

# Pacific Journal of Mathematics

**A NOTE ON CERTAIN BIORTHOGONAL POLYNOMIALS**

L. CARLITZ

## A NOTE ON CERTAIN BIORTHOGONAL POLYNOMIALS

L. CARLITZ

**Konhauser has introduced two polynomial sets  $\{Y_n^c(x; k)\}$ ,  $\{Z_n^c(x; k)\}$  that are biorthogonal with respect to the weight function  $e^{-x}x^c$  over the interval  $(0, \infty)$ . An explicit expression was obtained for  $Z_n^c(x; k)$  but not for  $Y_n^c(x; k)$ . An explicit polynomial expression for  $Y_n^c(x; k)$  is given in the present paper.**

1. Konhauser [2] has discussed two sets of polynomials  $Y_n^c(x; k)$ ,  $Z_n^c(x; k)$ ,  $n = 0, 1, \dots$ ,  $k = 1, 2, 3, \dots$ ,  $c > -1$ ;  $Y_n^c(x; k)$  is a polynomial in  $x$  while  $Z_n^c(x; k)$  is a polynomial in  $x^k$ . Moreover

$$(1) \quad \int_0^\infty e^{-x}x^c Y_n^c(x; k) x^{ki} dx = \begin{cases} 0 & (0 \leq i < n) \\ \neq 0 & (i = n) \end{cases}$$

and

$$(2) \quad \int_0^\infty e^{-x}x^c Z_n^c(x; k) x^i dx = \begin{cases} 0 & (0 \leq i < n) \\ \neq 0 & (i = n) . \end{cases}$$

For  $k = 1$ , conditions (1) and (2) reduce to the orthogonality conditions satisfied by the Laguerre polynomials  $L_n^c(x)$ .

It follows from (1) and (2) that

$$(3) \quad \int_0^\infty e^{-x}x^c Y_i^c(x; k) Z_j^c(x; k) dx = \begin{cases} 0 & (i \neq j) \\ \neq 0 & (i = j) . \end{cases}$$

The polynomial sets  $\{Y_n^c(x; k)\}$ ,  $\{Z_n^c(x; k)\}$  are accordingly said to be biorthogonal with respect to the weight function  $e^{-x}x^c$  over the interval  $(0, \infty)$ .

Konhauser showed that

$$(4) \quad Z_n^c(x; k) = \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + c + 1)}$$

As for  $Y_n^c(x; k)$ , he showed that

$$(5) \quad \begin{aligned} Y_n^c(x; k) &= \frac{k}{2i} \int_C \frac{e^{-xt}(t+1)^{c+kn}}{[(t+1)^k - 1]^{n+1}} dt \\ &= \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left\{ \frac{e^{-xt}(t+1)^{c+kn} t^{n+1}}{[(t+1)^{k+1} - 1]^{n+1}} \right\}_{t=0} . \end{aligned}$$

In the integral in (5),  $C$  may be taken as a small circle about the origin in the  $t$ -plane.

In the present note we give a generating function and an explicit polynomial expression for the polynomial  $Y_n^c(x; k)$ . Moreover we show that  $Y_n^c(x; k)$  can be identified with a polynomial studied recently by S. K. Chatterjea [1].

2. We apply the Lagrange expansion in the form [4, p. 125]

$$(6) \quad \frac{f(t)}{1 - w\phi'(t)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \left\{ \frac{d^n}{dt^n} [f(t)(\phi(t))^n] \right\}_{t=0},$$

where

$$w = \frac{t}{\phi(t)}, \quad \phi(t) = a_0 + a_1 t + \dots \quad (a_0 \neq 0).$$

Take

$$f(t) = \frac{e^{-xt}(t+1)^{ct}}{(t+1)^k - 1}, \quad \phi(t) = \frac{(t+1)^{kt}}{(t+1)^k - 1}.$$

Then we have

$$1 - w\phi'(t) = \frac{kt}{(t+1)(t+1)^k - 1},$$

so that

$$\frac{f(t)}{1 - w\phi'(t)} = e^{-xt}(t+1)^{c+1}.$$

Thus, by (5) and (6), we have

$$(7) \quad e^{-xt}(t+1)^{c+1} = \sum_{n=0}^{\infty} Y_n^c(x; k) \left( \frac{t}{\phi(t)} \right)^n.$$

If we put

$$w = \frac{t}{\phi(t)} = \frac{(t+1)^k - 1}{(t+1)^k} = 1 - (t+1)^{-k},$$

then

$$t = (1 - w)^{-1/k} - 1$$

and (7) becomes

$$(8) \quad (1 - w)^{-(c+1)/k} \exp \{-x[(1 - w)^{-1/k} - 1]\} = \sum_{n=0}^{\infty} Y_n^c(x; k) w^n.$$

In the next place, we have

$$\begin{aligned}
 & (1 - w)^{-(c+1)/k} \exp \{-x[(1 - w)^{-1/k} - 1]\} \\
 &= (1 - w)^{-(c+1)/k} \sum_{r=0}^{\infty} \frac{x^r}{r!} [(1 - w)^{-1/k} - 1]^r \\
 &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} (1 - w)^{-(s+c+1)/k} \\
 &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \sum_{n=0}^{\infty} \frac{((s+c+1)/k)_n}{n!} w^n \\
 &= \sum_{n=0}^{\infty} \frac{w^n}{n!} \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_n,
 \end{aligned}$$

where

$$(a)_n = a(a+1) \cdots (a+n-1), \quad (a)_0 = 1.$$

It therefore follows from (8) that

$$(9) \quad Y_n^c(x; k) = \frac{1}{n!} \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_n.$$

3. Chatterjea [1] has defined the polynomial

$$(10) \quad T_{k,n}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{xk} D^n(e^{\alpha+n} e^{-xk})$$

with  $k = 1, 2, 3, \dots$ . The case  $\alpha = 0$  had been discussed by Palas [3]. Chatterjea showed that (10) implies

$$(11) \quad T_{k,n}^{(\alpha)}(x) = \sum_{r=0}^{\infty} \frac{x^{kr}}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{\alpha+n+ks}{n}.$$

He also obtained operational formulas and a generating function for  $T_{k,n}^{(\alpha)}(x)$ . The assumption that  $k$  is a positive integer is not used in deriving (11).

If we replace  $k$  by  $k^{-1}$  and  $\alpha$  by  $k^{-1}\alpha$ , (10) becomes

$$T_{k^{-1},k}^{(-1\alpha)}(x) = \sum_{r=0}^n \frac{x^{kr}}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{k^{-1}(\alpha+s)+n}{n}.$$

On the other hand, since

$$\frac{1}{n!} \left(\frac{s+c+1}{k}\right)_n = \binom{k^{-1}(s+c+1)+n-1}{n},$$

(9) gives

$$Y_n^{c+k-1}(x^k; k) = \sum_{r=0}^n \frac{x^{kr}}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \binom{k^{-1}(s+c)+n}{n}.$$

It follows at once that

$$(12) \quad Y_n^{c+k-1}(x^k; k) = T_{k^{-1},n}^{(k^{-1}c)}(x),$$

or, if we prefer,

$$(13) \quad Y_n^{k\alpha+k-1}(x^k; k) = T_{k-1, n}^{(\alpha-1)}(x) .$$

4. It may be of interest to point out that a formula equivalent to (9) can be obtained without the use of the Lagrange expansion. In the integral representation (5), put

$$t = (1 + u)^{1/k} - 1 .$$

Then (5) becomes

$$Y_n^c(x; k) = \frac{1}{2\pi i} \int_C \frac{\exp\{-x[(1 - u)^{1/k} - 1]\}(1 + u)^{k^{-1}(c + 1) + n - 1}}{u^{n+1}} du ,$$

where  $C$  denotes a small circle about the origin in the  $u$ -plane. The numerator of the integral is equal to

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} (1 + u)^{k^{-1}(c+s+1) + n - 1} \\ & = \sum_{m=0}^{\infty} u^m \sum_{r=0}^m \frac{x^r}{r!} \sum_{s=0}^r (-1)^r \binom{r}{s} \binom{k^{-1}(c + s + 1) + n - 1}{m} . \end{aligned}$$

Taking  $m = n$ , we therefore get

$$(14) \quad Y_n^c(x; k) = \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^r \binom{r}{s} \binom{k^{-1}(c + s + 1) + n - 1}{n} .$$

Since

$$\binom{c + n - 1}{n} = \frac{(c)_n}{n!} ,$$

it is evident that (14) is identical with (9).

5. Making use of the explicit formulas (4) and (9), we can give a rather brief proof of (3). Indeed we have

$$\begin{aligned} J_{n,m} &= \int_0^{\infty} e^{-x} x^c Z_n^c(x; k) Y_m^c(x; k) dx \\ &= \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{\Gamma(kj + c + 1)} \\ &\quad \cdot \frac{1}{m!} \sum_{r=0}^m \frac{1}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s + c + 1}{k}\right)_m \cdot \int_0^{\infty} e^{-x} x^{kj+c+r} dx \\ &= \frac{\Gamma(kn + c + 1)}{n! m!} \sum_{j=0}^n (-1)^j \binom{n}{j} \\ &\quad \cdot \sum_{r=0}^m \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s + c + 1}{k}\right)_m \binom{kj + c + r}{r} . \end{aligned}$$

If  $f(x)$  is a polynomial of degree  $m$ , it is familiar that

$$f(x) = \sum_{r=0}^m \binom{x}{r} \Delta^r f(0),$$

where

$$\Delta^r f(0) = \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} f(s).$$

In particular, for

$$f(x) = \left( \frac{x + c + 1}{k} \right)_m,$$

we have

$$\begin{aligned} \left( \frac{x + c + 1}{k} \right)_m &= \sum_{r=0}^m \binom{x}{r} \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} \left( \frac{s + c + 1}{k} \right)_m \\ &= \sum_{r=0}^m \binom{x + r - 1}{r} \sum_{s=r}^m (-1)^s \binom{r}{s} \left( \frac{s + c + 1}{k} \right)_m. \end{aligned}$$

For  $x = -kj - c - 1$  this reduces to

$$(-j)_m = \sum_{r=0}^m \binom{kj + c + r}{r} \sum_{s=0}^r (-1)^s \binom{r}{s} \left( \frac{s + c + 1}{k} \right)_m.$$

Thus

$$\begin{aligned} J_{n,m} &= \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(-j)_m}{m!} \\ &= (-1)^m \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j}{m}. \end{aligned}$$

Since

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j}{m} = \binom{n}{m} \sum_{j=m}^n (-1)^j \binom{n-m}{j-m} = (-1)^m \binom{n}{m} (1-1)^{n-m}$$

it is evident that

$$(15) \quad J_{n,m} = \frac{\Gamma(kn + c + 1)}{n!} \delta_{nm}$$

in agreement with (3). In particular

$$J_{n,n} = \frac{\Gamma(kn + c + 1)}{n!}$$

as proved in [2].

A little more generally, we have

$$\begin{aligned}
 J'_{n,m} &= \int_0^\infty e^{-x} x_c Z_n^c(x; k) Y_n^{c'}(x; k) dx \\
 &= \frac{\Gamma(kn + c + 1)}{n! m!} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(-j - \frac{c - c'}{k}\right)_m \\
 &= (-1)^m \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j + a}{m},
 \end{aligned}$$

where  $a = (c - c')/k$ . It follows that

$$(16) \quad J'_{n,m} = \begin{cases} 0 & (n > m), \\ (-1)^{n+m} \frac{\Gamma(kn + c + 1)}{n!} \binom{a}{m - n} & (n \leq m). \end{cases}$$

Clearly (16) includes (15).

### REFERENCES

1. S. K. Chatterjea, *A generalization of Laguerre polynomials*, *Collectanea, Mathematica* **15** (1963), 285-292.
2. J. D. E. Konhauser, *Biorthogonal polynomials suggested by the Laguerre polynomials*, *Pacific J. Math.* **21** (1967), 303-314.
3. F. J. Palas, *A Rodrigues formula*, *Amer. Math. Monthly* **66** (1959), 402-404.
4. G. Pólya and G. Szegő, *Aufgaben und Lehrsatze aus der Analysis*, Vol. 1, Berlin, 1925.

Received May 6, 1966. Supported in part by NSF Grant GP-5174.

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN

Stanford University  
Stanford, California

J. DUGUNDJI

Department of Mathematics  
Rice University  
Houston, Texas 77001

J. P. JANS

University of Washington  
Seattle, Washington 98105

RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.



Duane W. Bailey, <i>On symmetry in certain group algebras</i> .....	413
Lawrence Peter Belluce and Surender Kumar Jain, <i>Prime rings with a one-sided ideal satisfying a polynomial identity</i> .....	421
L. Carlitz, <i>A note on certain biorthogonal polynomials</i> .....	425
Charles O. Christenson and Richard Paul Osborne, <i>Pointlike subsets of a manifold</i> .....	431
Russell James Egbert, <i>Products and quotients of probabilistic metric spaces</i> .....	437
Moses Glasner, Richard Emanuel Katz and Mitsuru Nakai, <i>Bisection into small annuli</i> .....	457
Karl Edwin Gustafson, <i>A note on left multiplication of semigroup generators</i> .....	463
I. Martin (Irving) Isaacs and Donald Steven Passman, <i>A characterization of groups in terms of the degrees of their characters. II</i> .....	467
Howard Wilson Lambert and Richard Benjamin Sher, <i>Point-like 0-dimensional decompositions of <math>S^3</math></i> .....	511
Oscar Tivis Nelson, <i>Subdirect decompositions of lattices of width two</i> .....	519
Ralph Tyrrell Rockafellar, <i>Integrals which are convex functionals</i> .....	525
James McLean Sloss, <i>Reflection laws of systems of second order elliptic differential equations in two independent variables with constant coefficients</i> .....	541
Bui An Ton, <i>Nonlinear elliptic convolution equations of Wiener-Hopf type in a bounded region</i> .....	577
Daniel Eliot Wulbert, <i>Some complemented function spaces in <math>C(X)</math></i> .....	589
Zvi Ziegler, <i>On the characterization of measures of the cone dual to a generalized convexity cone</i> .....	603