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Let X and Z be compact Hausdorff spaces, and let P be a linear subspace of $C(X)$ which is isometrically isomorphic to $C(Z)$. In this paper conditions, some necessary and some sufficient, are presented which insure that P is complemented in $C(X)$. For example if X is metrizable, P contains a strictly positive function, and the decomposition induced on X by P is lower semi-continuous then P is complemented in $C(X)$.

D. Amir has shown that not all such spaces P are complemented when X is metrizable ([1], see also R. Arens, [4]). However, R. Arens [4] has constructed a class of subspaces of $C(X)$ which are complemented. In § 2 we present classes of complemented subspaces which extend the class exhibited by R. Arens [Theorem 4, Lemma 5, Theorem 8]. A comparison of these results precedes Theorem 8.

Suppose that X is the Stone-Čech compactification of a locally compact completely regular space Y , Z is a compactification of Y which has first countable remainder, and P is the natural embedding of $C(Z)$ in $C(X)$. In § 3 we show that if P is complemented in $C(X)$, then Y is pseudo-compact. This theorem was proved by J. Conway [6] for the case in which Z is the one point compactification of Y .

By introducing the concept of weakly separating in § 2, we are paralleling the concept of a Choquet boundary. Related results and definitions are found in [22].

1. If A and B are subsets of a topological space, $\text{cl } A$ will denote the closure of A , and $A-B$ will denote the set of points which are in A but not in B . If E is a normed linear space, $S(E)$ and E^* denote the unit ball in E and the dual of E respectively. If K is a convex subset of a topological vector space, $\text{ext } K$ will represent the set of extreme points of K . If g and h are functions such that the range of g is contained in the domain of h , the composite of g and h will be written $h \circ g$. Finally, if X is a topological space and x is in X , the point evaluation functional associated with x is the linear functional x' defined on $C(X)$ by $x'(f) = f(x)$ for each f in $C(X)$. In this paper $C(X)$ will denote the Banach space of all bounded real-valued continuous functions on X normed with the supremum norm.

2. Let P be a subspace of a normed linear space E . We define $D(P) = \{b \text{ in } S(E^*): b \text{ restricted to } P \text{ is in } \text{ext } S(P^*)\}$. We say that P is *weakly separating* (with respect to E) if P separates the points

of $D(P)$ intersect $\text{ext } S(E^*)$, that is, if g and h are distinct points in this intersection, then there is a p in P such that $g(p) \neq h(p)$. Although we have stated the definition for an arbitrary normed linear space, we are mainly interested in the space $E = C(X)$, where X is a compact Hausdorff space. It follows readily from the definition that a subspace P of $C(X)$ is weakly separating if for any two distinct point evaluation functionals x' and y' whose restrictions to P have norm one, there is a p in P such that $|p(x)| \neq |p(y)|$. In particular, a subspace of $C(X)$ which contains the constants and separates the points of X , or a closed ideal in $C(X)$ is weakly separating.

LEMMA 1. *Let P be a subspace of E . The following are equivalent:*

- (i) P separates the members of $D(P)$
- (ii) P separates the members of $D(P)$ intersect $\text{ext } S(E^*)$
- (iii) $\text{ext } S(E^*)$ contains $D(P)$.

Proof. (iii) implies (i). If P does not separate the elements of $D(P)$, then there must exist distinct elements g and h in $D(P)$ such that the restriction of $g - h$ to P is the zero functional. It follows that $b = (1/2)(g + h)$ agrees with g and h on P . Hence b is in $D(P)$ but not in $\text{ext } S(E^*)$.

(ii) implies (iii). Now suppose that P separates the elements of $D(P)$ intersect $\text{ext } S(E^*)$. Let b be a point in $D(P)$. We are to prove that b is in $\text{ext } S(E^*)$. Let $K = \{k \text{ in } S(E^*): k \text{ agrees with } b \text{ on } P\}$. Clearly K is a convex set containing b . Also K is closed, and hence compact, in the weak* topology on E^* . By the Krien-Milman theorem, K has extreme points. We will show that $\text{ext } K$ is contained in $\text{ext } S(E^*)$. Suppose $k = (1/2)(g + h)$ where k is in $\text{ext } K$ and g and h are in $S(E^*)$. Thus for each p in P , $1/2h(p) + 1/2g(p) = k(p) = b(p)$. The restrictions of g and h to P both belong to $S(P^*)$, and the restriction of b is in $\text{ext } S(P^*)$. Therefore g and h agree with b on P and both must belong to K . Since k was assumed to be an extreme point of K , we have $g = h = k$. We conclude that $\text{ext } S(E^*)$ contains $\text{ext } K$. If b is the only point in K , then b must be in $\text{ext } S(E^*)$. Otherwise K must contain two distinct extreme points. Clearly P can not separate these two points of $D(P)$ intersect $\text{ext } S(E^*)$. This proves that (ii) implies (iii).

Since the fact that (i) implies (ii) is obvious, the proof is complete.

LEMMA 2. *If P is weakly separating in E , then the weak topology on $D(P)$ induced by P is equivalent to the weak topology induced by E .*

Proof. Clearly, the weak topology induced by P is coarser than the one induced by E . To prove the converse, suppose that g_i is a net of functionals in $D(P)$ which converge with respect to the weak topology induced by P to a functional g which is also in $D(P)$. If g_i does not converge to g with respect to the weak topology induced by E , there will exist a subnet which never intersects some neighborhood (in topology induced by E) of g . Since by Alaoglu's theorem $S(E^*)$ is compact, we may assume the existence of a further subnet g_j which converges to a functional h distinct from g . Since g_j is a subset of g_i , h must agree with g on P . Since the norm of h is less than or equal to one, h is in $D(P)$. Since P does not distinguish between g and h , the previous lemma contradicts the hypothesis that P is weakly separating. The lemma is proved.

In the following let X be a compact Hausdorff space.

LEMMA 3. *Let P be a weakly separating subspace of $C(X)$. The following are equivalent:*

- (i) *There is a projection of norm one of $C(X)$ onto P ,*
- (ii) *P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z ,*
- (iii) *There exist a closed subset Y of X such that P is isometrically isomorphic to $C(Y)$ via the restriction mapping.*

Furthermore, if P is weakly separating there can exist at most one projection of norm one of $C(X)$ onto P .

Proof. (i) *implies* (iii). Let L be a projection of norm one of $C(X)$ onto P . If x' is an evaluation functional in $D(P)$, then $x' \circ L$ is a functional in $S(C(X)^*)$ which agrees with x' on P . Since P is weakly separating in $C(X)$, $x' \circ L = x'$. Hence for each f in $C(X)$, Lf agrees with f on $\{x \text{ in } X: x' \text{ is in } D(P)\}$, and therefore on the closure Y of this set. With a simple application of the Tietze Extension Theorem, we see that the restriction map carries P onto $C(Y)$. Furthermore, this restriction mapping does not decrease the norm of points in P . For by Lemma 1 every functional in $D(P)$ can be expressed as either an evaluation functional of a point in Y or as the negative of such a functional, and for p in P , $\|p\| = \sup \{h(p): h \text{ in } D(P)\}$. We have shown that the restriction mapping is an isometric isomorphism of P onto $C(Y)$.

(ii) *implies* (i). Let Z be a compact Hausdorff space, and let L be an isometric isomorphism of P onto $C(Z)$. Let L' denote the adjoint of L . Since L is an isometric isomorphism, L' is an isometric isomorphism of $C(Z)^*$ onto P^* . Furthermore, L' restricted to $\text{ext } S(C(Z)^*)$ is a homeomorphism onto $\text{ext } S(P^*)$ with the weak topologies induced by $C(Z)$ and P respectively. Now for x in $\text{ext } S(P^*)$, let

$H(x)$ be the unique element in $\text{ext } S(C(X)^*)$ which agrees with x on P . For z in Z let $E(z)$ denote the evaluation functional of z . Now for f in $C(X)$ consider the function $f \circ H \circ L' \circ E(\cdot)$ defined on Z . By Lemma 2 this function is continuous. The map Q which carries f in $C(X)$ onto $L^{-1}(f \circ H \circ L' \circ E(\cdot))$ is a mapping of norm one of $C(X)$ into P . Furthermore, if p is in P , then $p \circ H \circ L' \circ E(z) = Lp(z)$, for all z in Z . Thus $p \circ H \circ L' \circ E(\cdot) = Lp$, and Q is a projection of $C(X)$ onto P .

It is evident that (iii) implies (ii).

To prove the second part of the lemma, suppose that H and L are two projections from $C(X)$ onto P , both of which have norm one. Let Y be the subset of X constructed in the proof that (i) implies (iii). For any f in $C(X)$, we have shown that Lf , Hf and f all agree on Y . It of course follows that $(H - L)(f)$ vanishes on Y . However, we have shown that the restriction mapping carries P isometrically onto $C(Y)$. Therefore, $(H - L)(f)$ must be the zero function, and $Hf = Lf$ for all f in $C(X)$. This completes the proof.

We will say that a subspace P of $C(X)$ has a *weakly separating quotient* if it has the property that for any two distinct points x and y in X such that $p(x) = -p(y)$ for every p in P , then the evaluation functional of x (or equivalently the evaluation functional of y) restricted to P is not an extreme point of $S(P^*)$.

REMARK. Each of the following properties on a subspace P of $C(X)$ imply that P has a weakly separating quotient:

- (i) P is weakly separating in $C(X)$,
- (ii) P contains a function which is strictly positive,
- (iii) for each p in P , $|p|$ is also in P .

A proof for the above remark is straightforward. In particular, any closed ideal in $C(X)$, or any subspace of $C(X)$ which contains the constants has a weakly separating quotient.

In order to state the next theorem we make a few more definitions. Let X be a Hausdorff space and let M be a partition of X into closed subsets. For x in X let $M(x)$ denote the member of M which contains x . Corresponding to the standard definitions we say that M is *lower semi-continuous* if $\{x \text{ in } X: M(x) \text{ intersect } U \text{ is non-empty}\}$ is an open set in X for every open set U in X .

If P is a linear space of bounded, continuous functions, then the P -partition of X is the partition associated with the following equivalence relation R . A couple (x, y) is in R if and only if $p(x) = p(y)$ for every p in P . Now let $K(P) = \cup \{K \text{ contained in } X: K \text{ is a member of the } P\text{-partition of } X, \text{ and } K \text{ contains more than one point of } X\}$. We will say that P has a *lower semi-continuous quotient* if the restriction of the P -partition to $\text{cl } K(P)$ is lower semicontinuous.

In the following let X denote a compact Hausdorff space, and let

P be a linear subspace of $C(X)$ which has a weakly separating quotient.

THEOREM 4. *If there is a projection of norm one of $C(X)$ onto P , then P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z . Conversely, suppose that X is metrizable, and that P has a lower semi-continuous quotient. If P is isometrically isomorphic to $C(Z)$, for some compact Hausdorff space Z , then there is a projection of $C(X)$ onto P which has norm less than or equal three.*

Proof. Let M denote the P -partition of X . Let X/M have the quotient topology, and let $M(\cdot)$ denote the natural mapping of X onto X/M . We observe that X/M is a compact Hausdorff space. Now let Q denote the linear subspace of $C(X)$ consisting of all functions that are constant on each closed subset of X which is a member of M . One can verify that P is contained in Q , and that the mapping which carries q in Q onto the function $q \circ M^{-1}(\cdot)$ in $C(X/M)$ is an isometric isomorphism of Q onto $C(X/M)$. The image P' of P under this mapping is a weakly separating subspace of $C(X/M)$ since P has a weakly separating quotient. If there is a projection of norm one from $C(X)$ onto P , then there certainly is a projection of norm one from $C(X/M)$ onto P' . By the preceding lemma, we conclude that P' , and hence P , is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z .

To prove the second part of the theorem, we assume that X is metrizable, P has a lower semi-continuous quotient, and that there is a compact Hausdorff space Z such that P is isometrically isomorphic to $C(Z)$. We maintain the same notation used directly above. Since P' is weakly separating in $C(X/M)$, and P is isometrically isomorphic to $C(Z)$, it follows from the preceding lemma that there is a projection of norm one from Q onto P . To complete the proof it will suffice to show that there is a projection from $C(X)$ onto Q which has norm less than or equal to three. We will prove a stronger result.

Let Y be a metric space. Let K be a partition of Y such that every member of K is a complete subset of Y . A member of K will be called a *plural set* if it contains two distinct points of Y . Let the restriction K' of K to the subset of Y ,

$$B = \text{cl} \cup \{A \text{ contained in } Y: A \text{ a plural set in } K\}$$

be lower semi-continuous. Assume also that B/K' is paracompact. Let Q denote the subspace of $C(Y)$ consisting of the functions which are constant on each member of K . We recall that by the notation we adopted, $C(Y)$ is the Banach space of all bounded continuous functions on Y . The following lemma establishes the theorem.

LEMMA 5. *There is a projection of $C(Y)$ onto Q which has norm less than or equal to three.*

Proof. In the usual manner we can embed B into the unit ball of $C(B)^*$. With the weak topology on $C(B)^*$ induced by $C(B)$, $C(B)^*$ is a locally convex space, B is embedded onto a homeomorphic image of itself, say B' , and the closed convex hull of compact subsets of B' are again compact. Let s denote the composite of the quotient mapping of B onto B/K' with the homeomorphism, h , between B and B' .

We now show that s^{-1} is a lower semi-continuous function carrying points in B/K' onto closed subsets of B' . Let U be an open set in B' . Let

$$W = \{y \text{ in } B/K': s^{-1}(y) \text{ intersect } U \text{ is not empty}\}.$$

To show that s^{-1} is lower semi-continuous we must show that W is open in B/K' . We note that $W = s(U)$. Now since K' is lower semi-continuous and $h^{-1} \circ s^{-1} \circ s \circ h(\cdot)$ carries a point b in B onto the member of K' which contains b , the set

$$V = \{b \text{ in } B: h^{-1} \circ s^{-1} \circ s \circ h(b) \text{ intersect } h^{-1}(U) \text{ is not empty}\}$$

is open in B . Hence $h(V) = \{b' \text{ in } B: h^{-1} \circ s^{-1} \circ s(b') \text{ intersect } h^{-1}(U) \text{ is not empty}\}$ is open in B' . Since this last set is $s^{-1} \circ s(U)$, $s^{-1} \circ s(U)$ is open. Since B/K' has the quotient topology induced by s , this implies that $s(U)$ —and hence W —is open in B/K' . Therefore s^{-1} is lower semi-continuous.

Now since B/K' is paracompact, and since there is a metric on B' (which induces an equivalent topology for B') for which the set $s^{-1}(y)$ is complete for each y in B/K' , we have satisfied the hypothesis for a selection theorem proved by E. Michael [20]. This theorem proves the existence of a continuous function t which carries B/K' into $C(B)^*$, and has property that $t(y)$ is contained in the closed convex hull of $s^{-1}(y)$ for each y in B/K' .

We now define a projection from $C(B)$ onto Q' the subspace of functions in $C(B)$ which are constant on members of K' . For f in $C(B)$, let Lf denote the function such that for each b in B ,

$$(Lf)(b) = [t(s \circ h(b))](f).$$

Since t is continuous on B/K' , Lf is a continuous function. Since $t(s \circ h(b))$ is in the closed convex hull of $s^{-1} \circ s \circ h(b)$, the norm of $t(s \circ h(b))$ does not exceed one. Thus the maximum of Lf over B does not exceed the maximum of f over B . Finally, one can verify that if q is in Q' , $Lq = q$, and that for each f in $C(B)$, Lf is in Q' . We have shown that L is a projection of norm one of $C(B)$ onto Q' .

Since Y is a metric space, there is an operator E of norm one from $C(B)$ into $C(Y)$ such that $R \circ Ef = f$ for every f in $C(B)$. Here R denotes the operator which assigns to each function in $C(Y)$ its restriction to B (R. Arens [3], also Dugundji [8]). Following a construction due to Arens [4], we define an operator J by $Jf = f + E(LRf - Rf)$. The proof of the lemma is completed by verifying that J is a projection of $C(Y)$ onto Q which has norm no greater than three.

In the following corollaries let X denote a compact Hausdorff space.

COROLLARY 6. *Let P be a finite dimensional subspace of $C(X)$ which has a weakly separating quotient. There is a projection of norm one from $C(X)$ onto P if and only if P has a basis $\{p_i\}_{i=1}^n$ such that $\|\sum_{i=1}^n c_i p_i\| = \max |c_i|$.*

COROLLARY 7. *$C(X)$ contains a weakly separating subspace of co-dimension n which has a projection of norm one if and only if X contains n isolated points.*

Proof. To prove the necessity of the condition, let L be a projection of norm one of $C(X)$ onto a weakly separating subspace P of co-dimension n in $C(X)$. Define $Y = \text{cl}\{x \text{ in } X: x' \circ L = x'\}$. We will show that $X - Y$ contains precisely n points. Since $X - Y$ is open, these points will be isolated. We observe that the range, Q , of $I - L$ has dimension n , and that if q is in Q , then q vanishes on Y . Since the functions in Q take all their nonzero values on $X - Y$, $X - Y$ must contain at least n points. If $X - Y$ contained $n + 1$ points, there would exist $n + 1$ open sets U_i in $X - Y$, and corresponding functions f_i of norm one which vanish off U_i . These functions span an $n + 1$ dimensional subspace of $C(X)$; hence there is a nonzero function f in this span that is also in P . But f vanishes on Y . By Lemma 3, the restriction map is an isometry of P onto $C(Y)$. Hence we arrive at the contradiction that f is the zero function.

If X contains n isolated points, the space of all functions in $C(X)$ which vanish on these n points is a weakly separating subspace of $C(X)$ (since this space is an ideal) of co-dimension n in $C(X)$. It is also clear there is a projection of norm one from $C(X)$ onto this subspace. The proof is completed.

REMARK. R. Arens [4] has constructed an example of two compact metric spaces X and Z such that $C(X)$ contains an isometric isomorphic copy of $C(Z)$ which has a weakly separating quotient, but which is not complemented in $C(X)$. Hence the assumption that P has a lower semi-continuous quotient cannot be simply omitted from the theorem,

(Also see Amir [1]).

The preceding theorem and lemma should be compared to Theorem 2.2 in (R. Arens [4]). Using the notation preceding the lemma, Professor Arens proved that under the following conditions there will exist a projection of norm less than or equal to three of $C(Y)$ onto Q :

- (i) K is a partition of Y into closed subsets
- (ii) Y and Y/K are metrizable
- (iii) the quotient map of Y onto Y/K is upper semi-continuous¹
- (iv) if $\{x_i\}$ is a sequence in Y such that each x_i belongs to a distinct plural set in K , then a member of K which contains a limit point of $\{x_i\}$ is a singleton.

Apropos to property (ii), A. H. Stone has proved ([23]) that a metrizable space is paracompact. Property (iv) above implies that K' is lower semi-continuous. In the special case that Y is a complete metric space, the preceding lemma contains the above theorem of Arens. If Y is compact, the previous theorem includes both of these results.

In the following, let Y be a metrizable space, and K a partition of Y satisfying properties (i), (iii), and (iv) above. For each K_i in K let P_i be a complemented subspace of $C(K_i)$ which contains the constants. Let L_i denote a projection of $C(K_i)$ onto P_i . We assume that $m = \sup \{\|L_i\|\} < \infty$. Finally, let Q denote the subspace of $C(Y)$ consisting of all functions q such that the restriction of q to K_i is a function in P_i .

THEOREM 8. *There is a projection of $C(Y)$ onto Q which has norm less than or equal to $2 + m$.*

Proof. For a set Z let $B(Z)$ denote the space of bounded functions on Z . Let $D = \cup \{K_i \text{ contained in } Y: K_i \text{ is a plural set in } K\}$. Let R and R_i denote the restriction map of $B(Y)$ onto $B(\text{cl } D)$ and of $B(Y)$ onto $B(K_i)$ respectively (K_i in K). Let E denote a linear mapping of $C(\text{cl } D)$ into $C(Y)$ such that E has norm one, and $R \circ E$ is the identity mapping on $C(\text{cl } D)$. Let H be the linear mapping of $C(Y)$ into $B(\text{cl } D)$ such that $R_i \circ H = L_i \circ R_i$ for all K_i in K . Let I denote the identity on $C(Y)$, and let $L = I + E \circ R(H - I)$. The proof consists of establishing that L is the desired projection. The variation of a function f defined on a set Z is $\text{var}(f) = \max_{z \text{ in } Z} f(z) - \min_{z \text{ in } Z} f(z)$.

We proceed by proving four assertions, the last of which establishes the theorem.

Assertion 1. If x_i is in K_i , K_i is in K , y is not in D and x_i con-

¹ Professor Arens has communicated that the assumption that the quotient mapping be upper semi-continuous had been inadvertently omitted from the statement of his theorem.

verges to y , then $\text{var}(R_i f)$ converges to zero for each f in $C(Y)$.

Assertion 2. $\|L_i \circ R_i f - R_i f\| \leq 1/2(1 + m) \text{var}(R_i f)$.

Assertion 3. If f is in $C(Y)$, Hf is in $C(\text{cl } D)$.

Assertion 4. The operator L is a projection from $C(Y)$ onto Q of norm at most $2 + m$.

If Assertion 1 is false it will be possible to find points z_i in K_i and a function f in $C(Y)$ such that for some r greater than zero, $f(x_i) - f(z_i)$ is greater than r . Since f is continuous, we may assume that there is a neighborhood N of y such that z_i does not belong to N . Put $Z = \{z_i\}$. Since the quotient map q of Y onto Y/K is, by hypothesis, closed $q(\text{cl } Z)$ is closed in Y/K . But $q(x_i) = q(z_i)$ is in $q(\text{cl } Z)$, and $q(x_i)$ converges to $q(y)$ by the continuity of q . Thus $q(y) = \{y\}$ is in $q(\text{cl } Z)$, and $\{y\} = q(z)$ for some z in $\text{cl } Z$. But $\text{cl } Z$ is contained in $Y - N$ so $z \neq y$. This contradicts the assumption that y is not in D .

To prove the second assertion, let $c = 1/2 \text{var}(R_i f)$. Since 1 is in P_i , $L_i \circ R_i 1 = 1$. Hence

$$\begin{aligned} \|L_i \circ R_i f - R_i f\| &= \|L_i \circ R_i(f - c) - R_i(f - c)\| \leq \|L_i - I\| \\ &\cdot \|R_i(f - c)\| \leq (m + 1)(1/2) \text{var}(R_i f) . \end{aligned}$$

To prove Assertion 3 let y be a point in $\text{cl } D$. We distinguish two cases. Case 1, y is in D . Let y be in the plural set K_i of the partition K . From the assumption of property (iv) it follows that there is an open set U containing K_i which meets no other plural set in K . Now let f be in $C(Y)$ and let N be a neighborhood of $Hf(y)$. Let V be a neighborhood of y such that $(L_i \circ R_i f)(V \cap K_i)$ is contained in N . Put $W = V \cap U$ and let x be an arbitrary point in W intersect $\text{cl } D$. Then x is in U , and x is in the closed set K_i . This shows that $W \cap \text{cl } D$ is contained in $K_i \cap V$. Hence on $W \cap \text{cl } D$, $Hf = L_i \circ R_i f$. Thus $Hf(W \cap \text{cl } D)$ is contained in $L_i \circ R_i f(K_i \cap V)$ which in turn is contained in N .

Case 2, y is not in D . In this case $\{y\}$ is in K , and $Hf(y) = f(y)$, since each P_i contains the constant functions. Let x_i converge to y . Then

$$|Hf(x_i) - Hf(y)| \leq |Hf(x_i) - f(x_i)| + |f(x_i) - f(y)| .$$

It is clear that $f(x_i)$ converges to $f(y)$. For the other term we use Assertions 1 and 2 above to write, with x_i in K_i (and K_i in K),

$$\begin{aligned} |Hf(x_i) - f(x_i)| &\leq |L_i \circ R_i f(x_i) - R_i f(x_i)| \\ &\leq (1/2)(m + 1) \text{var}(R_i f). \end{aligned}$$

Since this last term converges to zero, Hf is continuous at y .

To prove Assertion 4, we first observe that linearity and bound for L are obvious. If f is in $C(Y)$ we must show that Lf is in Q . Indeed,

$$R \circ L = R + R \circ H - R = R \circ H.$$

Hence

$$R_i \circ L = R_i \circ R \circ L = R_i \circ R \circ H = L_i \circ R_i$$

for each plural set K_i in K . Thus $R_i \circ Lf$ is in P_i for each plural set K_i in K . If K_i is a member of K which is not a plural set then, $R_i \circ Lf$ is in P_i trivially since P_i contains the constants.

Now we must show that if f is in Q then $Lf = f$. Since $R_i f$ is in P_i for all K_i in K , $R_i \circ Hf = L_i \circ R_i f = R_i f$. Thus $R \circ Hf = Rf$, and $Lf = f + E(Rf - Rf) = f$. This completes the proof of the theorem

REMARK. The assumption that Y is metrizable was used only to guarantee the existence of the linear mapping E . If we drop the hypothesis that Y is metrizable and assume outright the existence of a bounded linear mapping E from $C(\text{cl } D)$ into $C(Y)$ such that $R \circ E$ is the identity on $C(\text{cl } D)$, then the same proof establishes the existence of a projection from $C(Y)$ onto Q which has norm less than or equal to $1 + (m + 1) \|E\|$.

COROLLARY 9. *Let Y, K, K_i, P_i , and Q be as in the theorem. If each P_i has dimension less than n , then there is a projection of norm at most $n + 1$ from $C(Y)$ onto Q .*

3. Let X be a locally compact, Hausdorff space. A compactification of X is a compact Hausdorff space that contains X (a homeomorphic image of X) as a dense subspace. The Stone-Ćech compactification of X will be denoted by βX , and the one-point compactification will be denoted by pX .

If K is an arbitrary compactification of X , the linear mapping which carries a function in $C(K)$ onto the unique function in $C(\beta X)$ which agrees with it on X , is an isometric isomorphism of $C(K)$ into $C(\beta X)$. We will therefore assume that $C(\beta X)$ contains $C(K)$.

If Y is a closed subset of a compact Hausdorff space K , I_Y will denote the ideal of functions in $C(K)$ which vanish on Y . Let N denote the non-negative integers with the discrete topology. If K is

a compactification of X , the remainder of K (with respect to X) is the topological space $K - X$ equipped with the relative topology from K . In accordance with the usual terminology let $(m) = C(\beta N)$, $(c) = C(pN)$, and $(c_0) = I_{pN-N} = I_{\beta N-N}$, where the ideals are interpreted as subspaces of $C(pN)$ and $C(\beta N)$ respectively.

THEOREM 10. *Let K be a compactification of X which has a first countable remainder. If there is a bounded linear mapping of $C(\beta X)$ into $C(K)$ which acts as the identity on $I_{\beta X-X}$, then X is pseudocompact.*

We first will prove the following lemma.

LEMMA 11. *Let M be a compactification of N which has a first countable remainder. There does not exist a bounded linear mapping of (m) onto any subspace of $C(M)$ which contains (c_0) .*

Proof of lemma. Since N is both locally compact and the union of a countable family of compact sets, $M - N$ is a compact set which is the intersection of a countable family U of open sets in M . Let x be a point in $M - N$. Let V be a countable family of open sets in M whose intersections with $M - N$ form a basis for the neighborhood system for x in $M - N$. Let W be the countable family of open sets in M of the form u intersect v , where u is in U and v is in V . It is easy to see that the intersection of the members of W is the singleton containing x . A compactness argument shows that W is in fact a basis for the neighborhood system for x in M . Since N is first countable we have established that M is first countable. Hence M is sequentially compact.

There is a sequence of points in N , say J , which converges to some point k in M . Now suppose B is a subspace of $C(M)$ which contains (c_0) . The restriction of functions in B to J union $\{k\}$ carries B onto a Banach space which is either isometrically isomorphic to (c) or to (c_0) . In the former case since (c_0) is complemented in (c) , there is a bounded linear mapping of B onto (c_0) . In either case if there is a bounded linear mapping of (m) onto B , there is a bounded linear mapping, L , of (m) onto (c_0) . But no such mapping can exist. For since (c_0) is a separable Banach space and βN is extremally disconnected, L must be weakly compact (Grothendieck [14], p. 168, Cor. 1). Now an application of the open mapping theorem implies the false assertion that (c_0) is reflexive. This completes the proof of the lemma.

Proof of theorem. If X is not pseudocompact there is countable family of disjoint open sets V_i in X such that $\text{cl } \cup \{V_i\} = \cup \{\text{cl } V_i\}$. For each i let U_i be an open set such that $\text{cl } U_i \subseteq V_i$, let u_i be in U_i ,

and let f_i be a continuous function which vanishes off U_i and attains its norm of one at u_i . For a bounded sequence $x = (x_i, x_2 \dots)$ in (m) , let Ax be the unique function in $C(\beta X)$ which agrees with $\sum_{i=1}^{\infty} x_i f_i$ on X . The mapping A is an isometric isomorphism of (m) onto the range of A . Let L be the hypothesized mapping of the theorem, and let J carry a function in $C(\beta X)$ onto its restriction to $\text{cl}\{u_i\}$. Since $\text{cl}\{u_i\} - \{u_i\}$ is contained in $K - X$, $\text{cl}\{u_i\}$ is homeomorphic to a compactification M of N which has first countable remainder. Let G be the isometric isomorphism of $C(\text{cl}\{u_i\})$ onto $C(M)$ induced by this homeomorphism. The proof is completed by verifying that $G \circ J \circ L \circ A$ is a bounded linear mapping of (m) onto a subspace of $C(M)$ which contains (e_0) .

The case in which K is the one-point compactification of X was first proved by J. Conway ([6]). Examples to show that pseudocompactness of X is not sufficient to guarantee the existence of a projection from $C(\beta X)$ onto $I_{\beta X - X}$ have been constructed by J. Conway ([6]) and by A. Pełczyński and V. N. Sudakov ([21]).

COROLLARY 12. *Let X be an extremally disconnected, compact, Hausdorff space, and let P be a subspace of $C(X)$ which contains the constants and separates the points of X . If P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z , then the Šilov boundary of P is an extremally disconnected subset of X which has a pseudo-compact complement.*

Proof. Under the hypothesis of the corollary, the Šilov boundary of P is the set Y of Lemma 3. To show that Y is extremally disconnected, we intend to apply a theorem due to Nachbin (Trans. AMS, 68 (1950), 28-46, 1950), Goodner ([13]), Kelley ([11]) and others. A Banach space B is called injective if every Banach space which contains an isometric isomorphic copy B' of B , admits a projection of norm one onto B' . The theorem we wish to apply states that a Banach space is injective if and only if it is isometrically isomorphic to $C(Z)$, for a compact, extremally disconnected, Hausdorff space Z . Now $C(X)$ is injective and from Lemma 3 there is a projection of norm one from $C(X)$ onto P . From this it can be shown that $C(Y)$ is injective, and hence Y is extremally disconnected.

From Lemma 3 it follows that I_Y is complemented in $C(X)$. Let $G = X - Y$. Since $\text{cl} G$ is open in X , $I_{\text{cl} G - G}$ is complemented in $C(\text{cl} G)$. Since $\text{cl} G$ is extremally disconnected, it is the Stone-Čech compactification of G ([10], p. 69, Prob. 6M2). By the theorem, G is pseudocompact (in this case K is the one-point compactification of G), and the corollary is proved.

COROLLARY 13. *If X is a locally compact space such that βX has a first countable remainder, then X is pseudocompact.*

REMARK. Relevant to the last corollary, we observe that if Z is any compact Hausdorff space, there is a pseudocompact, locally compact space X such that $\beta X - X$ is homeomorphic to Z . For let y be a nonisolated point in βN and let $X = (\beta N - \{y\}) \times Z$. From results in ([11]) and ([10], 6M3) we have that X is pseudocompact, and $\beta X = \beta N \times Z$.

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