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BISECTION INTO SMALL ANNULI

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Moses Glasner, Richard Katz, and Mitsuru Nakai

In a Riemannian manifold the modulus of a relatively compact set with border consisting of two sets of components is introduced to measure its magnitude from the viewpoint of harmonic functions. The existence of a subdivision into two sets each having modulus arbitrarily close to one is established.

1. Let M be a Riemannian manifold, i.e. a connected orientable C^{∞} *n*-manifold that carries a metric tensor g_{ij} . Consider a bordered compact region $E \subset M$ whose border is the union of two nonempty disjoint sets α and β of components. We shall call the configuration (E, α, β) an *annulus*.

Let h be the harmonic function on E with continuous boundary values 0 on α and $\log \mu > 0$ on β such that

(1)
$$\int_{\alpha} *dh = 2\pi .$$

The number $\mu > 1$ is called the *modulus* of the annulus (E, α, β) and we set

$$\mu = \operatorname{mod}(E, \alpha, \beta) .$$

Let w be the harmonic measure of β with respect to E, i.e. the harmonic function on E with continuous boundary values 0 on α and 1 on β . By using Green's formula we obtain

(2)
$$\log \mu = \frac{2\pi}{D_E(w)},$$

where $D_E(w)$ denotes the Dirichlet integral $\int_E dw \wedge *dw$ of w over E.

An illustration of these concepts is obtained by taking the annulus $E = \{x \mid r \leq |x| \leq R\}$ in *n*-dimensional $(n \geq 3)$ Euclidean space. The harmonic measure of |x| = R with respect to E is

$$w = rac{|x|^{2-n} - r^{2-n}}{R^{2-n} - r^{2-n}}$$

and the modulus of (E, |x| = r, |x| = R) is given by

$$\log \mu = \pi^{1-(n/2)}(2-n) arGam(rac{n}{2})(R^{2-n}-r^{2-n})$$
 .

Note that $\mu > 1$, in a sense, measures the relative thickness of E and that $\mu \rightarrow 1$ as $R - r \rightarrow 0$.

Our result gains interest if we generalize the notion of annulus slightly. Let (E_j, α_j, β_j) $(j = 1, \dots, m)$ be annuli such that $E_i \cap E_j = \emptyset$ for $i \neq j$. Set $E = \bigcup_{j=1}^{m} E_j$, $\alpha = \bigcup_{j=1}^{m} \alpha_j$, $\beta = \bigcup_{j=1}^{m} \beta_j$. Then we shall also call the configuration (E, α, β) an annulus. The modulus $\mu = \mod (E, \alpha, \beta)$ and the harmonic measure of E with respect to β are defined exactly as for a connected annulus. Moreover, formula (2) is valid and consequently we have

(3)
$$\frac{1}{\log \mu} = \sum_{j=1}^{m} \frac{1}{\log \mu_j}$$
,

where $\mu_j = \mod(E_j, \alpha_j, \beta_j)$.

2. Let M be a noncompact Riemannian manifold throughout this number. A function which is positive and harmonic on M except for a fundamental singularity is called a *Green's function* if it majorizes no nonconstant positive harmonic functions on M. If a Green's functions exists, then M is called *hyperbolic*; otherwise it is called *parabolic*.

An increasing sequence (Ω_n) of bordered compact regions is called an *exhaustion* of M if $\bigcup \Omega_n = M$. Note that the configuration $(\Omega_{n+1} - \overline{\Omega}_n, \partial \Omega_n, \partial \Omega_{n+1})$ is an annulus and denote its modulus by μ_n .

The parabolicity of a noncompact Riemannian manifold M is characterized by the following

MODULAR CRITERION. There exists an exhaustion (Ω_n) of M with $\prod \mu_n = \infty$ if and only if M is parabolic.

In the 2-dimensional case this criterion has been established by Sario [5] and Noshiro [4] and their work can easily be generalized to arbitrary Riemannian manifolds (cf. Smith [7], Glasner [2]).

One naturally asks whether a convergent modular product has any bearing on the hyperbolicity of a manifold. The main result of this paper is that any annulus can be separated into two annuli each having modulus less than $1 + \varepsilon$. This clearly answers the question in the negative and also settles Problem 3 in Sario [6].

3. Suppose the annulus (E, α, β) has components (E_j, α_j, β_j) $(j = 1, \dots, m)$. Let γ_j be a hypersurface in E_j such that $E_j - \gamma_j = E'_j \cup E''_j$, $E'_j \cap E''_j = \emptyset$, and $(E'_j, \alpha_j, \gamma_j)$ and $(E''_j, \gamma_j, \beta_j)$ are annuli. Set $\gamma = \bigcup_{j=1}^m \gamma_j$. We shall call γ a bisecting surface of (E, α, β) . Also set $E' = \bigcup_{j=1}^m E'_j$ and $E'' = \bigcup_{j=1}^m E''_j$. We are now able to state the

THEOREM. Given an annulus (E, α, β) and $\varepsilon > 0$ there exists a bisecting surface γ of (E, α, β) such that

$$(\ 4\) \qquad \mod (E^{\,\prime},\,lpha,\,\gamma) < 1+arepsilon,\,\, \mathrm{mod}\,(E^{\,\prime\prime},\,\gamma,\,eta) < 1+arepsilon\,.$$

This was established by Sario [5] for doubly connected plane regions using Koebe's distortion theorem. All proofs for the 2-dimensional case known to the authors use either a distortion theorem, in essence, or an estimate (cf. Akaza-Kuroda [1]) obtained by means of Möbius transformations (Nakai-Sario [3]) which cannot be generalized to higher dimensions. Therefore, one is led to estimate the Dirichlet integral of the harmonic measure directly and the proof presented here seems to even give a more elementary proof for the 2-dimensional case.

4. Denote by $C(a, b) = C_{x_0}(a, b)$ the Euclidean cylinder

(5)
$$\sum_{j=1}^{n-1} (x^i - x_0^i)^2 < a^2, \ x_0^n < x^n < x_0^n + b$$
,

where a, b > 0 and $x_0 = (x_0^1, \dots, x_0^n)$ is a fixed point. Let $\mathfrak{F}(a, b)$ be the class of C^1 functions f on C(a, b) with continuous boundary values 0 on $\overline{C(a, b)} \cap \{x^n = x_0^n\}$ and 1 on $\overline{C(a, b)} \cap \{x^n = x_0^n + b\}$. Also denote by D^e the Dirichlet integral with respect to the Euclidean metric. We set s equal to the surface area of $\sum_{i=1}^{n-1} (x^i)^2 = 1, x^n = 0$ and state the

LEMMA. For every $f \in \mathfrak{F}(a, b)$,

(6)
$$D^{s}_{C(a,b)}(f) \ge \frac{sa^{n-1}}{b}$$

and equality holds for $f_0(x) = b^{-1}(x^n - x_0^n)$.

Clearly (6) is valid with equality for f_0 . To prove (6) for an arbitrary f we may assume $f \in C^1$ in a neighborhood of $\overline{C(a, b)}$. By Green's formula we have

$$D^{e}_{C(a,b)}(f-f_{\scriptscriptstyle 0},f_{\scriptscriptstyle 0})=\int_{\partial C(a,b)}(f-f_{\scriptscriptstyle 0})rac{\partial f_{\scriptscriptstyle 0}}{\partial n}ds=0$$
 ,

since $f - f_0 = 0$ on the upper and lower boundary of the cylinder and $(\partial f_0/\partial n) = 0$ on the side of the cylinder. Consequently Schwarz's inequality yields

$$D^{\,m{e}}_{C\,(a,\,b)}(f)\!\cdot\!D^{\,m{e}}_{C\,(a,\,b)}(f_{\,_0}) \geqq (D^{\,m{e}}_{C\,(a,\,b)}(f,\,f_{\,_0}))^2 = (D^{\,m{e}}_{C\,(a,\,b)}(f_{\,_0}))^2$$
 ,

which completes the proof.

5. We are ready to prove the main result. Take a point $x_0 \in \alpha$ and a point $y_0 \in \beta$. Let x^1, \dots, x^n be a local coordinate system at 460

 $x_0 = (x_0^1, \dots, x_0^n)$ valid in a neighborhood U of x_0 such that $U \cap \alpha$ is given by $x^n = x_0^n$ and x^n increases as x moves from α to E. Similarly, let y^1, \dots, y^n be a local coordinate system at $y_0 = (y_0^1, \dots, y_0^n)$ valid in a neighborhood V of y_0 such that $V \cap \beta$ is given by $y^n = y_0^n$ and y^n increases as y moves from β to E. Choose a constant c > 0 so small that

(7)
$$\sqrt{g} \mid U \cup V > \sqrt{c}$$

and also

$$(8) \qquad (g^{ij} \mid U \cup V) \xi_i \xi_j \ge \sqrt{c} \sum_{i=1}^n (\xi_i)^2$$

for every vector (ξ_1, \dots, ξ_n) . Now choose a > 0 sufficiently small to insure that $\sum_{i=1}^{n-1} (x^i - x_0^i) < a^2$ with $x^n = x_0^n$ and $\sum_{i=1}^{n-1} (y^i - y_0^i)^2 < a^2$ with $y^n = y_0^n$ are contained in $U \cap \alpha$ and $V \cap \beta$, respectively. Finally choose b > 0 so that

$$egin{aligned} (9) & 0 < b < rac{csa^{n-1}\log{(1+arepsilon)}}{2\pi}\,, \ & \overline{C_{x_0}(a,\,b)} - \{x^n = x_0^n\} \!\subset\! E, \quad \overline{C_{y_0}(a,\,b)} - \{y^n = y_0^n\} \!\subset\! E \end{aligned}$$

and

$$\overline{C_{x_0}(a, b)} \cap \overline{C_{y_0}(a, b)} = \varnothing$$
.

Now take a bisecting surface γ of (E, α, β) subject to the requirements

$$\gamma \cap (C_{x_0}(a, b) \cup C_{y_0}(a, b)) = \emptyset$$

and

$$\gamma \supset [\overline{C_{x_0}(a, b)} \cap \{x^n = x_0^n + b\}] \cup \overline{C_{y_0}(a, b)} \cap \{y^n = y_0^n + b\}].$$

Let w' (resp. w'') be the harmonic measure of γ (resp. β) with respect to E' (resp. E''). Since $E' \supset C_{x_0}(a, b)$, by using (7) and (8) we obtain

(10)
$$D_{E'}(w') > D_{C_{x_0}(a,b)}(w') \ge c D^{s}_{C_{x_0}(a,b)}(w') .$$

Hence by using (6) and (9) we have

$$rac{2\pi}{D_{\scriptscriptstyle E'}(w')} < \log{(1+arepsilon)}$$

and in view of (2) we conclude that

 $\mathrm{mod}\left(E^{\prime},\,lpha,\,\gamma
ight) <1+arepsilon$.

A similar consideration for E'' establishes (4).

References

1. T. Akaza and T. Kuroda, Module of annulus, Nagoya Math. J. 18 (1961), 37-41.

2. M. Glasner, Harmonic functions with prescribed boundary behavior in Riemannian spaces, Doctoral dissertation, University of California, Los Angeles, 1966.

3. M. Nakai and L. Sario, Classification theory (monograph, to appear).

4. K. Noshiro, Open Riemann surface with null boundary, Nagoya Math. J. 3 (1951), 73-79.

L. Sario, Modular criteria on Riemann surfaces, Duke Math. J. 20 (1953), 279-286.
 ——, Classification of locally Euclidean spaces, Nagoya Math. J. 25 (1965), 87-111.

7. S. Smith, *Classification of Riemannian spaces*, Doctoral dissertation, University of California, Los Angeles, 1965.

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