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If G is any monotone decomposition of R^3 , let H_G denote the union of the nondegenerate elements of G , and let P_G denote the projection map from R^3 onto the decomposition space R^3/G associated with G . Suppose that F and G are monotone decompositions of R^3 such that each of $\text{Cl}(P_F[H_F])$ and $\text{Cl}(P_G[H_G])$ is compact and 0-dimensional. Then F and G are *equivalent* decompositions of R^3 if and only if there is a homeomorphism h from R^3/F onto R^3/G such that

$$h[\text{Cl}(P_F[H_F])] = \text{Cl}(P_G[H_G]).$$

A necessary and sufficient condition for two decompositions to be equivalent is given. It is shown that there is a decomposition with only a countable number of nondegenerate elements which is equivalent to the dogbone decomposition, and several related results are obtained.

By introducing the idea of equivalent decompositions of R^3 , we are able to analyze in a precise way, a process that seems quite natural in the study of monotone decompositions of R^3 of the type we are considering. If F is a monotone decomposition of R^3 , the stipulation that $\text{Cl} P_F[H_F]$ be a compact 0-dimensional set is equivalent to the following condition: There is a sequence M_1, M_2, M_3, \dots of compact 3-manifolds-with-boundary in R^3 such that for each positive integer j , $M_{j+1} \subset \text{Int} M_j$ and g is a nondegenerate element of F if and only if g is a nondegenerate component of $\bigcap_{j=1}^{\infty} M_j$.

A process one finds useful in certain situations is one that involves a sequence f_1, f_2, f_3, \dots of homeomorphisms from R^3 onto R^3 such that (1) f_1 shrinks or stretches M_1 , (2) f_2 agrees with f_1 on $R^3 - M_1$ and shrinks or stretches M_2 , (3) f_3 agrees with f_2 on $R^3 - M_2$ and shrinks or stretches M_3 , and so on. The "new" decomposition has as its nondegenerate elements the nondegenerate components of

$$f_1[M_1] \cap f_2[M_2] \cap f_3[M_3] \cap \dots$$

We are able to show that under fairly mild restrictions, there exists such a sequence of homeomorphisms if and only if the original decomposition and the "new" one are equivalent in the sense of this paper.

We indicate some examples that illustrate these concepts. The first two examples give instances of previous applications of the ideas of this paper. The remaining ones are described in detail in the

present paper.

EXAMPLE 1. Meyer proved [10] that if C is a 3-cell in R^3 such that $\text{Bd } C$ is locally polyhedral except at points of an arc α on $\text{Bd } C$, then R^3/C is homeomorphic to R^3/α .

EXAMPLE 2. Bing described [6] a 2-sphere S in R^3 such that S is locally wild at each point of S and S bounds a 3-cell B in R^3 . Armentrout proved [1] that there is a 3-cell B' in R^3 such that $\text{Bd } B'$ is locally polyhedral except on a Cantor set on $\text{Bd } B'$ and R^3/B is homeomorphic to R^3/B' .

EXAMPLE 3. Suppose G is a monotone decomposition of R^3 such that there is a sequence M_1, M_2, M_3, \dots of compact 3-manifolds-with-boundary as described above. Suppose further that each component of each M_i is a 3-cell-with-handles. Then G is equivalent to a decomposition into 1-dimensional continua and one-point sets; see §7.

EXAMPLE 4. Bing's dogbone decomposition [5] is equivalent to a decomposition into one-point sets and at most countably many nondegenerate continua; see §4.

EXAMPLE 5. In §3 of [7], Bing described a point-like decomposition G of R^3 with only countably many nondegenerate elements such that R^3/G is not homeomorphic to R^3 . There exists a decomposition F of R^3 such that F is equivalent to G and F has uncountably many nondegenerate elements; see §5.

2. Notation and terminology. The statement that G is a *monotone* decomposition of R^3 means that G is an upper semi-continuous decomposition of R^3 into compact continua. A compact continuum K in R^3 is *point-like* if and only if $R^3 - K$ is homeomorphic to the complement, in R^3 , of a one-point set. A set M in R^3 is *cellular* if and only if there is a sequence C_1, C_2, C_3, \dots of 3-cells in R^3 such that for each i , $C_{i+1} \subset \text{Int } C_i$ and $M = \bigcap_{i=1}^{\infty} C_i$. For compact continua in R^3 , "point-like" and "cellular" are equivalent [12]. The statement that G is a *point-like* decomposition of R^3 means that G is a monotone decomposition of R^3 into point-like sets.

We shall use the notation and terminology introduced in the introduction.

If M is a 3-manifold-with-boundary, M need not be connected, and $\text{Bd } M$ and $\text{Int } M$ denote the boundary and interior, respectively, of M .

The statement that the subset K of R^3 is a *3-cell-with-handles* means that there is a finite collection C, C_1, C_2, \dots , and C_n of 3-cells such that if $i = 1, 2, \dots$, or n , $C_i \cap C$ is the union of two disjoint discs, and $C_i \cap C = (\text{Bd } C_i) \cap (\text{Bd } C)$, and if i and j are distinct, C_i and C_j are disjoint. Such a collection C, C_1, C_2, \dots , and C_n of 3-cells will be called a *standard decomposition* of K .

We shall use Cl to denote topological closure. If X is a subset of R^3 and ε is a positive number, then $V(X, \varepsilon)$ denotes the ε -neighborhood of X in R^3 .

Suppose G is a monotone decomposition of R^3 . Then M_1, M_2, M_3, \dots is a *defining sequence* for G if and only if M_1, M_2, M_3, \dots is a sequence such that (1) for each positive integer i , M_i is a compact 3-manifold-with-boundary such that $M_{i+1} \subset \text{Int } M_i$ and (2) g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^{\infty} M_i$. G has a defining sequence if and only if $\text{Cl } P_G[H_G]$ is a compact 0-dimensional set. G is *definable by 3-cells-with-handles* if and only if G has a defining sequence M_1, M_2, M_3, \dots such that for each positive integer i , each component of M_i is a 3-cell-with-handles. G is a *toroidal* decomposition of R^3 if and only if G has a defining sequence M_1, M_2, M_3, \dots such that for each positive integer i , each component of M_i is a solid torus (3-cell with one handle).

3. The existence of sequences of homeomorphisms. In this section we establish, under fairly weak conditions on the decompositions involved, the equivalence of two decompositions with the existence of a sequence of homeomorphisms h_1, h_2, h_3, \dots from R^3 to R^3 as indicated in the introduction.

A compact continuum M in R^3 is *semi-cellular* if and only if for each open set U in R^3 containing M , there is an open set V lying in U and containing M and such that each simple closed curve in V is null-homotopic in U . Every point-like compact continuum in R^3 is semi-cellular, since each such set is cellular. Each compact absolute retract in R^3 is semi-cellular. Since there exist noncellular arcs in R^3 , the two categories above are not identical. An example of a semi-cellular compact continuum in R^3 neither cellular nor an absolute retract may be obtained as follows: Let T_1, T_2, T_3, \dots be a sequence of solid tori (3-cells with one handle) in R^3 such that for each i , $T_{i+1} \subset \text{Int } T_i$, T_2 lies in T_1 as shown in Figure 1, T_3 lies in T_2 as T_2 lies in T_1 , and for each i , T_{i+1} lies in T_i as T_i lies in T_{i-1} . Then $\bigcap_{i=1}^{\infty} T_i$ is a continuum with the desired properties.

LEMMA 1. Suppose that F and G are monotone decompositions of R^3 such that $\text{Cl } P_F[H_F]$ and $\text{Cl } P_G[H_G]$ are compact 0-dimensional

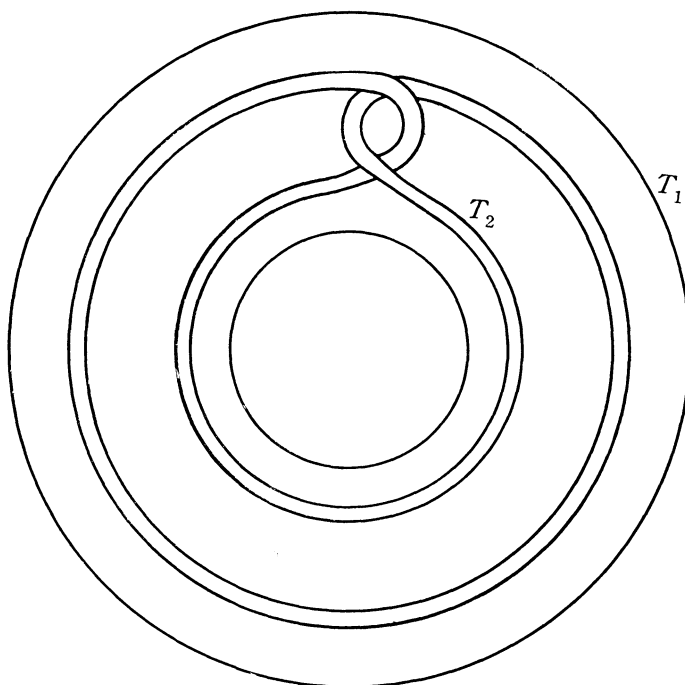


FIGURE 1.

sets. Suppose that M is a compact polyhedral 3-manifold-with-boundary, each component of which is a 3-cell-with-handles, such that $\text{Cl } H_F \subset \text{Int } M$. Suppose that each element of G is semi-cellular. Suppose that there is a homeomorphism h from R^3/F onto R^3/G such that $h[\text{Cl } P_F[H_F]] = \text{Cl } P_G[H_G]$. Let φ be the function from $R^3 - \text{Cl } H_F$ onto $R^3 - \text{Cl } H_G$ such that if $x \in (R^3 - \text{Cl } H_F)$, $\varphi(x) = P_G^{-1}hP_F(x)$. Then there is a homeomorphism f from R^3 onto R^3 such that

- (1) if $x \in R^3 - \text{Int } M$, $f(x) = \varphi(x)$ and
- (2) $f[M] = P_G^{-1}hP_F[M]$, and each component of $f[M]$ is a 3-cell-with-handles.

Proof. Let M_1, M_2, \dots , and M_n be the components of M . If $i = 1, 2, \dots$, or n , $\varphi|_{\text{Bd } M_i}$ is a homeomorphism, and thus $\varphi[\text{Bd } M_i]$ is a compact tame 2-manifold-with-boundary and $\varphi[\text{Bd } M_i]$ bounds a compact 3-manifold-with-boundary N_i in R^3 . Since $\bigcup_{i=1}^n \text{Bd } N_i$ is the boundary of the connected 3-manifold-with-boundary $\varphi[R^3 - \text{Int } M]$, the sets N_1, N_2, \dots , and N_n are mutually disjoint. Let N denote $\bigcup_{i=1}^n N_i$. It is not hard to see that N contains $\text{Cl } H_G$, φ takes $R^3 - \text{Int } M$ homeomorphically onto $R^3 - \text{Int } N$, and $\varphi[\text{Bd } M] = \text{Bd } N$. Therefore, in order to describe f , it is sufficient to construct, for each i , an extension of $\varphi|_{\text{Bd } M_i}$ to M_i .

Suppose then that $i = 1, 2, \dots$, or n . Since M_i is a 3-cell-with-

handles, there is a finite set $\{D_{i1}, D_{i2}, \dots, D_{im_i}\}$ of mutually disjoint polyhedral discs such that (1) if $j = 1, 2, \dots$, or m_i , $\text{Bd } D_{ij} \subset \text{Bd } M_i$ and $\text{Int } D_{ij} \subset \text{Int } M_i$ and (2) the closures of the components of

$$M - \bigcup_{i=1}^{m_i} D_{ij}, \quad C_{i1}, C_{i2}, \dots, C_{ik_i},$$

are polyhedral 3-cells such that if $t = 1, 2, \dots$, or k_i , $\text{Bd } C_{it}$ is the union of a punctured disc A_{it} and certain ones of the discs D_{i1}, D_{i2}, \dots , and D_{im_i} such that if D_{ij} and A_{it} intersect, $D_{ij} \cap A_{it} = \text{Bd } D_{ij}$ and is also a boundary curve of A_{it} . If $t = 1, 2, \dots$, or k_i , let S_{it} denote $\text{Bd } C_{it}$.

If $j = 1, 2, \dots$, or m_i , there is a polyhedral subdisc D'_{ij} of D_{ij} such that $D'_{ij} \subset \text{Int } D_{ij}$ and $(\text{Cl } H_F) \cap D_{ij} \subset \text{Int } D'_{ij}$. Let B_{ij} denote the annulus $D_{ij} - \text{Int } D'_{ij}$.

Now $P_G^{-1}hP_F[(\text{Cl } H_F) \cap D_{ij}]$ is compact and lies in $\text{Int } N_i$. Since each element of G is semi-cellular, there exists a finite collection $\{(U_1, V_1), (U_2, V_2), \dots, (U_r, V_r)\}$ of pairs of open sets in R^3 such that

(1) if $t = 1, 2, \dots$, or r , $V_t \subset U_t$, $U_t \subset \text{Int } N_i$, each simple closed curve in V_t is null-homotopic in U_t , and V_t is a union of elements of G , and

(2) each element of G that intersects $P_G^{-1}hP_F[D_{ij}]$ lies in some one of V_1, V_2, \dots , and V_r .

There is a triangulation T of D'_{ij} such that if σ is any 2-simplex of T , then for some t , $P_G^{-1}hP_F[\sigma] \subset V_t$. Let $\sigma_1, \sigma_2, \dots$, and σ_q denote the 2-simplexes of T .

Let $\langle x_{10}x_{11}x_{12} \rangle$ denote the 2-simplex σ_1 . Let y_{10}, y_{11} , and y_{12} be points of $P_G^{-1}hP_F(x_{10}), P_G^{-1}hP_F(x_{11})$, and $P_G^{-1}hP_F(x_{12})$, respectively. Since G is monotone, $P_G^{-1}hP_F[\langle x_{10}x_{11} \rangle]$ is a compact continuum and near it we can choose a polygonal arc $\langle y_{10}y_{11} \rangle$ such that if σ is any 2-simplex of T having $\langle x_{10}x_{11} \rangle$ as an edge and $P_G^{-1}hP_F[\langle x_{10}x_{11} \rangle] \subset V_s$, then $\langle y_{10}y_{11} \rangle \subset V_s$. It is to be true that if $\langle x_{10}x_{11} \rangle$ misses $\text{Cl } H_F$ then $\langle y_{10}y_{11} \rangle = P_G^{-1}hP_F[\langle x_{10}x_{11} \rangle]$. In a similar manner we choose polygonal arcs $\langle y_{11}y_{12} \rangle$ and $\langle y_{10}y_{12} \rangle$. We adjust these slightly near $\text{Cl } H_G$ so that if $\gamma_1 = \langle y_{10}y_{11} \rangle \cup \langle y_{11}y_{12} \rangle \cup \langle y_{10}y_{12} \rangle$, then γ_1 is a simple closed curve. Now for some t_1 , $P_G^{-1}hP_F[\sigma_1] \subset V_{t_1}$ and by construction $\gamma_1 \subset V_{t_1}$. Hence there is a polygonal singular disc τ_1 in U_{t_1} and bounded by γ_1 .

Corresponding to σ_2 , we construct γ_2 and τ_2 such that for some t_1, τ_2 is a polyhedral singular disc in U_{t_1} . It is to be the case that if a vertex of σ_2 belongs to σ_1 , we make the same choice for that vertex of σ_2 as was made for σ_1 , and similarly if an edge of σ_2 lies in σ_1 . In addition, if either a vertex or edge of σ_2 misses $\text{Cl } H_F$, then for the corresponding set in γ_2 , we use its image under φ and do not move it in adjusting to obtain γ_2 .

Continue this process. There result polyhedral singular discs

τ_1, τ_2, \dots , and τ_q in $\text{Int } N_i$ such that $\mathbf{U}_{t=1}^q \tau_t$ is a singular disc whose boundary is $\varphi[\text{Bd } D'_{ij}]$ and which lies in $\text{Int } N_i$. By applying Dehn's lemma [10] to the polyhedral singular disc $\varphi[B_{ij}] \cup (\mathbf{U}_{t=1}^q \tau_t)$, we see that there is a disc A'_{ij} such that $\text{Bd } A'_{ij} = \varphi[\text{Bd } D_{ij}]$ and $\text{Int } A'_{ij} \subset \text{Int } N_i$.

By well-known techniques it may be shown that there exist mutually disjoint discs A_{i1}, A_{i2}, \dots , and A_{im_1} such that for each j , $\text{Bd } A_{ij} = \varphi[\text{Bd } D_{ij}]$ and $\text{Int } A_{ij} \subset \text{Int } N_i$.

Recall that if $t = 1, 2, \dots$, or k_i , C_{it} is a 3-cell contained in M_i , $S_{it} = \text{Bd } C_{it}$, and A_{it} is the punctured disc $S_{it} - \mathbf{U}_{j=1}^{m_i} \text{Int } D_{ij}$. It is clear that if $D_{ij_1}, D_{ij_2}, \dots$, and $D_{ij_{w_t}}$ are those discs of D_{i1}, D_{i2}, \dots , and D_{im_i} whose boundaries are contained in A_{it} , then $\varphi[A_{it}] \cup (\mathbf{U}_{p=1}^{w_t} A_{ij_p})$ is a tame 2-sphere S'_{it} .

We can easily show that if s and t are distinct, then $\text{int } S'_{it}$ and $\text{int } S'_{is}$ are disjoint, where "int" denotes the interior, in E^3 , of a 2-sphere. Both $\text{int } S'_{it}$ and $\text{int } S'_{is}$ are contained in $\text{Int } N_i$. If S'_{it} and $\text{int } S'_{it}$ intersect, then some point of either $\varphi[A_{it}]$ or $\varphi[A_{is}]$ lies in $\text{Int } N_i$. This is a contradiction, so $\text{int } S'_{it}$ and $\text{int } S'_{is}$ are disjoint.

There is a homeomorphism θ_{i1} from S_{i1} onto S'_{i1} such that (1) $\theta_{i1}|A_{i1} = \varphi|A_{i1}$ and (2) if $D_{ij} \subset S_{i1}$, then $\theta_{i1}[D_{ij}] = A_{ij}$. There is a homeomorphism θ_{i2} from S_{i2} onto S'_{i2} such that (1) $\theta_{i2}|A_{i2} = \varphi|A_{i2}$, (2) if $D_{ij} \subset S_{i1} \cap S_{i2}$, then $\theta_{i2}|D_{ij} = \theta_{i1}|D_{ij}$, and (3) if $D_{ij} \subset S_{i2}$, $\theta_{i2}[D_{ij}] = A_{ij}$. If $t = 3, 4, \dots$, or k_i , there is a homeomorphism θ_{it} from S_{it} onto S'_{it} such that (1) $\theta_{it}|A_{it} = \varphi|A_{it}$, (2) if $s = 1, 2, \dots$, or $(t - 1)$ and $A_{ij} \subset S_{it} \cap S_{is}$, then $\theta_{it}|A_{ij} = \theta_{is}|A_{ij}$, and (3) if $D_{ij} \subset S_{it}$, $\theta_{it}[D_{ij}] = A_{ij}$.

If $t = 1, 2, \dots$, or k_i , there is a homeomorphism θ_{it}^* from C_{it} onto $(S'_{ij} \cup \text{int } S'_{ij})$ such that $\theta_{it}^*|S_{it} = \theta_{it}$. Now let φ_i be the function from M_i onto N_i defined as follows: If $x \in M_i$, let t be an integer such that $x \in C_{it}$, and let $\varphi_i(x)$ be $\theta_{it}^*(x)$. The function φ_i is well-defined because if $x \in C_{it} \cap C_{is}$, then $\theta_{it}^*(x) = \theta_{is}^*(x)$. It is easy to see that φ_i is a homeomorphism from M_i onto N_i and that $\varphi_i| \text{Bd } M_i = \varphi| \text{Bd } M_i$.

Now we are ready to define f . If $x \in R^3 - \text{Int } M$, then define $f(x)$ to be $\varphi(x)$. If $x \in M$, let i be the integer such that $x \in M_i$. Then define $f(x)$ to be $\varphi_i(x)$. It is easily seen that f is a homeomorphism from R^3 onto R^3 satisfying the conclusion of Lemma 1.

THEOREM 1. *Suppose that F and G are monotone decompositions of E^3 such that $\text{Cl } P_F[H_F]$ and $\text{Cl } P_G[H_G]$ are compact 0-dimensional sets. Suppose that F is definable by 3-cells-with-handles M_1, M_2, \dots . Suppose each element of G is semi-cellular. Then if F and G are equivalent decompositions, there exists a sequence f_1, f_2, f_3, \dots of homeomorphisms from R^3 onto R^3 such that (1) for each*

$$i, f_{i+1}|(R^3 - \text{Int } M_i) = f_i|(R^3 - \text{Int } M_i),$$

and (2) $f_1[M_1], f_2[M_2], f_3[M_3], \dots$ is a defining sequence for G .

Proof. Since F and G are equivalent, there is a homeomorphism h from R^3/F onto R^3/G such that $h[\text{Cl } P_F[H_F]] = \text{Cl } P_G[H_G]$. Let φ be the function from $R^3 - \text{Cl } H_F$ onto $R^3 - \text{Cl } H_G$ such that if

$$x \in (R^3 - \text{Cl } H_F), \quad \varphi(x) = P_G^{-1}hP_F(x).$$

Since F is definable by 3-cells-with-handles, there exists a defining sequence M_1, M_2, M_3, \dots for F such that for each positive integer i , each component of M_i is a 3-cell-with-handles. By Lemma 1, if i is any positive integer, there is a homeomorphism f_i from R^3 onto R^3 such that if $x \in E^3 - \text{Int } M_i$, $f_i(x) = \varphi(x)$. We will show that the sequence f_1, f_2, f_3, \dots satisfies the conclusion of Theorem 1.

Suppose i is any positive integer. Then $M_{i+1} \subset \text{Int } M_i$ since M_1, M_2, M_3, \dots is a defining sequence for H_F . Since

$$f_{i+1}|(R^3 - \text{Int } M_{i+1}) = \varphi|(R^3 - \text{Int } M_{i+1}),$$

then

$$f_{i+1}|(R^3 - \text{Int } M_i) = \varphi|(R^3 - \text{Int } M_i).$$

Since $f_i|(R^3 - M_i) = \varphi|(R^3 - \text{Int } M_i)$, it follows that

$$f_{i+1}|(R^3 - \text{Int } M_i) = f_i|(R^3 - \text{Int } M_i).$$

Suppose U is an open set in R^3 containing $\text{Cl } H_G$. Then $P_F^{-1}h^{-1}P_G[U]$ is open in R^3 and contains $\text{Cl } H_F$. Hence there is a positive integer n such that $M_n \subset P_F^{-1}h^{-1}P_G[U]$, and it follows that $P_G^{-1}hP_F[M_n] \subset U$. Since $f_n[M_n] = P_G^{-1}hP_F[M_n]$, $f_n[M_n] \subset U$. It is clear that for any i , $(\text{Cl } H_G) \subset f_i[M_i]$. Consequently, $f_1[M_1], f_2[M_2], f_3[M_3], \dots$ is a defining sequence for G . Hence Theorem 1 holds.

COROLLARY 1. *If F and G satisfy the hypothesis of Theorem 1, then G is definable by 3-cells-with-handles. If F is toroidal, so is G .*

Proof. We use the notation of Theorem 1. By Theorem 1, $f_1[M_1], f_2[M_2], f_3[M_3], \dots$ is a defining sequence for G . By Lemma 1, for each positive integer i , each component of $f_i[M_i]$ is a 3-cell-with-handles. Hence G is definable by 3-cells-with-handles. It is clear that if for each positive integer i , M_i is a solid torus, so is $f_i[M_i]$. Therefore, if F is toroidal, so is G .

THEOREM 2. *Suppose that F and G are monotone decompositions of R^3 such that $\text{Cl } P_F[H_F]$ and $\text{Cl } P_G[H_G]$ are compact 0-dimensional sets. Suppose that F has a defining sequence M_1, M_2, M_3, \dots and*

there exists a sequence $f_1, f_2 f_3, \dots$ of homeomorphisms from R^3 onto R^3 such that (1) for each $i, f_{i+1} | (R^3 - \text{Int } M_i) = f_i | (R^3 - \text{Int } M_i)$, and (2) $f_1[M_1], f_2[M_2], f_3[M_3], \dots$ is a defining sequence for G . Then F and G are equivalent.

Proof. We shall define a homeomorphism h from R^3/F onto R^3/G such that $h[\text{Cl } P_F[H_F]] = \text{Cl } P_G[H_G]$.

Suppose x is a point of R^3/F . Consider first the case where $x \in \text{Cl } P_F[H_F]$. Then $P_F^{-1}(x)$ is a one-point set and so there is a point y of R^3 such that $P_F(y) = x$. Further, $y \in \text{Cl } H_F$. Hence for some n_y , if $i > n_y$, $f_i(y) = f_{n_y}(y)$. Then define $h(x)$ to be the point $P_G f_{n_y}(y)$ of R^3/G .

Suppose $x \in \text{Cl } P_F[H_F]$. Then there is a sequence $M_{1j_1}, M_{2j_2}, M_{3j_3}, \dots$ such that for each k, M_{kj_k} is the component of M_k containing $P_F^{-1}(x)$. It is true, further, that $P_F^{-1}(x) = \bigcup_{k=1}^{\infty} M_{kj_k}$. Since $f_1[M_1], f_2[M_2], f_3[M_3], \dots$ is a defining sequence for H_G , then $\bigcap_{k=1}^{\infty} f_k[M_{kj_k}]$ is an element g_x of G . Define $h(x)$ to be the point z of R^3/G such that $P_G[g_x] = \{z\}$.

It is not hard to show, using the hypothesis, that h is a homeomorphism from R^3/F onto R^3/G such that $h[\text{Cl } P_F[H_F]] = \text{Cl } P_G[H_G]$.

THEOREM 3. *Suppose F and G are monotone decompositions of R^3 such that $\text{Cl } P_F[H_F]$ and $\text{Cl } P_G[H_G]$ are compact 0-dimensional sets. Suppose F is definable by 3-cells-with-handles and each element of G is semi-cellular. Then F and G are equivalent if and only if there exists a defining sequence M_1, M_2, \dots for F and a sequence f_1, f_2, f_3, \dots of homeomorphisms from R^3 onto R^3 such that (1) for each i ,*

$f_{i+1} | (R^3 - \text{Int } M_i) = f_i | (R^3 - \text{Int } M_i)$ and (2) $f_1[M_1], f_2[M_2], f_3[M_3], \dots$ is a defining sequence for G .

Theorem 3 is a corollary of Theorems 1 and 2.

We shall indicate now some conditions under which a monotone decomposition F of R^3 satisfies the hypothesis of Theorem 3 for F .

LEMMA 2. *Suppose that F is a monotone decomposition of R^3 such that $\text{Cl } P_F[H_F]$ is a compact 0-dimensional set. Then F is definable by 3-cells-with-handles provided it is true that if g is any element of F , g_0 is any subcontinuum of g embeddable in R^2 , and h is any embedding of g_0 in R^2 , then $h[g_0]$ does not separate R^2 . In particular, the condition stated holds provided g satisfies any one of the following:*

(1) g is tree-chainable (see [3] for definition).

- (2) g is snake-like (see [3] for definition).
- (3) g is a dendron.
- (4) g is an arc.

Lemma 2 may be established by the methods of [2].

4. The dogbone space. In this section it is proved that there is a decomposition F which is equivalent to the dogbone decomposition and such that F has only countably many nondegenerate elements. The notation and terminology of [5] will be used in this section.

LEMMA 3. Suppose f is a homeomorphism of A into R^3 , $B = f[A]$, $B_i = f[A_i]$, P_1, P_2, P_3, \dots , and P_m are disjoint horizontal planes in R^3 , there exist positive integers j and k such that $1 \leq j \leq k \leq m$ and B intersects only P_j, P_{j+1}, \dots , and P_k , and for each positive integer i , $1 \leq i \leq m$, each component of $B \cap P_i$ is a tame disc, $B \cap (\bigcup_{s < i} P_s)$ is contained in some component of $B - P_i$, $B \cap (\bigcup_{s > i} P_s)$ is contained in some component of $B - P_i$, and $B \cap P_i$ is contained in some component of $B - \bigcup_{s \neq i} P_s$. Then there exists a homeomorphism h of R^3 onto itself such that (1) h is point-wise fixed outside of B , (2) $h[B_1]$ intersects P_j, P_{j+1}, \dots , and P_k , (3) each of $h[B_2], h[B_3],$ and $h[B_4]$ intersects at most $k - j$ of the P_i 's, and (4) for $i = 1, 2, 3,$ or 4 , $h[B_i] \cap (\bigcup_{t=1}^m P_t)$ has the same properties as $B \cap (\bigcup_{t=1}^m P_t)$.

Proof. Adjust $\bigcup_{i=1}^4 f^{-1}[B_i] = \bigcup_{i=1}^4 A_i$ by a homeomorphism g of A onto itself such that g is fixed on the boundary and g carries $\bigcup_{i=1}^4 A_i$ to the positions indicated in Figure 2. Let h be fgf^{-1} . It can be assumed that $h[B_i]$ has small cross sectional diameter, $\bigcup_{i=1}^4 h[B_i]$

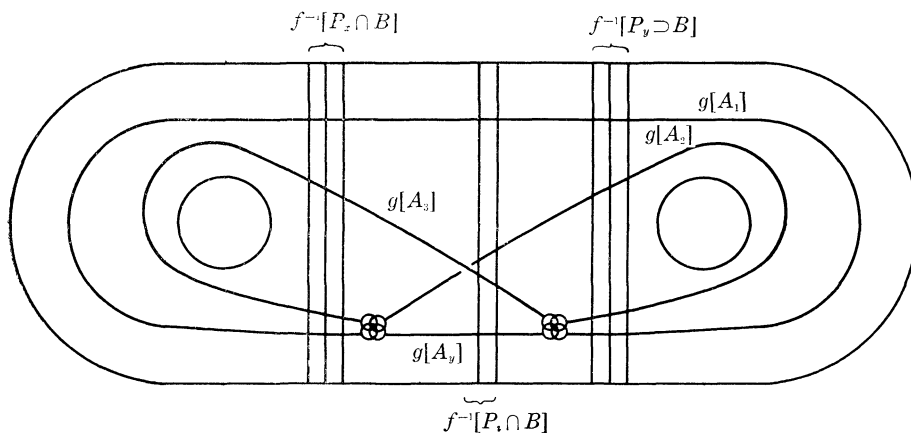


FIGURE 2.

and $\bigcup_{i=1}^m P_i$ are in relative general position, and each component of $(\bigcup_{i=1}^4 h[B_i]) \cap (\bigcup_{i=1}^m P_i)$ is a disc.

We will now construct the decomposition F . Let P_1, P_2, \dots , and P_r be horizontal planes which intersect A as shown in Figure 3. Apply Lemma 3 to A and P_1, P_2, \dots , and P_r to obtain a homeomorphism h_1 , and let $B_i = h_1[A_i]$. See Figure 3. Apply Lemma 3 to B_i and P_1, \dots , and P_r to obtain a homeomorphism h_2^i . Let $h_2 = h_2^1 h_2^2 h_2^3 h_2^4 h_1$ and let $B_{ij} = h_2[A_{ij}]$. This process is continued until $B_{ij\dots m}$ intersects at most one of the P_i 's. When $B_{ij\dots m}$ intersects only P_s , then discs are added to the collection of discs $B_{ij\dots m} \cap P_s$ so that the total collection cuts $B_{ij\dots m}$ in the same manner as P_1, \dots , and P_r cut up A , and so that each component of $B_{ij\dots m}$ in the complement of the collection of discs has diameter less than one half the diameter of B . A modified version of Lemma 3 is now applied to $B_{ij\dots m}$ and the collection of disjoint discs.

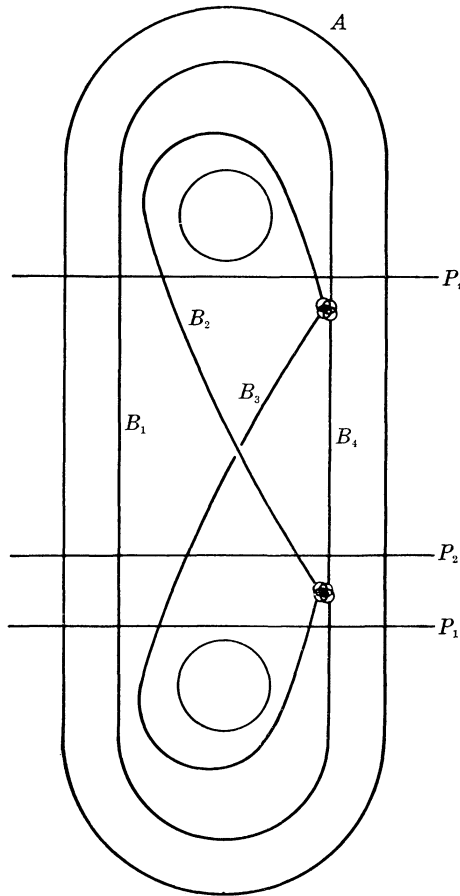


FIGURE 3.

F is the decomposition whose nondegenerate elements are the nondegenerate components of $A \cap (\cap B_i) \cap (\cup B_{ij}) \cap (\cup B_{ijk}) \cup \dots$. It is clear, using Theorem 2, that F is equivalent to the dogbone decomposition.

THEOREM 4. F has only countably many nondegenerate elements.

Proof. There is a one to one correspondence between the components of $A \cap (\cup B_i) \cap (\cap B_{ij}) \cap \dots$ and the set of all sequences into $\{1, 2, 3, 4\}$, where the sequence t corresponds to $A \cap B_{t(1)} \cap B_{t(1)t(2)} \cap \dots$. It will next be shown that f is a nondegenerate element of F if and only if the sequence corresponding to f converges to 1.

Suppose t is a sequence into $\{1, 2, 3, 4\}$, t converges to 1, and f corresponds to t . Then there exist disjoint discs E_1 and E_2 and an

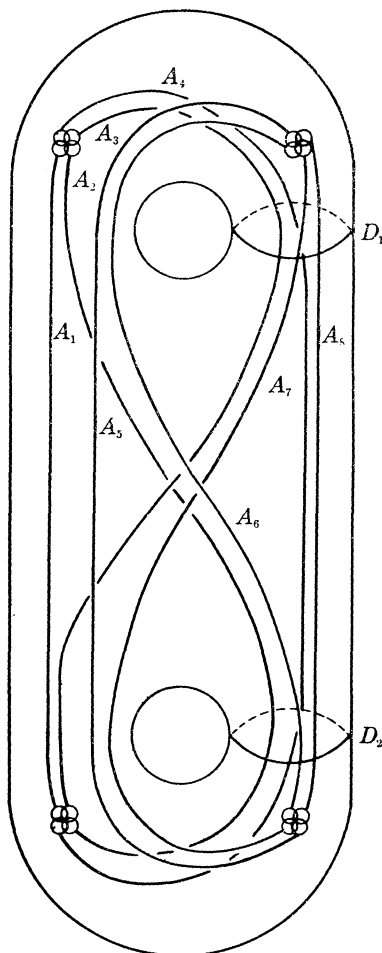


FIGURE 5.

integer m such that if $n \geq m$, then $B_{t(1)t(2)\dots t(n)}$ intersects E_1 and E_2 . Hence f is a nondegenerate element.

Suppose t is a sequence into $\{1, 2, 3, 4\}$ which does not converge to 1. Let $\{q_i\}$ be an increasing sequence such that for each i , $t(q_i) \neq 1$. Then $B_{t(1)t(2)\dots t(q_r)}$ intersects at most one of P_1, P_2, \dots , and P_r . For some n , $B_{t(1)\dots t(q_n)}$ intersects at most one of the discs used to define the homeomorphism $h_{t(q_r)+1}$. Hence $\lim_{n \rightarrow \infty} (\text{diam } B_{t(1)\dots t(n)})$ is zero.

EXAMPLE 6. There exists a point-like decomposition F such that if K is any point-like decomposition equivalent to F , then K has uncountably many nondegenerate elements. Let A be a solid double torus and let A_1, A_2, \dots, A_r , and A_s be solid double tori embedded in A as shown in Figure 4. Inside each of the A_i 's eight double tori are embedded like the A_i 's are in A , etc. Suppose K is equivalent to F and let $A'_{i_j \dots i_m}$ correspond to $A_{i_j \dots i_m}$. Let D'_1 and D'_2 be disjoint discs in A' which are embedded in A' in the same manner as D_1 and D_2 are embedded in A . See Figure 4. It follows from the arguments in [5] that two of the A'_i 's intersect both D'_1 and D'_2 , and inside each of those two of the A'_{i_j} 's intersect both D'_1 and D'_2 , etc. It follows that K has uncountably many nondegenerate elements.

5. A decomposition not equivalent to the dogbone. In this section G will denote the point-like decomposition of R^3 described by Bing in [7], and the notation and terminology of that paper will be used. It will be proved that any point-like decomposition equivalent to G has at least one nondegenerate element which is not locally connected. Let T_0 denote a round solid torus in R^3 . Let T_{00} and T_{01} be disjoint solid tori embedded in the interior of T_0 as shown in Figure 5. Inside each T_{0i} two tori are embedded, etc. G is the decomposition of R^3 whose nondegenerate elements are the nondegenerate components of $T_0 \cap (\cup T_{0i}) \cap (\cup T_{0ij}) \cap \dots$. G has countably many nondegenerate elements, each of which is indecomposable.

Property P. Suppose T is a solid torus. A disc D has Property P with respect to T if and only if D is a polyhedral disc in general position with respect to T and $\text{Bd } D$ is a simple closed curve on $\text{Bd } T$ which circles $\text{Bd } T$ meridionally.

Property A. A collection of sets $\{T, D_1, \dots, D_n\}$ has Property A if and only if (1) T is a solid torus, (2) for $1 \leq i \leq n$, D_i is a disc which has Property P with respect to T and no proper subdisc of D_i has Property P with respect to T , (3) if $i \neq j$ then D_i and D_j are disjoint, and (4) if C is a longitudinal curve on $\text{Bd } T$ which intersects

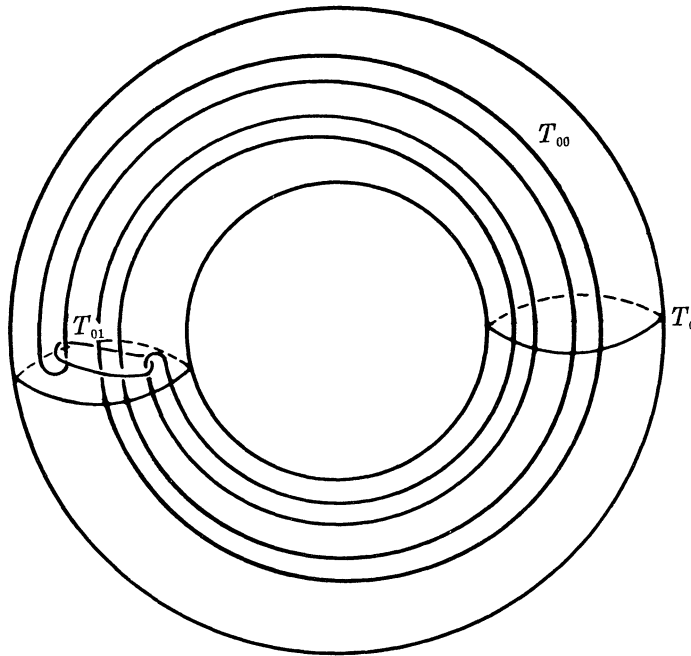


FIGURE 5.

each $\text{Bd } D_i$ in a single point q_i , then the ordering of the q_i 's on C is $q_1 q_2 \cdots q_n q_1$.

Suppose $\{T, D_1, \dots, D_n\}$ has Property A. A collection $\{Q_1, \dots, Q_n\}$ is a division of T determined by $\{T, D_1, \dots, D_n\}$ if and only if for each i , $1 \leq i < n$, if D'_i denotes the component of $D_i - (R^3 - T)$ which contains $\text{Bd } D_i$, for $1 \leq i < n$, Q'_i denotes the component of $T - (\cup D'_i)$ whose closure intersects both D'_i and $D'_{(i+1)}$, $Q_i = \bar{Q}'_i$, and Q_n is the closure of the component of $T - (D'_1 \cup D'_n)$ which is disjoint from Q'_1 .

LEMMA 4. *If $\{T_0, D_1, \dots, D_n\}$ has Property A and $\{Q_1, \dots, Q_n\}$ is a division of T_0 determined by $\{T_0, D_1, \dots, D_n\}$, then there exist an integer i and discs E_1, E_2, \dots , and E_m such that (1) $i = 0$ or 1, (2) $\{T_{0i}, E_1, \dots, E_m\}$ has Property A, (3) if $\{R_1, \dots, R_m\}$ is the division of T_{0i} determined by $\{T_{0i}, E_1, \dots, E_m\}$, then there exist integers i_1, i_2, \dots , and i_{2n} , such that $1 \leq i_1 < i_2 < \dots < i_{2n} \leq m$ and for some*

$$t, 1 \leq t \leq n, R_{i_1} \subset Q_t, R_{i_2} \subset Q_{t-1}, \dots, R_{i_t} \subset Q_1, R_{i_{t+1}} \subset Q_n, \dots, R_{i_n} \\ \subset Q_{t+1}, R_{i_{n+1}} \subset Q_{t+1}, \dots, R_{i_{2n}} \subset Q_t,$$

and (4) if $R_{i_k} \subset Q_j$, then E_{i_k} is contained in one of D_j and D_{j+1} and $E_{i_{k+1}}$ is contained in the other.

Proof. Consider the universal covering space for T_0 . It is represented by Figure 6 where it appears that T_0 has been rolled out onto a cylinder. It follows from the proof of [7, Th. 5] that for some k , either each center for T_{00} intersects two adjacent copies of D_k in the universal covering space, or each center for T_{01} intersects two adjacent copies of D_k in the universal covering space. Assume each center for T_{00} intersects two adjacent copies of D_k in the universal covering space and let $i = 0$. Let C be a center for T_{00} such that $C \cap (\cup D_j)$ is a finite set, and if C' is a center for T_{00} , then $C' \cap (\cup D_j)$ contains at least as many elements as $C \cap (\cup D_j)$. This last condition implies that if $r \in C \cap D_j$, then there is a subdisk E of D_j which has Property P with respect to T_{00} and $E \cap C = r$. It can be assumed without loss of generality that T_{00} is polyhedral and $\text{Bd } T_{00}$ and $\cup D_j$ are in relative general position.

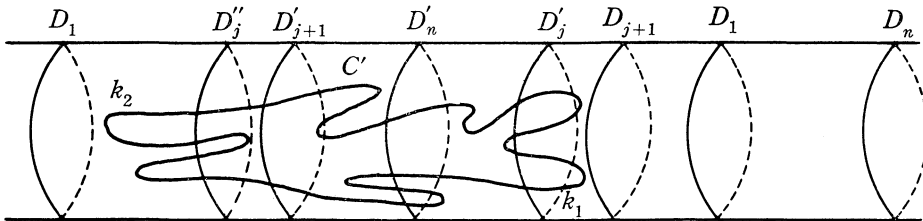


FIGURE 6.

Let C' denote one of the copies of C in the universal covering space as shown in Figure 6. Assume that one of the copies of D_j , say D'_j , is the rightmost one of the copies of the D_k 's that intersect C' . Let D'_j be the first copy of D_j to the left of D'_j and let D'_k be the first copy of D_k to the right of D'_j . Let t be $j - 1$ if $2k \leq j \leq n$ or t be n if $j = 1$. Let k_1 be a point in C' to the right of D'_j and let k_2 be a point in C' to the left of D'_j . Let A be an arc in C' from k_1 to k_2 and B be the arc in C' from k_2 to k_1 which intersects A only in the end points. Let r'_2 be the first point of A in D'_{j-1} and let r'_1 be the last point of $A \cap D'_j$ preceding r'_2 . Let r'_4 be the first point of A in D'_{j-2} and let r'_3 be the last point of $A \cap D'_{j-1}$ preceding r'_4 . Continue this procedure to obtain points $r'_5, r'_6, \dots, r'_{2n-1}$, and r'_{2n} . Let r'_{2n+2} be the first point of B in D'_{j+1} and let r'_{2n+1} be the last point of $B \cap D'_j$ preceding r'_{2n+2} . Continue this to get $r'_1, r'_2, \dots, r'_{4n-1}$ and r'_{4n} . Let r_i be the point in C corresponding to r'_i .

The r_i 's have the ordering $r_1 r_2 \dots r_{4n} r_1$ on C , and determine disks E_1, \dots , and E_{4n} on T_{00} . It can be assumed that each of the E_i 's is a subdisk of $\cup D_k$, each has property P with respect to T_{00} , no proper subdisk of E_i has property P with respect to T_{00} , if $r_i = r_{i+1}$ then $E_i = E_{i+1}$, otherwise the E_i 's form a disjoint collection, and finally

$E_j \cap C = r_j$. If the collection $\{E_1, \dots, E_{4n}\}$ is reindexed to give a disjoint collection $\{E_1, \dots, E_m\}$ then clearly $m \geq 2n$ and there exist integers i_1, \dots , and i_{2n} which satisfy the conclusion of the lemma.

THEOREM 5. *If F is a point-like decomposition equivalent to G , then some nondegenerate element of F is not locally connected.*

Proof. By Theorem 1 and Corollary 1, F is a toroidal decomposition of R^3 and there exists a sequence of homeomorphisms $\{h_i\}_{i=0}^\infty$ of homeomorphisms such that h_i is from R^3 onto R^3 , if $j > k$, then $h_j|_{R^3 - \cup T_{0i_1 \dots i_k}} = h_k$, and the nondegenerate elements of F are the nondegenerate components of $h_0[T_0] \cap h_1[\cup T_{0i}] \cap \dots$.

Let D_1 and D_2 be disjoint discs, each of which has Property P with respect to $h_0[T_0]$, and such that no proper subdisc of either D_1 or D_2 has property P with respect to $h_0[T_0]$. Then $h_1^{-1}[D_1]$ and $h_1^{-1}[D_2]$ are discs, each of which has Property P with respect to T_0 , and no proper subdisc of either has Property P with respect to T_0 . Let R_1 and R_2 be the division of T_0 determined by $\{T_0, h_1^{-1}[D_1], h_1^{-1}[D_2]\}$.

By Lemma 4, there exist an integer t_1 in $\{0, 1\}$ and disks E_{11}, E_{12}, \dots , and $E_{1m(1)}$ such that, $\{T_{0t_1}, E_{11}, \dots, E_{1m(1)}\}$ has Property A , and if $\{R_{11}, \dots, R_{1m(1)}\}$ is a division of T_{0t_1} determined by $\{T_{0t_1}, E_{11}, \dots, E_{1m(1)}\}$, then there exist integers j_{11} and j_{12} , $j_{11} < j_{12}$, such that, $R_{1j_{11}}$ and $R_{1j_{12}}$ are contained in R_1 , $E_{1j_{11}}$ and $E_{1j_{12}}$ are contained in one of $h^{-1}[D_1]$ and $h^{-1}[D_2]$, and $E_{1(j_1+1)}$ and $E_{1(j_2+1) \bmod m(1)}$ are contained in the other. Then $\{T_{0t_1}, h_2^{-1}h_1[E_{11}], \dots, h_2^{-1}h_1[E_{1m(1)}]\}$ has Property A , and by applying Lemma 4 again, there exist an integer t_2 in $\{0, 1\}$ and discs E_{21}, \dots , and $E_{2m(2)}$ such that $\{T_{0t_1 t_2}, E_{21}, \dots, E_{2m(2)}\}$ has Property A , and if $\{R_{21}, \dots, R_{2m(2)}\}$ is a division determined by $\{T_{0t_1 t_2}, E_{21}, \dots, E_{2m(2)}\}$, then there exist integers $j_{21} < j_{22} < j_{23} < j_{24}$ such that

$$R_{2j_{21}} \subset R_{1j_{11}}, R_{2j_{22}} \subset R_{2j_{12}}, R_{2j_{23}} \subset R_{1j_{12}}, \quad \text{and} \quad R_{2j_{24}} \subset R_{1j_{11}}.$$

Continuing this process by induction it follows that

$$(h_0[T_0] \cap h_1[T_{0t_1}] \cap h_2[T_{0t_1 t_2}] \cap \dots) - (D_1 \cup D_2)$$

has an infinite number of components, each of which intersects both D_1 and D_2 , and hence is not locally connected.

In fact, countably many of the nondegenerate elements fail to be locally connected. To see this let v_i denote $(t_i + 1) \bmod 2$. Let D_{21} and D_{22} be disjoint discs which have Property P with respect to $h_1[T_{0v_1}]$ and repeat the above argument. Similarly for each of $h_2[T_{0t_1}], h_3[T_{0t_1 t_2 v_3}], h_4[T_{0t_1 t_2 v_3 v_4}]$, etc.

COROLLARY 2. *There does not exist a point-like decomposition*

F equivalent to G such that each nondegenerate element of F is an arc.

COROLLARY 3. *The decomposition G is not equivalent to the dogbone decomposition.*

EXAMPLE 7. There does exist a decomposition F equivalent to G such that some nondegenerate element of F is an arc.

Construction of F . Let T_{00} and T_{01} be embedded in T_0 as shown in Figure 7. If this pattern is used at each stage, then $T_{01} \cap T_{011} \cap T_{0111} \cap \dots$ is an arc.

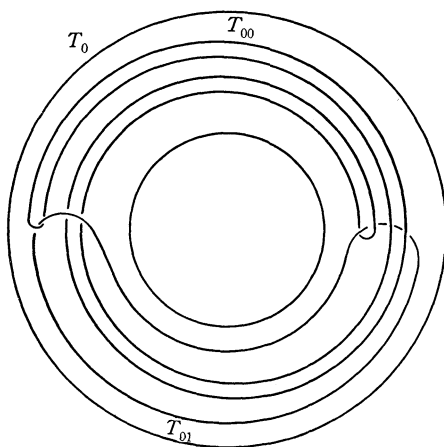


FIGURE 7.

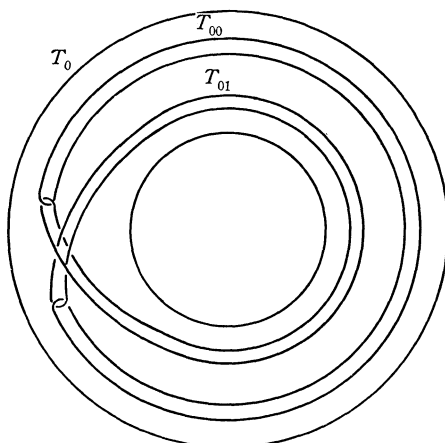


FIGURE 8.

EXAMPLE 8. There exists a decomposition F equivalent to G such that F has uncountably many nondegenerate elements and each is an indecomposable continuum.

Construction of F . Let T_{00} and T_{01} be embedded in T_0 as shown in Figure 8. This pattern is used at each stage.

6. Tamely finnable 3-cells. In this section we show that if a 3-cell C in R^3 is tamely finnable, then there is a 3-cell C' in R^3 with a flat spot on its boundary such that the decomposition of R^3 whose only nondegenerate element is C is equivalent to the decomposition of R^3 whose only nondegenerate element is C' . A 3-cell C in R^3 is *tamely finnable* if and only if there exists a tame disc D in R^3 such that $D \cap C$ is an arc α and $\alpha \subset (\text{Bd } D) \cap (\text{Bd } C)$. The statement that $\text{Bd } C$ has a *flat spot* means that $\text{Bd } C$ contains a polyhedral

disc. We begin by describing several sets and functions which will be used in the proof.

Let R be the 3-cell $\{(x, y, z) : |x| \leq 1, |y| \leq 2, |z| \leq 1\}$, R^+ be $(R \cap \{(x, y, z) : y \geq 0\})$, and R^- be $(R \cap \{(x, y, z) : y \leq 0\})$. For each subset X of R , let X^+ denote $X \cap R^+$ and X^- denote $X \cap R^-$.

If P and Q are points in R^3 , let $[P, Q]$ denote the straight line interval from P to Q . Let D_1 be $\{(x, y, z) : x^2 + y^2 \leq 1\}$, D_2 be

$$\bigcup \{(x, 0, 0), (x, y, 1) : x^2 + y^2 = 1\},$$

D_3 be

$$\bigcup \{(x, y, 0), (x, y, 1) : x^2 + y^2 = 1\},$$

D_4 be

$$\bigcup \{(x, y, 0), (x, 0, -1) : x^2 + y^2 = 1\},$$

and D^5 be $\{(x, y, 0) : x^2 + y^2 \leq 1\}$.

Let K be the 3-cell bounded by $D_1 \cup D_2$, L be $\text{Cl}(R - K)$, M be the 3-cell bounded by $D_1 \cup D_3 \cup D_4$, and N be $\text{Cl}(R - M)$; see Figure 9.

Let g_1 be a homeomorphism of L^+ onto N^+ such that g_1 is fixed on $\text{Bd } L^+ \cap \text{Bd } R$, $g_1[\{(x, 0, 0) : -1 \leq x \leq 1\}]$ is $\{(x, y, 0) : x^2 + y^2 = 1, y \geq 0\}$ and g_1 moves points only along lines parallel to the y -axis. Let g_2 be a homeomorphism of L^- onto N^- such that g_2 is fixed on

$$\text{Bd } L^- \cap \text{Bd } R, \quad g_2[\{(x, 0, 0) : -1 \leq x \leq 1\}]$$

is $\{(x, y, 0) : x^2 + y^2 = 1, y \leq 0\}$ and g_2 moves points only along lines parallel to the y -axis.

Let M' be $M \cap \{(x, y, z) : z \geq 0\}$ and let g_3 be a continuous function from M' onto K such that $g_3[D_5] = \{(x, 0, 0) : -1 \leq x \leq 1\}$, g_3 is the identity on D_1 , $g_3|D_3^+ = g_1^{-1}$, $g_3|D_3^- = g_2^{-1}$, and g_3 is a homeomorphism on $(M' - D_5)$.

THEOREM 6. *Let C be a 3-cell in R^3 such that C is tamely finable. Then there exists a 3-cell C' in R^3 such that C' has a flat spot and the decomposition of R^3 whose only nondegenerate element is C is equivalent to the decomposition of R^3 whose only nondegenerate element is C' .*

Proof. Let C be the 3-cell and D be a tame disc such that $D \cap C$ is an arc α lying on $\text{Bd } D \cap \text{Bd } C$. There exists a homeomorphism h of R^3 onto itself such that (1) $h[\alpha] = \{(x, 0, 0) : -1 \leq x \leq 1\}$, (2) $h[D] = \{(x, 0, z) : |x| \leq 1, 0 \leq z \leq 1\}$, and (3) $h[\text{Bd } C - \alpha]$ and $K \cup (R^+ \cap R^-)$ are disjoint.

Let F be a homeomorphism from $R - (K \cup (R^+ \cap R^-))$ onto $R - M$

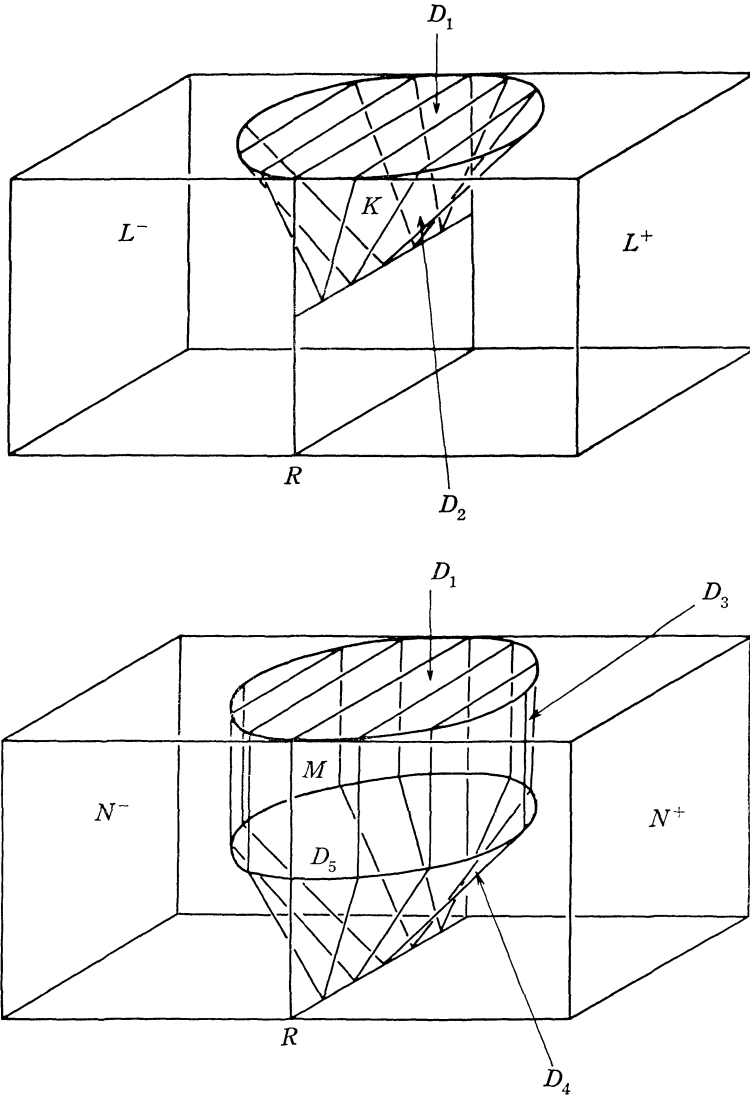


FIGURE 9.

such that if $x \in [R - (K \cup (R^+ \cap R^-))]^+$, $F(x) = g_1(x)$, and if $x \in [R - (K \cup (R^+ \cap R^-))]^-$, $F(x) = g_2(x)$. Extend F to $R^3 - R$ such that if $x \in (R^3 - R)$, $F(x) = x$. Let S be $D_5 \cup Fh[\text{Bd } C - \alpha]$.

It is easily seen that S is a 2-sphere in R^3 which bounds a 3-cell C' and $\text{Bd } C'$ has a flat spot, the disk D_5 . It remains to show that the decomposition of R^3 corresponding to C and C' respectively are equivalent.

We will define a function Φ from R^3 onto itself such that $|\Phi[C'] = h[C]$ and $\Phi|_{\text{Ext } C'}$ is a homeomorphism. If $P \in R^3 - M$, $\Phi(P) = F^{-1}(P)$. If $P \in M'$, $\Phi(P) = g_3(P)$. If $P \in M \cap \{(x, y, z) : z \leq 0\}$ and $P = (x, y, z)$, let $\Phi(P)$

be $(x, 0, z)$.

Now the function $h^{-1}\phi$ is a continuous function from R^3 onto itself which maps C' onto C and is a homeomorphism outside C' . It follows that the corresponding decompositions are equivalent.

COROLLARY 4. *If C is a 3-cell in R^3 and C is tamely finnable, then there exists a disc D in R^3 such that the decomposition of R^3 whose only nondegenerate element is C is equivalent to the decomposition of R^3 whose only nondegenerate element is D .*

Proof. This follows from Theorem 3 of [10].

The statement that K is a *crumpled cube* means that K is homeomorphic to the union of a 2-sphere and its interior in R^3 .

THEOREM 7. *If K is a crumpled cube in R^3 , there exists a 3-cell C in R^3 such that the decomposition of R^3 whose only nondegenerate element is K is equivalent to the decomposition of R^3 whose only nondegenerate element is C .*

Proof. Apply Theorem 2 of [8].

7. Improving elements of decompositions. Suppose K is a 3-cell-with- n -handles in R^3 and $C, C_1, C_2, C_3, \dots, C_n$ is a standard decomposition of K . If i is a positive integer less than or equal to n , let $D_{i,1}$ and $D_{i,2}$ be the two components of $C \cap C_i$. Let p be an element of $\text{Int } C$, and if i and j are integers, $1 \leq i \leq n, 1 \leq j \leq 2$, let $p_{i,j}$ be an element of $\text{Int } D_{i,j}$. Let T be $\{(x, y, z) \in R^3: x^2 + y^2 \leq 1, |z| \leq 1\}$. If i is a positive integer less than or equal to n , there is a homeomorphism f_i of C_i onto T such that $f_i[D_{i,1}] = \{(x, y, z): x^2 + y^2 \leq 1, z = 1\}$, $f_i[D_{i,2}] = \{(x, y, z): x^2 + y^2 \leq 1, z = -1\}$, $f_i(p_{i,1}) = (0, 0, 1)$ and $f_i(p_{i,2}) = (0, 0, -1)$. Let α_i be $f_i^{-1}\{(0, 0, z): |z| \leq 1\}$.

Let f be a homeomorphism of C onto the unit ball $\{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$ such that $f(p) = (0, 0, 0)$. If i and j are integers, $1 \leq i \leq n, 1 \leq j \leq 2$, let $b_{i,j}$ be the straight line interval from $f(p_{i,j})$ to $f(p)$, and let $\beta_{i,j}$ be $f^{-1}[b_{i,j}]$. Let S be $(\bigcup_{i=1}^n \alpha_i) \cup (\bigcup_{i=1}^n \bigcup_{j=1}^2 \beta_{i,j})$. We will call S a *special spine* of K .

A *partition* of K is a finite collection \mathcal{P} of subsets of K such that (1) if $Q \in \mathcal{P}$, Q is a 3-cell, (2) if $Q \in \mathcal{P}$ and $Q \subset C$, then $Q = C$, (3) if $Q \in \mathcal{P}$ and there is a positive integer i less than or equal to n such that $Q \subset C_i$, then there exist real numbers a and b such that $-1 \leq a \leq b \leq 1$ and $f_i[Q] = \{(x, y, z): a \leq z \leq b\} \cap T$, (4) if $Q_1 \in \mathcal{P}, Q_2 \in \mathcal{P}, Q_1 \neq Q_2$, and $Q_1 \cap Q_2 \neq \emptyset$, then $Q_1 \cap Q_2$ is a disc on $\text{Bd } Q_1 \cap \text{Bd } Q_2$, and (5) $\bigcup \{Q: Q \in \mathcal{P}\} = K$.

If K is a polyhedral cell with handles in R^3 , S is a special spine

of K , and $\varepsilon > 0$, there is clearly a homeomorphism h of R^3 onto itself and a partition \mathcal{S} of K such that (1) if $x \in (R^3 - V(K, \varepsilon))$, $h(x) = x$, (2) $h[K] \subset V(S, \varepsilon)$, and (3) if $Q \in \mathcal{S}$, $(\text{diam } h[Q]) < \varepsilon$.

THEOREM 8. *Suppose F is an upper semi-continuous decomposition of R^3 and F is definable by 3-cells-with-handles. Then there exists an upper semi-continuous decomposition G of R^3 such that F is equivalent to G and each nondegenerate element of G is one dimensional.*

Proof. Since F is definable by 3-cells-with-handles, there exists a defining sequence M_1, M_2, M_3, \dots for F such that for each positive integer k , each component of M_k is a 3-cell-with-handles. Let $C_{k,1}, C_{k,2}, \dots, C_{k,n_k}$ be the 3-cells-with-handles which are the components of M_k , and if j is a positive integer less than or equal to n_k , let $S_{k,j}$ be a special spine of $C_{k,j}$.

Let ε_1 be a positive number such that $\varepsilon_1 < 1$ and $V(C_{1,1}, \varepsilon_1), V(C_{1,2}, \varepsilon_1), \dots$, and $V(C_{1,n_1}, \varepsilon_1)$ are mutually disjoint sets. For each positive integer j less than or equal to n_1 , there exist a partition $P_{1,j}$ of $C_{1,j}$ and a homeomorphism $h_{1,j}$ of $V(C_{1,j}, \varepsilon_1)$ onto itself such that (1) if $x \in V(C_{1,j}, \varepsilon_1) - V(C_{1,j}, \varepsilon_1/2)$, $h_{1,j}(x) = x$, (2) $h_{1,j}[C_{1,j}] \subset V(S_{1,j}, \varepsilon_1)$, and (3) if $Q \in \mathcal{S}_{1,j}$, $(\text{diam } h_{1,j}[Q]) < \varepsilon_1$. Let h_1 be a homeomorphism of R^3 onto itself such that if $x \in \bigcup_{i=1}^{n_1} V(C_{1,i}, \varepsilon_1)$, $h_1(x) = x$, and if i is a positive integer less than or equal to n_1 and $x \in V(C_{1,i}, \varepsilon_1)$, $h_1(x) = h_{1,i}(x)$.

Let δ_1 be $\min \{(\text{diam } h_1[Q]): Q \in (\bigcup_{i=1}^{n_1} P_{1,i})\}$ and let ε_2 be a positive number such that $\varepsilon_2 < \min \{\delta_1/2, 1/2\}$ and $V(h_1[C_{2,1}], \varepsilon_2), V(h_1[C_{2,2}], \varepsilon_2), \dots$, and $V(h_1[C_{2,n_2}], \varepsilon_2)$ are mutually disjoint sets each one of which is contained in $h_1[V(M_1, 1/2)]$. For each positive integer j less than or equal to n_2 , there exist a partition $\mathcal{S}_{2,j}$ of $C_{2,j}$ and a homeomorphism $h_{2,j}$ of $V(h_1[C_{2,j}], \varepsilon_2)$ onto itself such that (1) if

$$x \in V(h_1[C_{2,j}], \varepsilon_2) - V(h_1[C_{2,j}], \varepsilon_2/2), h_{2,j}(x) = x,$$

(2) $h_{2,j}[h_1[C_{2,j}]] \subset V(h_1[S_{2,j}], \varepsilon_2)$, (3) if $Q_1 \in \mathcal{S}_{2,j}$, there exists an element Q_2 of $\bigcup_{i=1}^{n_1} \mathcal{S}_{1,i}$ such that $h_{2,j}h_1[Q_1] \subset h_1[Q_2]$, and (4) if

$$Q \in \mathcal{S}_{2,j}, (\text{diam } h_{2,j}h_1[Q]) < \varepsilon_2.$$

Let h_2 be a homeomorphism of R^3 onto itself such that if

$$x \in \bigcup_{i=1}^{n_2} V(h_1[C_{2,i}], \varepsilon_2), h_2(x) = x,$$

and if i is a positive integer less than or equal to n_2 and

$$x \in V(h_1[C_{2,i}], \varepsilon_2), h_2(x) = h_{2,i}(x).$$

Continue in this manner obtaining a sequence h_1, h_2, h_3, \dots of homeomorphisms of E^3 onto itself. Let h be $\lim_{n \rightarrow \infty} (h_n h_{n-1} \dots h_1)$, and let G be $\{h[f]: f \in F\}$. It is easily seen that G is an upper semi-continuous decomposition of E^3 such that F and G are equivalent. The fact that each nondegenerate element g of G is one-dimensional can be seen by noticing that g intersects the boundaries of the images of the elements of the partitions in a 0-dimensional set.

THEOREM 9. *Let G be a monotone upper semi-continuous decomposition of R^3 such that G has only countably many nondegenerate elements, and each nondegenerate element is tame (relative to the usual triangulation of R^3). Then there exists a homeomorphism h of R^3 onto itself such that if $g \in G$, $h[g]$ is polyhedral.*

Proof. Let g_1, g_2, g_3, \dots denote the nondegenerate elements of G . Let ε_1 be a positive number such that $\varepsilon_1 < 1/2$. Since g_1 is tame, it follows from Theorem 9 of [4] that there exists a homeomorphism h_1 of R^3 onto itself such that if $x \in R^3 - V(g_1, \varepsilon_1/4)$, $h_1(x) = x$, if $x \in R^3$, $d(x, h_1(x)) < \varepsilon_1/4$, and $h_1[g_1]$ is polyhedral.

Let ε_2 be a positive number such that

$$\varepsilon_2 < (\varepsilon_1/2), \quad V(h_1[g_2], \varepsilon_2) \subset h_1[V(g_2, 1/2^2)],$$

and $V(h_1[g_2], \varepsilon_2) \cap h_1[g_1] = \emptyset$. There exists a homeomorphism h_2 of R^3 onto itself such that if $x \in R^3 - V(h_1[g_2], \varepsilon_2/4)$, $h_2(x) = x$, if $x \in R^3$ then $d(h_2(x), x) < \varepsilon_2/4$, and $h_2 h_1[g_2]$ is polyhedral.

If n is a positive integer and h_1, h_2, \dots , and h_{n-1} are chosen, let ε_n be a positive number such that

$$\varepsilon_n < 1/2^n, \quad V(h_{n-1} \dots h_1[g_n], \varepsilon_n) \subset h_{n-1} \dots h_1[V(g_n, 1/2^n)],$$

and

$$V(h_{n-1} \dots h_1[g_n], \varepsilon_n) \cap \left(\bigcup_{i=1}^{n-1} h_{n-1} \dots h_1[g_i] \right) = \emptyset.$$

There exists a homeomorphism h_n of R^3 onto itself such that if

$$x \in R^3 - V(h_{n-1} \dots h_1[g_n], \varepsilon_n/4),$$

then $h_n(x) = x$, if $x \in R^3$ then $d(h_n(x), x) < \varepsilon_n/4$, and $h_n \dots h_1[g_n]$ is polyhedral.

Let h be $\lim_{n \rightarrow \infty} h_n h_{n-1} \dots h_1$. h is the uniform limit of continuous functions, thus h is continuous. It follows from Theorem C_2 of [9] that h is onto R^3 .

To show that h is one-to-one, let x and y be distinct points of R^3 . If x and y belong to the same element of G , then clearly $h(x) \neq h(y)$.

Suppose $x \in g_x$ and $y \in g_y$ and $g_x \neq g_y$ where g_x and g_y are elements of G .

Since G is upper semi-continuous, there exists a positive integer N such that if n is an integer greater than N and $x \in V(g_n, 1/2^n)$, then $y \notin V(g_n, 1/2^n)$, and if n is an integer greater than N and $y \in V(g_n, 1/2^n)$, then $x \notin V(g_n, 1/2^n)$.

Now for each positive integer n , let U_n be

$$(h_{n-1} \cdots h_1)^{-1}[V(h_{n-1} \cdots h_1)[g_n], \varepsilon_n/4].$$

Then $U_n \subset V(g_n, 1/2^n)$. If for each positive integer n neither x nor y belongs to U_n , then

$$h(x) = h_N \cdots h_1(x), h(y) = h_N \cdots h_1(y)$$

and

$$h(y) \neq h(x).$$

Suppose there exists a positive integer n such that $n > N$ and $x \in U_n$. Then

$$x \in V(g_n, 1/2^n), y \in V(g_n, 1/2^n)$$

and

$$h_{n-1} \cdots h_1(x) \in V(h_{n-1} \cdots h_1[g_n], \varepsilon_n/4).$$

Since

$$\begin{aligned} V(h_{n-1} \cdots h_1[g_n], \varepsilon_n) &\subset h_{n-1} \cdots h_1[V(g_n, 1/2^n)], \\ h_{n-1} \cdots h_1(y) &\notin h_{n-1} \cdots h_1[V(g_n, 1/2^n)] \end{aligned}$$

and

$$d(h_{n-1} \cdots h_1(y), h_{n-1} \cdots h_1(x)) \geq \varepsilon_n.$$

Then

$$d(h(x), h_{n-1} \cdots h_1(x)) \leq \varepsilon_n/2, d(h(y), h_{n-1} \cdots h_1(y)) \leq \varepsilon_{n+1}/2 < \varepsilon_n/2$$

and

$$d(h(x), h(y)) \neq 0.$$

Thus $h(x) \neq h(y)$. Hence h is one-to-one.

To show that h^{-1} is continuous, suppose there exists a sequence $x_n \rightarrow x$ such that $h^{-1}(x) \neq h^{-1}(x)$. Picking a subsequence if necessary, it can be assumed that there exists a positive number ε such that for each i , $h^{-1}(x_i) \notin V(h^{-1}(x), \varepsilon)$.

Since h is bounded, $\{h^{-1}(x_i): i \in J\}$ is bounded and $\text{Cl}\{h^{-1}(x_i): i \in J\}$ and $\{h^{-1}(x)\}$ are disjoint compact sets. Hence for some

$$\delta > 0, d(h[\text{Cl}\{h^{-1}(x_i): i \in J\}], h^{-1}(x)) > \delta.$$

Then for each positive integer i , $d(x_i, x) > \delta$. This is a contradiction. Thus h^{-1} is continuous.

COROLLARY 5. *If F is an upper semi-continuous decomposition of R^3 into tame 3-cells and points, then there exists an upper semi-continuous G into polyhedral 3-cells and points such that F is equivalent to G .*

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