+1)$, let $u_0(m) = u_1(m) = f_0(m)$; and for $g(r_0+1) \leqq m < g(r_0+2)$, let $u_i(m) = f_i(m)$.

If $\phi(t) \neq a_j$ for any j, then we want one of the other functions to vary on this interval, so we define u_0 and u_1 exactly as above *except* that for $v(t+1) \leqq m < g(m_0)$, both $u_0(m)$ and $u_1(m)$ are equal to the expression with x_{t^*}.

The assumptions we made concerning the values of $u_i(m)$ for $m < v(t+1)$ and $i < 2$ remain true when $t+1$ is replaced by $t+2$. We thus proceed to get functions u_0, u_1, v such that (u_0, u_1, v) is clearly a special triple and satisfies (i).

We need only show that (u_0, u_1, v) satisfies (ii) and (iii) (C). As for the first, we must show that at each stage of the construction of u_0, u_1, v the choices made are made recursively in f_0, f_1, g and that this is done uniformly (with respect to the stages.) The latter is clear since what is done between $v(n+1)$ and $v(n+2)$ depends only on $\phi(n)$. As to the former, we choose, for given s and y, an n, a u, and a v according to statement (j'). We show that this is done recursively in f_0, f_1, g. Indeed (j') can be written in the form $(s)(y)(En)(Eu)(Ew)(Ey')(Ey'')Q(s, y, n, u, w, y', y'')$ where y' and y'' are to be considered as Godel numbers of computations for $\{e\}^u(n)$ and $\{e\}^w(n)$ and the predicate Q is recursive in f_0, f_1, g. To make our choice we look successively at quintuples (n, u, w, y', y'') until we find one that works. Thus with the stipulation that where we said above "choose n, u, w minimal" we meant "choose n, u, w so that (n, u, w, y', y'') is minimal," it is clear that u_0, u_1, v are recursive in f_0, f_1, g.

To show that (iii) (C) holds, we suppose that $h \in F(u_0, u_1, v)$ and that $\{e\}^{h_-}(m)$ is defined for all m. We must show that h_- is recursive in $f_0, f_1, g, \{e\}^{h_-}$. Fix $t \geqq 0$. We indicate how to obtain the values of $h_-(m)$ for $v(t+1) \leqq m < v(t+2)$ from $f_0, f_1, g, \{e\}^{h_-}$, and the values of $h_-(m)$ for $m < v(t+1)$. Note that the values of $h_-(m)$ for $m < v(1)$ are obtained quite easily from f_0, f_1, and g.

Now since $h \in F(u_0, u_1, v)$ it follows that for each $j < t+1$ there is an a such that $(m)(v(j) \leqq m < v(j+1) \,.\!\rightarrow\!.\, h_-(m) = u_{a-}(m))$. For each j, let c_j be the least a for which this holds. Let $i = 2^{t+1}$ if $(j)(j < t+1 \,.\!\rightarrow\!.\, c_j = 0)$ and otherwise let $i = c_0 2^0 + c_1 2^1 + \cdots + c_t 2^t$.

Consider now the definition of x_i and y_i in Case 3. Clearly $s(m) = y(m) = h_-(m)$ for $m < v(t+1)$. Also $\{e\}^u(n)$ and $\{e\}^w(n)$ are defined and are unequal, and u, w, and n can be computed from f_0, f_1, g and the values of $h_-(m)$ for $m < v(t+1)$; this is so because Q is recursive. Now either u or w is an initial segment of h_-, and knowing which one determines what $h_-(m)$ is on all of the interval $v(t+1) \leqq m < v(t+2)$; but $\{e\}^{h_-}(n)$ can be only one of $\{e\}^u(n)$ and $\{e\}^w(n)$, so that knowing $\{e\}^{h_-}$ determines what h_- is on the interval.

This completes the proof of Lemma 2.

THEOREM. *Let T be the upper semi-lattice of finite subsets of N. Then T can be embedded as an initial segment in the upper semi-lattice of degrees of recursive unsolvability.*

Proof. If $\{a_0 < a_1 < \cdots < a_{k-1}\}$ is a finite subset of N then we shall have $\underline{h}_{a_0 \cdots a_{k-1}}$ correspond to it, where h is the function we are about to construct. Thus in particular $\underline{0}$ will correspond to ϕ. In order that this be an embedding of the finite subsets of N as an initial segment of the degrees it is necessary and sufficient that:

(i) if $\{a_0, \cdots, a_{k-1}\}$ and $\{b_0, \cdots, b_{r-1}\}$ are distinct sets, then $\underline{h}_{a_0 \cdots a_{k-1}} \neq \underline{h}_{b_0 \cdots b_{r-1}}$;

(ii) if $\{a_0, \cdots, a_{k-1}\}$ is included in $\{b_0, \cdots, b_{r-1}\}$, then $\underline{h}_{a_0 \cdots a_{k-1}} \leqq \underline{h}_{b_0 \cdots b_{r-1}}$;

(iii) if $\underline{d} < \underline{h}_{a_0 \cdots a_{k-1}}$, then for some $j, 0 \leqq j \leqq k-1$, we have $\underline{d} \leqq \underline{h}_{a_0 \cdots \hat{a}_j \cdots a_{k-1}}$.

Note that (i) includes the statement that h_i is not recursive, and together with (iii) implies that h_i is of minimal degree. Note too that we need do nothing to guarantee that (ii) holds; it follows from the definition of $h_{a_0 \cdots a_{k-1}}$.

We define a sequence $(f_0^\alpha, f_0^\alpha, g^\alpha)$ of special triples, each a contraction of (or equal to) the preceding one, such that for each α, f_0^α, f_1^α, and g^α are recursive.

Indeed define f_0^0, f_1^0, g^0 as follows:

$$\begin{aligned} g^0(0) &= 0 \qquad g^0(1) = 1 \\ g^0(n+2) &= (\mu t)(Er)[t = p^r_{\phi(n)} + 1 \wedge g^0(n+1) < t] \\ f_0^0(m) &= 0 \qquad \text{for all } m \\ f_1^0(m) &= 1 \qquad \text{if } g^0(n+1) \leqq m < g(n+2) \wedge (Er)(m = p^r_{\phi(n)}) \\ &= 0 \qquad \text{otherwise} . \end{aligned}$$

Then clearly (f_0^0, f_1^0, g^0) is a special triple and f_0^0, f_1^0, and g^0 are recursive functions.

We proceed inductively as follows:

Case 1. $\alpha + 1$ is not of the form $2^e 3^A 5^B 7^t$, where A and B represent $\{a_0, \cdots, a_{k-1}\}$ and $\{b_0, \cdots, b_{r-1}\}$ in some standard coding, $a_0 < \cdots < a_{k-1}$, $b_0 < \cdots < b_{r-1}$, and $t < 2$. In this case we set $f_0^{\alpha+1} = f_0^\alpha$, $f_1^{\alpha+1} = f_1^\alpha$, and $g^{\alpha+1} = g^\alpha$.

Case 2. The hypothesis of Case 1 is false and $t = 0$. In this case we want to guarantee that if $\{a_0, \cdots, a_{k-1}\}$ is not included in $\{b_0, \cdots, b_{r-1}\}$, then $h_{a_0 \cdots a_{k-1}}$ is not recursive in $h_{b_0 \cdots b_{r-1}}$ with Godel number e. Of course if $\{a_0, \cdots, a_{k-1}\}$ is included in $\{b_0, \cdots, b_{r-1}\}$, then we do not want to do this (good thing—we can't) so we proceed as in Case 1.

Let j be minimal such that $a_j \notin \{b_0, \cdots, b_{r-1}\}$; we will use the fact that $(f_0^\alpha, f_1^\alpha, g^\alpha)$ is a special triple to find an interval in which the a_j'th function varies but for each i the b_i'th function does not; we then compute a value for $\{e\}^{h_{-}}(p^x_{a_j})$ (where $_{-}$ of course stands for $b_0 \cdots b_{r-1}$) and choose for $h(p^x_{a_j})$ the opposite value. The end result is that for each $h \in F(f_0^{\alpha+1}, f_1^{\alpha+1}, g^{\alpha+1})$, h_{a_j} is not recursive in $h_{b_0 \cdots b_{r-1}}$ with Godel number e. Once this has been done for every e, we conclude that $h_{a_0 \cdots a_{k-1}}$ is not recursive in $h_{b_0 \cdots b_{r-1}}$, for h_{a_j} is recursive in $h_{a_0 \cdots a_{k-1}}$ but not in $h_{b_0 \cdots b_{r-1}}$.

To do this we let

$$\begin{aligned} n_0 = (\mu n)(Em)[g^\alpha(m+1) \leqq n < g^\alpha(m+2) \wedge (Et)(n = p^t_{\phi(m)}) \wedge \\ a_j = \phi(m) \wedge f_0^\alpha(n) \neq f_1^\alpha(n)] ; \end{aligned}$$

n_0 exists since $(f_0^\alpha, f_1^\alpha, g^\alpha)$ is a special triple.

Subcase 1. $\sim(Ew)[w \in F^*(f_0^\alpha, f_1^\alpha, g^\alpha) \wedge \{e\}^{w_{-}}(n_0)$ is defined]. In this case let s be defined by $s(m) = f_0^\alpha(m)$ for $0 \leqq m < g^\alpha(2)$ and use Lemma 1 to get a special triple $(f_0^{\alpha+1}, f_1^{\alpha+1}, g^{\alpha+1})$ satisfying the induction requirements. Note particularly, here and in the subsequent cases, that since these functions are recursive in $f_0^\alpha, f_1^\alpha, g^\alpha$ which are assumed to be recursive, we have that $f_0^{\alpha+1}, f_1^{\alpha+1}, g^{\alpha+1}$ are recursive.

Subcase 2. $(Ew)[w \in F^*(f_0^\alpha, f_1^\alpha, g^\alpha) \wedge \{e\}^{w}\text{-}(n_0)$ is defined]. Let v be the least such w and let

$$\begin{aligned} t_0 = (\mu t)[t \in F^*(f_0^\alpha, f_1^\alpha, g^\alpha) \wedge lht > n_0 \wedge lht \geqq lhv \wedge \\ (t)_{n_0} \neq \{e\}^{v}\text{-}(n_0) \wedge (x)_{0 \leqq x < lhv}(P_{b_0}(x) \vee \cdots \vee P_{b_{r-1}}(x) .\!\rightarrow\!. (t)_x = (v)_x)] \,. \end{aligned}$$

Now apply Lemma 1 with t_0 as the initial segment to obtain $(f_0^{\alpha+1}, f_1^{\alpha+1}, g^{\alpha+1})$.

Case 3. The hypothesis of Case 1 is false and $t = 1$. In this case we wish to guarantee that if d is recursive in $h_{a_0 \cdots a_{k-1}}$ with Godel number e, then either $h_{a_0 \cdots a_{k-1}}$ is recursive in d or d is recursive in some $h_{a_0 \cdots \hat{a}_j \cdots a_{k-1}}$. But to do this we need only apply Lemma 2 with e and $a_0, \cdots, a_{k-1}$, to get $(f_0^{\alpha+1}, f_1^{\alpha+1}, g^{\alpha+1})$. (Note the role that the recursiveness of f_0^α, f_1^α, and g^α play here.)

This is the end of the construction. It is clear that there is exactly one function h in the intersection of the sequence of special triples and that this function has exactly the properties we want. The enterprising soul will see that in fact we have embedded T in the set of degrees $\leqq \underline{0}''$.

COROLLARY. *If B is finite, then $P(B)$ and any initial segment of $P(B)$ is embeddable as an initial segment in the upper semi-lattice of degrees of unsolvability.*

(This result has been obtained independently by J. Shoenfield.)

I expect that any finite partially ordered set with a least element can be embedded as an initial segment of the degrees and that this can be shown using methods not much more complicated than these. Indeed, at first glance, or even second glance, it might seem that slight modifications of the above construction give a proof of the conjecture. This is not so; indeed, technical difficulties appear out of nowhere, and they even make it impossible to construct a degree which is greater than exactly three distinct minimal degrees, and nothing else except $\underline{0}$. It remains to be seen whether these technical difficulties are really essential difficulties, or whether some simple trick will enable them to disappear.

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