Pacific Journal of Mathematics

ON A CLASS OF CONVOLUTION TRANSFORMS

ZEEV DITZIAN

Vol. 25, No. 1 September 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 1, 1968

ON A CLASS OF CONVOLUTION TRANSFORMS

ZEEV DITZIAN

In this paper the convolution transform

$$(1.1)\qquad f(x)=\int_{-\infty}^{\infty} G(x-t)\varphi(t)dt \equiv (G^*\varphi)(x)$$

whose kernel $G(t)$ is the Fourier transform of $[E(iy)]^{-1}$ where $E(s)$ is defined by

$$(1.2)\qquad E(s) = e^{bs}\prod_{k=1}^{\infty}(1-s/a_k)\exp(sRe\, a_k^{-1}),$$

$$Reb = b \text{ and } \sum |a_k|^{-2} < \infty$$

will be studied. An inversion theory similar to that achieved when a_k of (1.2) are real will be obtained. The results will show that under certain rather weak conditions, an infinite subsequence $a_{k(i)}$ of a_k can satisfy

$$\min\{|\arg a_{k(i)}|, |\arg - a_{k(i)}|\} \geqq \frac{\pi}{4}.$$

Classes of transforms will be introduced that allow the occurrence of $\min\{|\arg a_k|, |\arg - a_k|\} \geqq \pi/4$ for all k.

We hope this will partly answer a problem set by Dauns and Widder [1] in Remark 1, page 441.

The inversion operator $P_m(D)$ is defined by

$$(1.3)\qquad P_m(D) = \exp((b-b_m)D)\prod_{k=1}^{m}\left(1-\frac{D}{a_k}\right)\exp\left(\left(Re\frac{1}{a_k}\right)D\right)$$

where $D \equiv d/dx$, $\exp(kD)f(x) = f(x+k)$ and $\lim_{m\to\infty} b_m = 0$.

The inversion formula will be

$$(1.4)\qquad \lim_{i\to\infty} P_{m(i)}(D)f(x) = \varphi(x).$$

This inversion formula was achieved under general conditions on $\varphi(x)$ in the case a_k were real by I. I. Hirschman and D. V. Widder in a series of papers and in their book, "The convolution transform" [7]. Hirschman and Widder [6] also found a slightly changed version of (1.4) when $\sum_{r=1}^{\infty}(\mathrm{Im}\, a_k/\mathrm{Re}\, a_k)^2 < \infty$. A. O. Garder [5] showed that if $a_{2k-1} = \bar{a}_{2k}$ then arg a_{2k} can tend to 0 or π slower than is required in [6]. Dauns and Widder [1] showed that if $a_{2k-1} = -a_{2k}$, $0 \leqq \mathrm{Re}\, a_{2k-1} \in \uparrow$ and $|\arg a_{2k-1}| < (\pi/4) - \eta$, where η is independent of k, then (1.4) can be achieved.

It will be noted that in [1] and [5] the a_k's were in a special order. The order of the a_k's, though having no influence on $E(s)$, may be quite important when treating (1.4) as discussed with some examples in [2] and [4].

We shall define class $A(2)$ (that will depend also on the order of the a_k's). The sequence $\{a_k\}$ belongs to class $A(2)$ if $\operatorname{Re} a_k \neq 0$,

$$\sum_{k=1}^{\infty} (\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \infty , \tag{1.5}$$

$$(1-\theta)(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1} > 0 \tag{1.6}$$

for $k > k_0$ for some $\theta, 0 < \theta < 1$ where θ is independent of k, and

$$\frac{(\operatorname{Im}\{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})} a_{2k-1}^{-1} a_{2k}^{-1}\})^2 |a_{2k-1} a_{2k}|^2}{|a_{2k-1}|^2 + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}} < 1 - \eta \tag{1.7}$$

for $k \geqq k_1$ for some $\eta, 0 < \eta < 1$ where η is independent of k.

A transform belongs to $A(2)$ if there is an order under which $\{a_n\} \in A(2)$. Class $A(2)$ includes the transforms of [1], [5] and [6].

LEMMA 1.1. $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / \operatorname{Re} a_k)^2 < \infty$ *implies* $\{a_k\} \in A(2)$ (and the order does not matter).

Proof. $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / \operatorname{Re} a_k)^2 < \infty$ implies $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / |a_k|)^2 < \infty$ which implies $\sum_{k=1}^{\infty} (\operatorname{Im} a_k^{-1})^2 / |a_k|^{-2} < \infty$ which implies (1.5). To prove that a_k satisfies (1.6) and (1.7) is not difficult.

REMARK. The inversion operator introduced by Hirschman and Widder [6] was slightly different from (1.4) but since

$$\sum_{k=1}^{\infty} \{(\operatorname{Re} a_k)^{-1} - \operatorname{Re} a_k^{-1}\} = \sum_{k=1}^{\infty} \frac{(\operatorname{Im} a_k)^2}{|a_k|^2 \operatorname{Re} a_k} < \infty ,$$

the difference is a change in b and b_m without changing $\lim_{m \to \infty} b_m = 0$.

LEMMA 1.2. *Let* $a_{2k-1} = -a_{2k}$, *let* $\operatorname{Re} a_{2k} > 0$ *and* $|\arg a_{2k}| < (\pi/4) - \eta_1$ *for* $k > k_2$, *where* η_1 *satisfies* $0 < \eta_1 < \pi/4$ *and* η_1 *is independent of* k, *then* $\{a_k\} \in$ *class* $A(2)$.

Proof. It is easy to see that the sum in (1.5) is equal to zero and the right side of (1.7) is equal to zero. $|\arg a_{2k}| < (\pi/4) - \eta_1$ implies (1.6), with $\theta = 1 - 2(\operatorname{Sin}((\pi/4) - \eta_1))^2$, for $k > k_2$.

This shows that the transforms treated in [1] are included in class $A(2)$.

LEMMA 1.3. *Let* $a_{2k-1} = \overline{a_{2k}}$ *and let* $\min\{|\arg a_{2k}|, |\arg - a_{2k}|\} < (\pi/4) - \eta_2$ *for* $k \geqq k_2$ *where* η_2, $0 < \eta_2 < \pi/4$, *is independent of* k, *then* $\{a_k\} \in A(2)$.

Proof. It is easy to see that the sum in (1.5) and the right side of (1.7) are equal to zero. One can show that $\min\{|\arg a_{2k}|, |\arg - a_{2k}|\} < (\pi/4) - \eta_2$ implies (1.6) with $\theta = 1 - 2(\mathrm{Sin}\,((\pi/4) - \eta_2))^2$ for $k \geqq k_2$.

Lemma 1.3 shows that the transforms treated by A. O. Garder [5] belong to class $A(2)$. Some cases which do not belong to class $A(2)$ will be treated, among them will be the case when $a_{2k-1} = -a_{2k}$ and $\min\{|\arg a_{2k}|, |\arg - a_{2k}|\} = \pi/4$ (see Remark 2, [1], p. 442) where estimates different from those achieved for class $A(2)$ will be obtained.

For the definition of $G(t)$

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[E(s)]^{-1}e^{st}dt \tag{1.8}$$

we have to assume that the integral on the right converges.

For the convergence of (1.8) we shall have to estimate $E(iy)$ and to these estimates the various classes correspond.

2. Estimates for $E_{2m}(s)$ when $\{a_k\} \in$ class $A(2)$. In previous papers (see [1] and [6] for example) it was found useful and important to estimate $E_m(s)$ which is defined by

$$E_m(s) = e^{b_m s}\prod_{k=m+1}^{\infty}(1 - s/a_k)\exp(s\,\mathrm{Re}\,a_k^{-1})\ . \tag{2.1}$$

In order to estimate $E_m(s)$ we shall estimate one term first.

LEMMA 2.1. *Let $\{a_k\} \in$ class $A(2)$ then for $k \geqq K$*

$$\begin{aligned} &|(1 - iy/a_{2k-1})(1 - iy/a_{2k})|^2 \\ &\quad \geqq (1 + \alpha y^2/|a_{2k-1}|^2)(1 + \alpha y^2/|a_{2k}|^2)(1 - \alpha^{-1}[(\mathrm{Im}\,(a_{2k-1}^{-1} \\ &\qquad + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})])\ . \end{aligned} \tag{2.2}$$

where $0 < \alpha < 1$ and α is independent of k. (α does depend on θ and η of the definition of class $A(2)$).

Proof. By a simple calculation we get

$$\begin{aligned} I_k \equiv |(1 - iy/a_{2k-1})(1 - iy/a_{2k})|^2 &= 1 + 2y\,\mathrm{Im}\,(a_{2k-1}^{-1} + a_{2k}^{-1}) \\ &+ y^2\{|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4\,\mathrm{Im}\,a_{2k-1}^{-1}\,\mathrm{Im}\,a_{2k}^{-1}\} \\ &+ 2y^3\,\mathrm{Im}\,\{a_{2k-1}^{-1}a_{2k}^{-1}\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})}\} + y^4\,|a_{2k-1}|^{-2}\,|a_{k2}|^{-2}\ . \end{aligned}$$

We assume $K \geqq k_1$ and therefore by (1.7) we get

$$\frac{(\mathrm{Im}\,\{\overline{(a_{2k-1}^{-1}+a_{2k}^{-1})}(a_{2k-1}^{-1}\cdot a_{2k}^{-1})\})^2}{\left[\left(1-\frac{\eta}{2}\right)(|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2}+4\,\mathrm{Im}\,a_{2k-1}^{-1}\,\mathrm{Im}\,a_{2k}^{-1})\right]\cdot\left[\left(1-\frac{\eta}{2}\right)|\,a_{2k-1}\cdot a_{2k}\,|^{-2}\right]}$$
$$< (1-\eta)\Big/\left(1-\eta+\frac{\eta^2}{4}\right) < 1-\frac{\eta^2}{4}\,.$$

It is easy to see that $y^2(A+2By+Cy^2) \geqq 0$ whenever $A>0$, $C>0$ and $B^2<AC$. We substitute

$$A-\left(1=\frac{\eta}{2}\right)\{|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2}+4\,\mathrm{Im}\,a_{2k-1}^{-1}\,\mathrm{Im}\,a_{2k}^{-1}\}\,,$$
$$B=\mathrm{Im}\,\{\overline{(a_{2k-1}^{-1}+a_{2k}^{-1})}a_{2k-1}^{-1}a_{2k}^{-1}\} \qquad \text{and}$$
$$C=\left(1-\frac{\eta}{2}\right)|\,a_{2k-1}a_{2k}\,|^{-2}\,.$$

We use (1.6), (1.7) and the above calculation to show that, for $k>\max(k_0,k_1)$, $A>0$, $C>0$ and $B^2>AC$. By omitting $y^2(A+2By+Cy^2)$ from the right side of the equation defining I_k we obtain

$$\begin{aligned} I_k &\geqq 1+2y\,\mathrm{Im}\,(a_{2k-1}^{-1}+a_{2k}^{-1}) \\ &\quad +\frac{\eta\theta}{2}y^2(|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2})+\frac{\eta y^4}{2}|\,a_{2k-1}a_{2k}\,|^{-2} \end{aligned}\tag{2.3}$$

by minimum consideration

$$\begin{aligned} &1+2y\,\mathrm{Im}\,(a_{2k-1}^{-1}+a_{2k}^{-1}) \\ &\quad +\frac{\eta\theta}{4}y^2(|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2}) \geqq 1-\frac{\frac{4}{\eta\theta}(\mathrm{Im}\,(a_{2k-1}^{-1}+a_{2k}^{-1}))^2}{(|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2})} \end{aligned}\tag{2.4}$$

the last term tends to 1 for large k because of (1.5). Using (2.3), (2.4) and letting the coefficients of y^2 and y^4 be smaller, we obtain (2.2) with $\alpha=\eta\theta/4$.

LEMMA 2.2. *Suppose $\{a_k\}\in$ class $A(2)$. Then for $k>K$ there exist A and B, $0<A<B<1$ independent of k (but they depend on η and θ) so that for any r, $r<\min(|\,a_{2k-1}\,|, |\,a_{2k}\,|)$, we shall have:*

(a) *For $|\,\sigma\,|\leqq Ar$ and $|\,y\,|\leqq Br$*

$$\begin{aligned} H_k(\sigma) &\equiv |\,(1-(\sigma+iy)/a_{2k-1})(1-(\sigma+iy)/a_{2k})\,|^2\exp(2\sigma\,\mathrm{Re}\,(a_{2k-1}^{-1}+a_{2k}^{-1})) \\ &\geqq 1-2\alpha^{-1}\frac{\mathrm{Im}\,(a_{2k-1}^{-1}+a_{2k}^{-1})}{|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2}}-\frac{r^2}{4}(|\,a_{2k-1}\,|^{-2}+|\,a_{2k}\,|^{-2}) \\ &\quad -4\sigma^2(\mathrm{Re}\,(a_{2k-1}^{-1}+a_{2k}^{-1}))^2\,. \end{aligned}$$

(b) *For $|\,\sigma\,|\leqq Ar$ and $|\,y\,|\geqq Br$*

$$H_k(\sigma) \geqq \left(1 + \frac{\alpha}{4} y^2 \,|\, a_{2k-1} |^{-2}\right)\left(1 + \frac{\alpha}{4} y^2 \,|\, a_{2k} |^{-2}\right) \times \left(1 - \frac{2}{\alpha} \frac{(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}}\right),$$

where α is that of Lemma 2.1.

Proof. By a simple calculation

$$\begin{aligned} \left|\left(1 - \frac{\sigma + iy}{a_{2k-1}}\right)\left(1 - \frac{\sigma + iy}{a_{2k}}\right)\right|^2 &= \left|\left(1 - \frac{iy}{a_{2k-1}}\right)\left(1 - \frac{iy}{a_{2k}}\right)\right|^2 \\ &- 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) + [\sigma^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2} + 4 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1}) \\ &+ \sigma^4 \,|\, a_{2k-1}a_{2k} |^{-2} + 2\sigma^2 y^2 \,|\, a_{2k-1}a_{2k} |^{-2} - 4\sigma y \operatorname{Im}(a_{2k-1}^{-1}a_{2k}^{-1}) \\ &- 2(\sigma^2 + y^2)\sigma \operatorname{Re}\{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})}a_{2k-1}^{-1}a_{2k}^{-1}\} \\ &+ 2\sigma^2 y \operatorname{Im}\{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})}a_{2k-1}^{-1}a_{2k}^{-1}\}] \equiv I_k - 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) + J_k \,. \end{aligned}$$

For the estimation of J_k we shall recall that

$$|\,(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1}a_{2k}^{-1} \,| \leqq 2(|\, a_{2k-1} |^{-3} + |\, a_{2k} |^{-3}) \tag{2.5}$$

and

$$\begin{aligned} |\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2} + 4 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1} &\geqq -2 \,|\operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1} | \\ &\geqq -(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) \,. \end{aligned} \tag{2.6}$$

To prove (a) assume $|\,\sigma\,| \leqq Ar$, $|\,y\,| \leqq Br$. Using (2.5) and (2.6) and dropping positive terms we obtain for $A < B$

$$\begin{aligned} J_k &\geqq (-A^2 - |\, 2AB)r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) \\ &\quad + (-4(A^2 + B^2)A - 4A^2B)r^3(|\, a_{2k-1} |^{-3} + |\, a_{2k} |^{-3}) \\ &\geqq (-3B^2 - 12B^3)r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) \,. \end{aligned}$$

Choosing $A < B$ and (for instance) $B = 3^{-2}$ and using Lemma 2.1 with $y = 0$ we obtain

$$\begin{aligned} H_k(\sigma) &\geqq \left(1 - \frac{\alpha^{-1}(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}} - \frac{1}{9} r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) \right. \\ &\quad \left. - 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1})\right) \exp(2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1})) \\ &\geqq 1 - 2\alpha^{-1}(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2[|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}]^{-1} \\ &\quad - \frac{1}{4} r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) - 4\sigma^2(\operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 \,. \end{aligned}$$

(The coefficients in the above estimation are not the best but they are convenient). To prove (b) (for which we are free to choose A, $A < B$) we recall that for $A \leqq \beta B$, $0 < \beta < 1$ and $|\,\sigma\,| < Ar$ we have

$$\begin{aligned}
&|2\sigma^2 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1})| \leqq \beta^2 B^2 r^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) , \\
&|2\sigma^2 y \operatorname{Im} \{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1} \cdot a_{2k}^{-1}}\}| \\
&\qquad \leqq \sigma^2 y^2 |a_{2k-1}a_{2k}|^{-2} + 2\sigma^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) , \\
&|2y^2\sigma \operatorname{Re} \{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1}a_{2k}^{-1}}\}| \\
&\qquad \leqq \beta y^4 |a_{2k-1}a_{2k}|^{-2} + 2\beta y^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})
\end{aligned}$$

and

$$\begin{aligned}
&|2\sigma^3 \operatorname{Re} \{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1}a_{2k}^{-1}}\}| \\
&\qquad \leqq \sigma^4 |a_{2k-1}a_{2k}|^{-2} + 2\sigma^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) .
\end{aligned}$$

Choosing β so that $5\beta^2 + 4\beta < \alpha/4$ and K so that

$$\alpha^{-1}(\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \leqq \frac{1}{4}$$

for $k \geqq K$ we obtain by the above estimations

$$\begin{aligned}
H_k(\sigma) &\leqq \left(1 + \frac{\alpha}{2}y^2 |a_{2k-1}|^{-2}\right)\left(1 + \frac{\alpha}{2} |a_{2k}|^{-2}\right)\Big(1 - \frac{1}{\alpha}(\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 , \\
&\times[|a_{2k-1}|^{-2} + |a_{2k}|^{-2}]^{-1} - 2\sigma \operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1})\Big) \cdot \exp (2\sigma \operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1})) \\
&\leqq \left(1 + \frac{3\alpha}{8}y^2 |a_{2k-1}|^{-2}\right)\left(1 + \frac{3\alpha}{8}y^2 |a_{2k}|^{-2}\right)\Big(1 - \frac{2}{\alpha}(\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 , \\
&\times[|a_{2k-1}|^{-2} + |a_{2k}|^{-2}]^{-1}\Big) \cdot (1 - 4\sigma^2(\operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2) .
\end{aligned}$$

Since $\beta < \alpha \cdot 4^{-2}$, $\beta^2 < \alpha \cdot 4^{-4}$ we obtain (b) easily.

Define S_m and $S_m^{(l)}$ (see [7], [2] and [4]) by

$$S_m = \sum_{k=m+1}^{\infty} |a_k|^{-2} \tag{2.7}$$

$$S_m^{(l)} = S_m - \max_{k(1)<\cdots<k(l)} \sum_{i=1}^{l} |a_{k(i)}|^{-2} . \tag{2.8}$$

Define also r_m by

$$r_m = \min_{k>m} |a_k| \tag{2.9}$$

One can easily see that $S_m^{(0)} = S_m$ and $S_m^{(1)} = S_m - r_m^{-2}$.

THEOREM 2.3. *Let $\{a_k\} \in$ class $A(2)$, then for $m \geqq K$, $|\sigma| \leqq AS_{2m}^{-1/2}$, and $b_{2m} = 0$ we have*

$$|E_{2m}(\sigma + iy)| \geqq \sqrt{2}/2 . \tag{2.10}$$

(A being that of Lemma 2.2.)

Proof. To prove (2.10) we use Lemma 2.2(a) whose conditions are satisfied since $S_{2m} > r_{2m}^{-2}$, $S_{2m}^{-1/2} < r_{2m} = \min_{k>2m} |a_k|$. We also recall that for $A_n > 0$ and $\sum_{n=m+1}^{\infty} A_n < 1/2$ we have

$$\prod_{n=m+1}^{\infty} (1 - A_n) \geqq 1 - \sum_{n=m+1}^{\infty} A_n \geqq \frac{1}{2}\,.$$

Remembering that for large m

$$2\alpha^{-1} \sum_{k=m+1}^{\infty} (\mathrm{Im}\,(a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|\,a_{2k-1}|^{-2} + |\,a_{2k}|^{-2}) < \frac{1}{8}$$

and

$$\begin{aligned} 4\sigma^2 \sum_{k=m+1}^{\infty} (\mathrm{Re}\,(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 &\leqq 8A^2 S_{2m} \sum_{k=m+1}^{\infty} (|\,a_{2k-1}|^{-2} + |\,a_{2k}|^{-2}) \\ &\leqq 8A^2 < 8^{-1} \end{aligned}$$

and using Lemma 2.2(a) we conclude the proof of (2.10) in the case where $|\,\sigma\,| \leqq S_{2m}^{-1/2}$ and $|\,y\,| \leqq BS_{2m}^{-1/2}$. Using Lemma 2.2(b), (2.10) in the case where $|\,\sigma\,| \leqq AS_{2m}^{-1/2}$, $|\,y\,| \geqq BS_{2m}^{-1/2}$ follows by an argumentation similar to that used in the first part. Then:

THEOREM 2.4. *Let* $\{a_k\} \in A(2)$, $b_{2m} = 0$, *then for* $m \geqq k$, $|\,\sigma\,| \leqq AS_{2m}^{-1/2}$ *and* $|\,y\,| \geqq BS_{2m}^{-1/2}$ *we have*

$$\begin{aligned} |\,E_{2m}(\sigma + iy)\,| &\geqq \frac{3}{4}\Big(1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \cdot \Big(\frac{\alpha}{4}\Big)^n \cdot \prod_{l=0}^{n-1} S_{2m}^{(l)}\Big)^{1/2} \\ &\geqq \frac{3}{4}\Big(1 + \frac{1}{n!} y^{2n} \Big(\frac{\alpha}{4}\Big)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}\Big)^{1/2}\,. \end{aligned} \tag{2.11}$$

Proof. Using (1.5) we can choose, by the method in the proof of Theorem 2.3, m so that

$$\sum_{k=m+1}^{\infty} \Big(1 - \frac{2}{\alpha}[(\mathrm{Im}\,(a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|\,a_{2k-1}|^{-2} + |\,a_{2k}|^{-2})]\Big) \geqq \frac{9}{16} \tag{2.12}$$

(9/16 can be replaced of course by any $1 - \varepsilon$).

$$\sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \Big(\frac{\alpha}{2}\Big)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}$$

converges for all y since $S_{2m} = S_{2m}^{(0)} > S_{2m}^{(1)} > \cdots > S_{2m}^{(l)}$. By Lemma 2.2 and (2.12) we have

$$|\,E_{2m}(\sigma + iy)\,| \geqq \frac{3}{4}\Big(1 + \sum_{n=1}^{\infty} y^{2n} \Big(\frac{\alpha}{4}\Big)^n \sum_{\substack{k(1)>2m \\ k(i)<k(i+1)}} |\,a_{k(1)} \cdots a_{k(n)}|^{-2}\Big)^{1/2}\,.$$

But we have

$$I(n, m) \equiv \sum_{\substack{2m<k(1)\\ k(i)<k(i+1)}} |a_{k(1)} \cdots a_{k(n)}|^{-2} = \frac{1}{n!} \sum_{\substack{k(i)>2m\\ k(i)\neq k(j), j\neq i}} |a_{k(1)} \cdots a_{k(n)}|^{-2}$$
$$\geqq \frac{1}{n!} \sum_{\substack{k(i)>2m\\ k(i)\neq k(j), j\neq i}} \left(S_{2m} - \sum_{r=1}^{n-1} |a_{k(r)}|^{-2}\right) |a_{k(1)} \cdots a_{k(n-1)}|^{-2}$$
$$\geqq \frac{1}{n!} S_{2m}^{(n-1)} \sum_{\substack{k(i)>2m\\ k(i)\neq k(j), j\neq i}} |a_{k(1)} \cdots a_{k(n-1)}|^{-2} .$$

Since

$$\sum_{k(i)>2m} |a_{k(i)}|^{-2} = S_{2m} = S_{2m}^{(0)} ,$$

by induction $I(n, m) \geqq 1/n! . \prod_{l=0}^{n-1} S_{2m}^{(l)}$, which concludes the proof of the theorem.

THEOREM 2.5. *Let* $\{a_k\} \in A(2)$, $b_{2m} = 0$, *and* σ *satisfies* $\operatorname{Re} a_k \neq \sigma$ *for all* $k > n$, *then for* $p, n = 0, 1, 2, \cdots$ there exist $k_1(p, \sigma, n)$ *and* $k_2(p, \sigma, n)$ *so that*

$$|E_{2n}(\sigma + i\tau)|^2 \geqq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} . \tag{2.13}$$

Proof. Since $S_{2m} = o(1)m \to \infty$ we can choose m so that $AS_{2m}^{-1/2} \geqq \sigma$ (for A of Theorems 2.3 and 2.4). Combining Theorems 2.3, 2.4 and the fact that $|\prod_{k=2n+1}^{2m} (1 - \sigma + i\tau/a_k))e^{\sigma \operatorname{Re} a_k^{-1}}| \geqq \delta$ whenever $\operatorname{Re} a_k \neq \sigma$, we obtain (2.13).

3. Estimates for $E_{m(i)}(s)$ in special cases when $\{a_k\} \notin A(2)$. In this section we shall estimate $E_{2m}(s)$ in case $\{a_k\}$ does not necessarily belong to $A(2)$ but $a_{2k-1} = -a_{2k}$ or $a_{2k-1} = \bar{a}_{2k}$ and some other conditions are satisfied.

First we prove some lemmas concerning the above mentioned cases.

LEMMA 3.1. *Let* a *be a complex number* $\operatorname{Re} a \neq 0$, *then for all real* y *and* $q \geqq 1$

$$I(a) = \left|\left(1 - \frac{iy}{a}\right)\left(1 + \frac{iy}{a}\right)\right|^2 = \left|\left(1 - \frac{iy}{a}\right)\left(1 - \frac{iy}{\bar{a}}\right)\right|^2$$
$$\geqq \begin{cases} 1 - q\left(\dfrac{\operatorname{Re} a^2}{|a|^2}\right)^2 + \left(1 - \dfrac{1}{q}\right)y^4 |a|^{-4} \\ 1 + y^4 |a|^{-4} \qquad \operatorname{Re} a^2 \geqq 0 . \end{cases} \tag{3.1}$$

Proof. Simple calculation yields

$$\left|\left(1 - \frac{iy}{a}\right)\left(1 - \frac{iy}{a}\right)\right|^2 = 1 - q\left(\frac{\operatorname{Re} a^2}{|a|^2}\right)^2$$
$$+ \left(\sqrt{q}\frac{\operatorname{Re} a^2}{|a|^2} + \frac{1}{\sqrt{q}}y^2 |a|^{-2}\right)^2 + \left(1 - \frac{1}{q}\right)y^4 |a|^{-4}$$

from which (3.1) is immediate.

LEMMA 3.2. *Let a be complex number,* $\operatorname{Re} a \neq 0$, *then*

$$\tag{3.2} \begin{aligned} &\left|\left(1 - \frac{\sigma + iy}{a}\right)\left(1 + \frac{\sigma + iy}{a}\right)\right|^2 \\ &\quad = I(a) + 2\sigma^2(|a|^2 - 2(\operatorname{Re} a)^2 |a|^{-4}) + \sigma^4 |a|^{-4} \\ &\qquad + 2\sigma^2 y^2 |a|^{-4} + 4\sigma y (\operatorname{Im} a^2) |a|^{-4} , \end{aligned}$$

$$\tag{3.3} \begin{aligned} &\left|\left(1 - \frac{\sigma + iy}{a}\right)\left(1 - \frac{\sigma + iy}{\bar{a}}\right)\right|^2 \\ &\quad = I(a) - 4\sigma \operatorname{Re} a |a|^{-2} + \sigma^2(2|a|^{-2} + 4(\operatorname{Re} a)^2 |a|^{-4}) \\ &\qquad + \sigma^4 |a|^{-4} + 2\sigma^2 y^2 |a|^{-4} - 4(\sigma^2 + y^2)\sigma |a|^{-4} \operatorname{Re} a , \end{aligned}$$

where $I(a)$ is defined in Lemma 3.1.

Proof. The proof is a corollary of the proof of Lemma 2.2 combined with Lemma 3.1.

LEMMA 3.3. *Let* $\operatorname{Re} a \neq 0$, *then for* $K > 1$ *there exists* A *and* B, *independent of* a, $0 < A < B < 1$ *such that for* $r < |\operatorname{Re} a|$ *we have:*

(a) *For* $|\sigma| \leqq Ar$ *and* $|y| \leqq Br$

$$\tag{3.4} \begin{aligned} &\left|\left(1 - \frac{\sigma + iy}{a}\right)\left(1 + \frac{\sigma + iy}{a}\right)\right|^2 \\ &\quad \geqq 1 - K^{-1} r^2 |a|^{-2} - (\min(0, (\operatorname{Re} a^2)\cdot|a|^{-2}))^2 . \end{aligned}$$

(b) *For* $|\sigma| \leqq Ar_1 \leqq Ar$, $|y| \geqq Br$ *and* $\delta > 0$

$$\tag{3.5} \begin{aligned} &\left|\left(1 - \frac{\sigma + iy}{a}\right)\left(1 + \frac{\sigma + iy}{a}\right)\right|^2 \\ &\quad \geqq \left(1 + \frac{1}{4} y^4 |a|^{-4}\right)(1 - 2(\min(0, \operatorname{Re} a^2/|a|^2))^2 \\ &\qquad - K^{-1}(r^2 |a|^{-2} + r_1^2 |a|^{-1-\delta} + |a|^{-2+2\delta})) . \end{aligned}$$

Proof. To prove (3.4) we use (3.2) and (3.1) with $q = 1$ and obtain the result by choosing B so that $6B^2 < K^{-1}$, and dropping some positive terms.

To prove (3.5) we use

$$4\sigma y (\operatorname{Im} a^2) |a|^{-4} \geqq -4 |\sigma y| |a|^{-2} \geqq -\left(\frac{1}{\beta^2} y^2 |a|^{-(3-\delta)} + 2\beta^2 \sigma^2 |a|^{-(1+\delta)}\right)$$

and

$$-\frac{1}{\beta^2}y^2\,|\,a\,|^{-(3-\delta)} + \frac{1}{4}y^4\,|\,a\,|^{-4} \geqq -\frac{4}{\beta^4}\,|\,a\,|^{-(2-2\delta)}\ .$$

Choosing $4/\beta^4 \leqq 1/K$ or $\beta \geqq \sqrt[4]{4K}$ and A so that $2\beta^2 A^2 < K^{-1}$ or $A^2 < 1/4K\sqrt{K}$ or $A < 1/2K$ one can conclude the proof by using Lemma 3.1 (choosing there $q = 2$ in case $\operatorname{Re} a^2 < 0$) and dropping some positive terms.

LEMMA 3.4. *Let* $\operatorname{Re} a \neq 0$, *then for* $K > 1$ *there exist* A *independent of* a, $0 < A < 1$, *such that for* $r < |\operatorname{Re} a|$ *and* $|\,\sigma\,| \leqq Ar$ *we have*

$$\begin{aligned}(3.6)\qquad &\left|\left(1-\frac{\sigma+iy}{a}\right)\left(1-\frac{\sigma+iy}{\bar{a}}\right)\right|^2 \exp\left(4\sigma \operatorname{Re} a/|\,a\,|^2\right)\\ &\geqq \left(1+\frac{1}{4}y^4\,|\,a\,|^{-4}\right)\left(1-2(\min(0, \operatorname{Re} a^2/|\,a\,|^2))^2 - K^{-1}r^2\,|\,a\,|^{-2}\right)\ .\end{aligned}$$

Proof. Using (3.3) of Lemma 3.2, Lemma 3.1 with $q = 3/2$, the estimations

$$\begin{aligned}&-4\sigma^3\,|\,a\,|^{-4}\operatorname{Re} a \geqq -\sigma^4\,|\,a\,|^{-4} - 4\sigma^2(\operatorname{Re} a)^2\,|\,a\,|^{-4}\ ,\\ &-4y^2\sigma\,|\,a\,|^{-4}\operatorname{Re} a \geqq -\frac{1}{2^5}y^4\,|\,a\,|^{-4} - 4^3\sigma^2\,|\,a\,|^{-2}\end{aligned}$$

and dropping some positive terms we obtain

$$\begin{aligned}&\left|\left(1-\frac{\sigma+iy}{a}\right)\left(1-\frac{\sigma+iy}{\bar{a}}\right)\right|^2 \geqq 1 + \left(\frac{1}{3}-\frac{1}{2^4}\right)y^4\,|\,a\,|^{-4}\\ &\quad -\frac{3}{2}(\min(0, \operatorname{Re} a^2/|\,a\,|^2))^2 - 4^3A^2r^2\,|\,a\,|^{-2} - 4\sigma(\operatorname{Re} a)\,|\,a\,|^{-2}\ .\end{aligned}$$

Choosing A so that $4^3A^2 < 1/4K$, which implies

$$-4^3A^2r^2\,|\,a\,|^{-2} > -\frac{1}{4K}r^2\,|\,a\,|^{-2}\ ,\qquad 4\,|\,\sigma\,|\,|\,a\,|^{-1} < \frac{1}{4}\ ,$$

and

$$\exp\left(4\sigma(\operatorname{Re} a)\,|\,a\,|^{-2}\right) \geqq 1 + 4\sigma(\operatorname{Re} a)\,|\,a\,|^{-2} - 4^2\sigma^2\,|\,a\,|^{-2}$$

from which (3.6) follows.

We shall define now two classes of convolution transforms by the function $E(s)$ and the sequence $\{a_k\}$.

DEFINITION 3.1. $\{a_k\} \in$ class $B(2, \delta)$ if

$$E(s) = \prod_{k=1}^{\infty} (1 - s^2 a_k^{-2}) , \tag{3.7}$$

$$\sum_{\mathrm{Re}\, a_k^2 < 0} | a_k |^{-4} (\mathrm{Re}\, a_k^2)^2 < \infty \tag{3.8}$$

and

$$\sum_{k=1}^{\infty} | a_k |^{-1-\delta} < \infty , \qquad \sum_{k=1}^{\infty} | a_k |^{-2+\delta} < \infty \qquad \text{for some } \delta > 0 . \tag{3.9}$$

DEFINITION 3.2. $\{a_k\} \in B(2)$ if there is $\delta > 0$ so that $\{a_k\} \in B(2, \delta)$.

DEFINITION 3.3. $\{a_k\} \in$ class $C(2)$ if

$$E(s) = \prod_{k=1}^{\infty} (1 - s a_k^{-1})(1 - s \cdot \bar{a}_k^{-1}) , \tag{3.10}$$

if condition (3.8) is satisfied and $\sum | a_k |^{-2} < \infty$.

REMARK. $S_{2m} = 2 \sum_{k=m+1}^{\infty} | a_k |^{-2}$ in case of class $B(2)$ and $C(2)$. We have to introduce some more notations before being able to prove the estimation on $E(s)$ for transforms of classes $B(2)$ and $C(2)$.

$$Q_m = \sum_{k=m+1}^{\infty} | a_k |^{-4} . \tag{3.11}$$

$$Q_m^{(j)} = Q_m - \max_{m < k(1) < \cdots < k(j)} \left\{ \sum_{r=1}^{j} | a_{k(r)} |^{-4} \right\} . \tag{3.12}$$

We shall state the estimations for classes $B(2)$ and $C(2)$ together and then outline the proofs.

THEOREM 3.5. *If $\{a_k\} \in B(2, \delta)$, then for $m \geqq M$ and some A and B we have:*

(a) $| \sigma | \leqq A S_{2m}^{-1/2}$, $| y | \leqq B S_{2m}^{-1/2}$ *imply*

$$| E_{2m}(s) | \geqq 3/4 . \tag{3.13}$$

(b) $| \sigma | \leqq A (\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta})^{-1/1+\delta}$ *and* $| y | \geqq B S_{2m}^{-1/2}$ *imply*

$$| E_{2m}(s) | \geqq \frac{3}{4} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{4n} \prod_{l=0}^{n-1} Q_m^{(l)} \right)^{1/2} . \tag{3.14}$$

THEOREM 3.6. *If $\{a_k\} \in C(2)$ then for $m \geqq M$ there exists an A so that for $| \sigma | \leqq A S_{2m}^{-1/2}$ (3.14) is valid.*

Proof of Theorems 3.5 *and* 3.6. The proof follows the proof of Theorems 2.3 and 2.4 Using Lemmas 3.3 and 3.4 we have to choose

$r = S_{2m}^{-1/2}$ and $r_1 = (2\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta})^{-1/(1+\delta)}$ (r_1 necessary only in proving Theorem 3.5 from Lemma 3.3). Obviously $r_1 < \min_{k>m} | a_k |$, $r \leqq \min_{k>m} | a_k |$. Also we have

$$\begin{aligned}\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta} &\geqq \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)\Big(\min_{k>m} | a_k |\Big)^{1-\delta} \\ &\geqq \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)^{(1+\delta)/2} \cdot \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)^{(1-\delta)1/2}\Big(\min_{k>m} | a_k |\Big)^{1-\delta} \\ &\geqq \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)^{(1+\delta)/2} .\end{aligned}$$

This implies

$$r_1 = \Big(2\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta}\Big)^{-1/(1+\delta)} \leqq (S_{2m})^{-1/2} \leqq r .$$

Choose m and K so that $\sum_{k>m} (\min (0, \operatorname{Re} a_k^2/| a_k |^2))^2 < \varepsilon_1$, $1/K < \varepsilon_1$ (K of Lemmas 3.3 and 3.4) and, for proving Theorem 3.5, $\sum_{k=m+1}^{\infty} | a_k |^{-2+2\delta} < \varepsilon_1$. The choice $\varepsilon_1 \leqq 1/16$ will yield the number 3/4 in (3.13) (every $1 - \eta$ could be achieved by ε_1 small enough) and the coefficient 3/4 in (3.14).

To complete the proof we have to show

$$\prod_{k=m+1}^{\infty}\Big(1 + \frac{1}{4}y^4 | a_k |^{-4}\Big) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}y^{4n}\prod_{l=0}^{n-1} Q_m^{(l)} ,$$

the proof of which follows stepwise that of Theorem 2.4.

The classes in this section are not included in $A(2)$ since (1.6) may fail to be valid. The estimates in this section are weaker in the case where the transforms are also $A(2)$.

THEOREM 3.7. *Let* $\{a_k\} \in B(2)$ *or* $C(2)$. *Then for* σ *satisfying* $\operatorname{Re} a_k \neq 0$ *for all* $k > n$, *and for* $p, n = 0, 1, 2 \cdots$ *there exist* $k_1(p, \sigma, n)$ *and* $k_2(p, \sigma, n)$ *such that when* $\sigma \neq \operatorname{Re} a_k$

$$| E_{2n}(\sigma + i\tau) |^2 \geqq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} . \tag{3.15}$$

Proof. Deduced from Theorems 3.5 and 3.6 as Theorem 2.5 is deduced from Theorem 2.4 and 2.3.

4. Estimates for $G_m(t)$. We define $G_m(t)$, in the usual manner, by

$$G_m(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[E_m(s)]^{-1}e^{st}ds , \qquad G_0(t) = G(t) . \tag{4.1}$$

We define also:

(4.2) $$\alpha(m) = \max\{\operatorname{Re} a_k, -\infty \mid \operatorname{Re} a_k < 0 \text{ and } k > m\} .$$

(4.3) $$\beta(m) = \min\{\operatorname{Re} a_k, \infty \mid \operatorname{Re} a_k > 0 \text{ and } k > m\} .$$

We recall that in the cases $\{a_k\} \in A(2)$, $\{a_k\} \in B(2)$ and $\{a_k\} \in C(2)$ we have

(4.4) $$| E_{2n}(\sigma + i\tau) |^2 \geqq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} ,$$

for $n, p = 0, 1, 2 \cdots$ and $\alpha(2n) < \sigma < \beta(2n)$.

THEOREM 4.1. *Let $E_n(s)$, $P_n(D)$ and $G_n(t)$ be defined by* (2.1), (1.3) *and* (4.1); *let* (4.4) *be satisfied for $m(l)$, a subsequence of m, then:*

A. *For any σ satisfying $\sigma(m(l)) < \sigma < \beta(m(l))$ we have*

(4.5) $$G_{m(l)}(t) = P_{m(l)}(D)G(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}[E_{m(l)}(s)]^{-1}e^{st}ds .$$

B. *Suppose in case $\alpha(m(l)) \neq -\infty$ that $a_{k(1,1)} = \cdots = a_{k(1,m_1+1)}$, $a_{k(2,1)} = \cdots = a_{k(2,m_2+1)}, \cdots, a_{k(r,1)} = \cdots = a_{k(r,m_2+1)}$ are all with indices greater than $m(l)$ and $\alpha(m(l)) = \operatorname{Re} a_{k(1,1)} = \operatorname{Re} a_{k(2,1)} = \cdots = \operatorname{Re} a_{k(r,1)}$, then*

(4.6) $$\frac{d^v}{dt^v}G_{m(l)}(t) = \sum_{i=1}^{r}\frac{d^v}{dt^v}\{P_i(t)e^{ta_{k(i,1)}}\} + 0(e^{k(t)}) \qquad t \to \infty$$

where $p_i(t)$ are polynomials of order m_i and k is any real number satisfying

$$\max\{\operatorname{Re} a_k, -\infty \mid k > m(l), \operatorname{Re} a_k < \alpha(m(l))\} < k < \alpha(m(l)) .$$

C. *Suppose $\alpha(m(l)) = -\infty$, then*

(4.7) $$\frac{d^v}{dt^v}G_{m(l)}(t) = 0(e^{kt}) \qquad t \to \infty \text{ for any real } k, k < 0 .$$

D. *Suppose in case $\beta(m(l)) \neq \infty$ that $a_{r(1,1)} = \cdots = a_{r(1,m_1+1)}$, $\cdots a_{r(j,1)} = \cdots = a_{r(j,m_j+1)}$ are all with indices greater than $m(l)$ and $\beta(m(l)) = \operatorname{Re} a_{r(1,1)} = \cdots \operatorname{Re} a_{r(j,1)}$, then*

(4.8) $$\frac{d^v}{dt^v}G_{m(l)}(t) = \sum_{i=1}^{j}\frac{d^v}{dt^v}\{q_i(t)e^{ta_{r(i,1)}}\} + 0(e^{kt}) \qquad t \to -\infty$$

where $q_i(t)$ are polynomials of order m_i and k is a real number satisfying $\beta(m(l)) < k < \min\{\operatorname{Re} a_k, \infty \mid k > m(l), \operatorname{Re} a_k > \beta(m(l))\}$.

E. *Suppose $\beta(m(l)) = \infty$, then*

(4.9) $$\frac{d^v}{dt^v}G_{m(l)}(t) = 0(e^{kt}) \qquad t \to -\infty$$

where k is any real positive number.

F. *For $\alpha(m(l)) < \operatorname{Re} s < \beta(m(l))$ we have*

$$\frac{1}{E_{m(l)}(s)} = \int_{-\infty}^{\infty} e^{st} G_{m(l)}(t)dt \tag{4.10}$$

which implies

$$1 = \int_{-\infty}^{\infty} G_{m(l)}(t)dt \ . \tag{4.11}$$

Proof. The proof follows the method used in Hirschman and Widder's book "The convolution transform" [6, p. 108]. Formula (4.4), that was proved for class $A(2)$, $B(2)$ and $C(2)$, is used here instead of the theorems on $E_m(s)$ in [6].

The following result will estimate $G_{2m}(t)$ in the case when m is large near the point $t = 0$ as well as when $|t| \to \infty$.

THEOREM 4.2. *Let $\{a_k\} \in A(2)$ and suppose that for some n $S_{2m}^{(n+1)} \geqq L_n S_{2m}$ where $L_n > 0$ is independent of m, then there exist $M(n) > 0$ and $A > 0$ such that*

$$| G_{2m}^{(n)}(t) | \leqq M(n) S_{2m}^{-(n+1)/2} \exp(-A \cdot S_{2m}^{-1/2} | t |) \ . \tag{4.12}$$

Proof. By Theorem 4.1.A we have

$$G_{2m}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-(\sigma+iy)t}}{E_{2m}(\sigma + iy)} dy$$

and therefore

$$G_{2m}^{(n)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(\sigma + iy)^n e^{-(\sigma+iy)t}}{E_{2m}(\sigma + iy)} dy \ .$$

Remembering that $S_{2m}^{(n+1)} \geqq L_n S_{2m}$ implies $S_{2m}^{(k)} \geqq L_n S_{2m}$ for $k \leqq n + 1$, and using Theorems 2.3 and 2.4 we obtain, choosing $\sigma = AS_{2m}^{-1/2}$ for the case $t > 0$,

$$\begin{aligned} | G_{2m}^{(n)}(t) | &\leqq \frac{1}{2\pi} \exp(-AS_{2m}^{-1/2} t) \Big\{ \int_{-BS_{2m}^{-1/2}}^{BS_{2m}^{-1/2}} \frac{(|\sigma| + |y|)^n}{|E_{2m}(\sigma + iy)|} dy \\ &+ \int_{|y| \geqq BS_{2m}^{-1/2}} \frac{(|\sigma| + |y|)^n dy}{|E_{2m}(\sigma + iy)|} \Big\} \leqq \exp(-AS_{2m}^{-1/2} t) \Big\{ \frac{\sqrt{2}}{2\pi} (A + B)^n 2BS_{2m}^{-(n+1)/2} \\ &+ 2 \sum_{k=0}^{n} \binom{n}{k} S_{2m}^{-k/2} \frac{2}{3\pi} \int_{BS_{2m}^{-1/2}}^{\infty} \frac{y^{n-k} dy}{\left(1 + y^{2(n+2)} \frac{1}{(n+2)!} (L_{(n)})^{n+1} S_{2m}^{n+2} \left(\frac{\alpha}{4}\right)^{n+2}\right)^{1/2}} \Big\} \\ &\leqq M(n) S_{2m}^{-(n+1)/2} \exp(-AS_{2m}^{-1/2} t) \ . \end{aligned}$$

The result for $t<0$ is achieved choosing $\sigma = -AS_{2m}^{-1/2}$.

REMARK. When $a_{2k-1} = -a_{2k}$ we have $S_{2m}^{(1)} \geqq (1/2)S_{2m}$ and therefore Theorem 4.2 for $n=0$ includes Lemma 2.4 of [1, p. 432]. Whenever the connection between pair is $0<\theta_1 \leqq |a_{2k-1}/a_{2k}| \leqq \theta_2 < \infty$, where θ_1, θ_2 are fixed for all m, we have $S_{2m}^{(1)} \geqq L_1 S_{2m}\, L_1 > 0$. But in case of $n=0$ the restriction $S_{2m}^{(1)} \geqq L_1 S_{2m}$ is not necessary as is proved by the following.

THEOREM 4.3. *Let $\{a_k\} \in A(2)$, then for some $A>0$ we have*

$$|G_{2m}(t)| \leqq MS_{2m}^{-1/2} \exp(-AS_{2m}^{-1/2}|t|)\ . \tag{4.12}$$

Proof. Following the proof of Theorem 4.2 and using Theorem 2.4 we have for $t>0$ ($t<0$ can be treated similarly)

$$|G_{2m}(t)| \leqq \exp(-AS_{2m}^{-1/2}t)\left\{\frac{\sqrt{2}}{\pi}BS_{2m}^{-1/2} + \frac{4}{3\pi}\int_{BS_{2m}^{-1/2}}^{\infty}\frac{dy}{\left(1+\frac{1}{2}y^4 S_{2m}S_{2m}^{(1)}\left(\frac{\alpha}{4}\right)^2\right)^{1/2}}\right\}.$$

$$\begin{aligned}\int_{BS_{2m}^{-1/2}}^{\infty}\frac{dy}{(1+Ly^4S_{2m}S_{2m}^{(1)})^{1/2}} &= S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\int_{(B/L^{1/4})(S_{2m}/S_{2m}^{(1)})^{1/4}}^{\infty}\frac{1}{(1+y^4)^{1/2}}dy\\
&\leqq 2^d S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\int_{B_1(S_{2m}/S_{2m}^{(1)})^{1/4}}^{\infty}\frac{dy}{1+y^2}\\
&\leqq 2S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\lim_{\zeta\to\infty}(\operatorname{arc} tg\zeta - \operatorname{arc} tgB_1(S_{2m}/S_{2m}^{(1)})^{1/4})\\
&\leqq 2S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\lim_{\zeta\to\infty}\operatorname{arc} tg\left\{\frac{1-B_1(S_{2m}/S_{2m}^{(1)})^{1/4}\zeta}{\zeta^{-1}+B_1(S_{2m}/S_{2m}^{(1)})^{1/4}}\right\} \leqq B_2S_{2m}^{-1/2}\ .\end{aligned}$$

From this the proof can be easily concluded.

Lemma 3.2A of [1, p. 434] is generalized by Theorem 4.2 in case $S_{2m}^{(2)} \geqq L_2S_{2m}$ for some L_2. Case B is covered only in part. The following theorem generalizes Lemma 3.2B [1, p. 434].

THEOREM 4.4. *Let $a_k \in A(2)$, $b_{2m}=0$ and suppose $0<\theta_1<|a_{2k}/a_{2k-1}|<\theta_2<\infty$ where θ_1, θ_2 are independent of k and $|\mathrm{Re}\, a_k|/|a_k|>\eta$, then for some $A_1>0$ and M_1 we have:*

$$|G'_{2m}(t)| \leqq M_1S_{2m}^{-1}\exp(-A_1S_{2m}^{-1/2}|t|)\ . \tag{4.13}$$

Proof. Let us split the proof into two cases

(a) $$S_{2m} - \max_{k>m} (|a_{k-1}|^{-2} + |a_{2k}|^{-2}) \geqq \frac{1}{K} S_{2m}$$

and

(b) $$S_{2m} - \max_{k>m} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \frac{1}{K} S_{2m} .$$

In case (a) (4.13) was proved by Theorem 4.2 for any arbitrary K. We shall choose $K > 2$. To prove (4.13) in case (b) we define k_0 by

$$\max_{k>m} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) = |a_{2k_0-1}|^{-2} + |a_{2k_0}|^{-2} .$$

(In case (b) the choice of k_0 is unique.) Define $g^*_{k_0}(t)$ and $G_{2m+2}(t)$ by:

(4.14) $$g^*_{k_0}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [(1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0})]^{-1} e^{-iyt} dy .$$

(4.15) $$G^*_{2m+2}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [(1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0})](E_{2m}(iy))^{-1} e^{-iyt} dy .$$

By [9, p. 255] we have

$$G_{2m}(t) = g^*_{k_0}(t) * G^*_{2m+2}(t) .$$

One can calculate $g^*_{k_0}(t)$:

$$g^*_{k_0}(t) = \frac{a_{2k_0-1}a_{2k_0}}{a_{2k_0-1} - a_{2k_0}} \begin{cases} e^{a_{2k_0}t} & t \geqq 0 \\ e^{a_{2k_0-1}t} & t < 0 \end{cases}$$

when $\operatorname{Re} a_{2k_0-1} > 0$, $\operatorname{Re} a_{2k_0} < 0$.

$$g^*_{k_0}(t) = \begin{cases} \dfrac{a_{2k_0-1}a_{2k_0}}{a_{2k_0} - a_{2k_0-1}} [e^{a_{2k_0-1}t} - e^{a_{2k_0}t}] & t < 0 \\ 0 & t > 0 \end{cases}$$

when $\operatorname{Re} a_{2k_0-1} > 0$, $\operatorname{Re} a_{2k_0} > 0$, $a_{2k_0} \neq a_{2k_0-1}$.

$$g^*_{k_0}(t) = \begin{cases} -a^2_{2k_0} t e^{a_{2k}t} & t < 0 \\ 0 & t > 0 \end{cases}$$

when $a_{2k_0} = a_{2k_0-1}$, $\operatorname{Re} a_{2k_0} > 0$.

Either $g^*_{k_0}(t)$ or $g^*_{k_0}(-t)$ is of the above form.

$$G^*_{2m+2}(t - \operatorname{Re} (a^{-1}_{2k_0+1} + a^{-1}_{2k_0}))$$

satisfies the assumptions of Theorem 4.3 with $S^*_{2m+2} = S_{2m} - |a_{2k_0-1}|^2 - |a_{2k_0}|^2$ and therefore

$$| G^*_{2m+2}(t) | \leqq M(S^*_{2m+2})^{1/2} \exp(-AS^{*-1/2}_{2m+2} | t + \operatorname{Re}(a^{-1}_{2k_0+1} + a^{-1}_{2k_0}) |) .$$

Integrating by parts

$$\begin{aligned} G'_{2m}(t) &= \int_{-\infty}^{\infty} g^*_{k_0}(u) \frac{d}{dt} G^*_{2m+2}(t-u)du \\ &= \left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\}\left(\frac{d}{du} g^*_{k_0}(u)\right) G^*_{2m+2}(t-u)du . \end{aligned}$$

Since

$$\theta_1^2 | a_{2k_0-1} |^2 \leqq | a_{2k_0} |^2 \quad \text{and} \quad | a_{2k_0} |^2 \leqq \theta_2^2 | a_{2k_0-1} |^2$$

we have

$$(\theta_1^{-2} + 1) | a_{2k_0} |^{-2} \geqq \frac{1}{2} S_{2m} \quad \text{and} \quad (\theta_2^2 + 1) | a_{2k_0-1} |^{-2} \geqq \frac{1}{2} S_{2m} ;$$

therefore

$$\max(| a_{2k_0} |, | a_{2k_0-1} |) \leqq [(2\theta_1^{-2} + 2)^{1/2} + (2\theta_2^2 + 2)^{1/2}] S_{2m}^{-1/2} = R_1 S_{2m}^{-1/2} .$$

By the same method $(\theta_2^{-2} + 1) | a_{2k_0} |^{-2} \leqq S_{2m}$ and $(\theta_1^2 + 1) | a_{2k_0-1} |^2 \leqq S_{2m}$, from which we deduce

$$\begin{aligned} | \operatorname{Re} a_{2k_0} | \geqq \eta | a_{2k_0} | &\geqq \eta(\theta_2^{-2} + 1)^{-1/2} S_{2m}^{-1/2} , \\ | \operatorname{Re} a_{2k_0-1} | &\geqq \eta(\theta_1^2 + 1)^{-1/2} S_{2m}^{-1/2} \end{aligned}$$

and

$$\min(| \operatorname{Re} a_{2k_0} |, | \operatorname{Re} a_{2k_0-1} |) \geqq R_2 S_{2m}^{-1/2} > 0$$

where

$$R_2 = \eta \cdot \min((\theta_2^{-2} + 1)^{-1/2}, (\theta_1^2 + 1)^{-1/2}) .$$

One has to estimate $G_{2m}(t)$ for different cases of $g^*_{k_0}(t)$ of which the case where $\operatorname{Re} a_{2k_0} > 0$, $\operatorname{Re} a_{2k_0-1} > 0$ and $a_{2k_0} \neq a_{2j-1}$ will be done here. The other cases are similar and simpler.

$$\frac{dg^*(u)}{du} = \begin{cases} a_{2k_0-1} a_{2k_0} \dfrac{(a_{2k_0-1} \exp(a_{2k_0-1}u) - a_{2k_0} \exp(a_{2k_0}u))}{a_{2k_0} - a_{2k_0-1}} & u < 0 \\ 0 & u > 0 . \end{cases}$$

Let us recall from [8, p. 203] that if $f'(t)$ is continuous and $f(t)$ is complex valued, then

$$\frac{f(a) - f(b)}{a - b} = \lambda f'(t_1) + (1 - \lambda) f'(t_2) \qquad t_1, t_2 \in (a, b) \ 0 < \lambda < 1$$

from which it is obvious that

$$\frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2} = \lambda f'(\zeta_3) + (1 - \lambda) f'(\zeta_4) \qquad 0 < \lambda < 1$$

where $\zeta_i = \alpha_i \zeta_1 + (1 - \alpha_i)\zeta_2, 0 \leqq \alpha_i \leqq 1$ and $i = 3, 4$. Substituting $f(\zeta) = \zeta e^\zeta$, $f'(\zeta) = e^{\zeta u} + \zeta u e^{\zeta u}$, we obtain the following estimate for $(d/du)g^*(u)$ when $u < 0$:

$$\left| \frac{dg^*(u)}{du} \right| \leqq | a_{2k_0-1} a_{2k_0} | (\exp (R_{k_0} u) + \max (| a_{2k_0-1} |, | a_{2k_0} |) | u | \exp (R_{k_0} u))$$

where $R_{k_0} = \min (\operatorname{Re} a_{2k_0-1}, \operatorname{Re} a_{2k_0})$. Therefore we obtain

$$\begin{aligned} | G'_{2m}(t) | = | a_{2k_0-1} a_{2k_0} | \int_{-\infty}^{0} & \exp (R_{k_0} u)\{1 + | u | \max (| a_{2k_0-1} |, | a_{2k_0} |)\} \\ & \cdot S^{*-1/2}_{2m+2} \exp (-A S^{*-1/2}_{2m+2} | t - u + \operatorname{Re} (a^{-1}_{2k_0-1} + a^{-1}_{2k_0}) |) du \,. \end{aligned}$$

Using relations among S^*_{2m+2}, S_{2m}, a_{2k_0} and a_{2k_0-1} one obtains

$$\exp (-A S^{*-1/2}_{2m+2} | t - u + \operatorname{Re} (a^{-1}_{2k_0-1} + a^{-1}_{2k_0}) |) \leqq M_2 \exp (-A S^{-1/2}_{2m+2} | t - u |) \,.$$

Using this and the definition of R_1 and R_2 one derives

$$\begin{aligned} | G'_{2m}(t) | \leqq M_2 R_1^2 S^{-1}_{2m} \int_{-\infty}^{0} & \exp (R_2 S^{-1/2}_{2m} u)\{1 + | u | R_1 S^{-1/2}_{2m}\} S^{*-1/2}_{2m+2} \\ & \cdot \exp (-A S^{*-1/2}_{2m+2} | t - u |) du \,. \end{aligned}$$

We have to distinguish two cases $t < 0$ and $t \geqq 0$. Let us prove first the theorem in case $t < 0$:

$$\begin{aligned} | G'_{2m}(t) | \leqq\ & M_2 R_1^2 S^{-1}_{2m} \exp (-A t S^{*-1/2}_{2m+2}) \int_{-\infty}^{t} \{1 - u R_1 S^{-1/2}_{2m}\} S^{*-1/2}_{2m+2} \\ & \cdot \exp \{(R_2 S^{-1/2}_{2m} + A S^{*-1/2}_{2m+2}) u\} du \\ & + M_2 R_1^2 S^{-1}_{2m} \exp (A t S^{*-1/2}_{2m+2}) \int_{t}^{0} \{1 - u R_1 S^{-1/2}_{2m}\} S^{*-1/2}_{2m+2} \\ & \cdot \exp \{(R_2 S^{-1/2}_{2m} - A S^{*-1/2}_{2m+2}) u\} du \,. \end{aligned}$$

Choosing K so that $A S^{*-1/2}_{2m+2} > 2 R_2 S^{-1/2}_{2m}$ we have

$$| G'_{2m}(t) | \leqq M_1 S^{-1}_{2m} \exp (R_2 t S^{-1/2}_{2m}) \,.$$

For $t > 0$

$$\begin{aligned} | G'_{2m}(t) | \leqq\ & M_2 R_1^2 S^{-1}_{2m} \exp (-A t S^{*-1/2}_{2m+2}) \cdot \int_{-\infty}^{0} \{1 - u R_1 S^{-1/2}_{2m}\} \cdot S^{*-1/2}_{2m+2} \\ & \cdot \exp \{(R_2 S^{-1/2}_{2m} + A S^{*-1/2}_{2m+2}) u\} du \leqq M_1 S^{-1}_{2m} \exp (-A t S^{*-1/2}_{2m+2}) \\ & \leqq M_1 S^{-1}_{2m} \exp (-R_2 S^{-1/2}_{2m} t) \,. \end{aligned}$$

Estimations similar to those achieved in Theorem 4.2 for $\{a_k\} \in B(2, \delta)$ and $\{a_k\} \in C(2)$ are developed in the following theorems.

THEOREM 4.6. *Let* $\{a_k\} \in B(2, \delta)$ *and* $Q_m^{(j)} \geqq L(j)Q_m$ *for some* j, *then there exist* $A > 0$ *and* $M > 0$ (*independent of* m) *so that for* $k \leqq 2j$:

$$(4.14) \qquad |G_{2m}^{(k)}(t)| \leqq MQ_m^{-k/4} \exp\left(-A\left(2\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta}\right)^{-1/(1+\delta)} |t|\right).$$

THEOREM 4.7. *Let* $\{a_k\} \in C(2)$ *and* $Q_m^{(j)} \geqq L(j)Q_m$ *for some* j, *then there exist* $A > 0$ *and* $M > 0$ (*independent of* m) *so that for* $k \leqq 2j$:

$$(4.15) \qquad G_{2m}^{(k)}(t) \leqq MQ_m^{-k/4} \exp(-AS_{2m}^{-1/2}|t|).$$

One can note that in case $k = 0$ no condition of the form $Q_m^{(j)} \geqq L(j)Q_m$ is needed.

Proof of Theorems 4.6 *and* 4.7. Using Theorems 3.5 and 3.6 (for Theorems 4.6 and 4.7 respectively) we obtain by Theorem 4.1

$$|G_{2m}^{(k)}(t)| \leqq \left| e^{-\sigma t}\int_{\sigma+i\infty}^{\sigma-i\infty} \frac{(\sigma+iy)^k e^{-iyt}}{E_{2m}(\sigma+iy)} dy \right|, \qquad \beta(2m) < \sigma < \alpha(2m).$$

Using the fact that $Q_m^{-1/4} < ((1/2)S_{2m})^{-1/2}$, as $S_m^2 > Q_m$ (which is achieved by dropping many positive terms) and recalling that

$$S_{2m}^{-1/2} < \left(2\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta}\right)^{1/1+\delta},$$

we obtain

$$Q_m^{-1/4} < \left(\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta}\right)^{1/1+\delta}.$$

The completion of the proof is similar to the proof of Theorem 4.2.

5. **Some inversion theorems.** In this section we shall show that inversion formulae can be given for $\{a_k\} \in A(2)$, $\{a_k\} \in B(2, \delta)$ and $\{a_k\} \in C(2)$.

THEOREM 5.1. *Suppose*: (1) $G(t)$ *and* $E(s)$ *are defined by* (1.2) *and* $\{a_k\} \in A(2)$.

(2) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$

(3) *For some* M *and* K, $|\varphi(t)| \leqq Ke^{M|t|}$, *where* $M < \min|\operatorname{Re} a_n|$.

(4) $b_{2m} = o(1) \quad m \uparrow \infty$.

Then

$$(5.1) \qquad \lim_{m\to\infty} P_{2m}(D)f(x) = \lim_{m\to\infty} \exp((b - b_{2m})D) \prod_{k=1}^{m}\left(1 - \frac{D}{a_{2k-1}}\right)\left(1 - \frac{D}{a_{2k}}\right).$$

$\exp((\mathrm{Re}\, a_{2k-1}^{-1} + a_{2k}^{-1})D)f(x) = \varphi(x)$ *at any point of continuity of* $\varphi(t)$.

Proof. By steps following those of [1; p. 433]

$$|P_{2m}(D)f(x) - \varphi(x)| \leqq \sup_{|t|<\delta} |\varphi(x-t) - \varphi(x)| \int_{-\infty}^{\infty} |G_{2m}(t)|\, dt + M_0 \int_{|t|>\delta} |G_{2m}(t)|\, e^{M|t|} dt\ .$$

Using Theorem 4.3, the conditions of which are satisfied by the kernel $G_{2m}(t + b_m)$, choosing m so big that $|b_m| < \delta/2$ and $AS_{2m}^{-1/2} > 4M$, we conclude the proof of the theorem.

THEOREM 5.2. *Suppose: Assumptions* (1) *and* (2) *of Theorem* 5.1 *are satisfied*

(3) *For* $\alpha(t) = \int_0^t \varphi(u)du$ *there exist positive* M *and* K *so that* $|\alpha(t)| \leqq ke^{M|t|}$ *where* $M < \mathrm{Min}\, |\mathrm{Re}\, a_k|$.

(4) $b_{2m} = o(S_{2m}^{1/2})$ $m \to \infty$.

(5) $\int_0^h [\varphi(x+y) - \varphi(x)]dy = o(h)$ $h \to 0$.

(6) *Either* $S_{2m}^{(2)} \geqq L(2)S_{2m}$ *or* $0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty$ *and* $|\mathrm{Re}\, a_k/|a_k|| > \eta$.

Then $\lim_{m\to\infty} P_{2m}(D)f(x) = \varphi(x)$.

Proof. Integrating by parts and since $\int_{-\infty}^{\infty} G_{2m}(t)dt = 1$ we obtain

$$|P_{2m}(D)f(x) - \varphi(x)| > \int_{|x-t|\leqq\delta} |\beta(t)|\, |G'_{2m}(x-t)|\, dt + \int_{|x-t|\geqq\delta} |G'_{2m}(x-t)|\, |\beta(t)|\, dt\ ,$$

where $\beta(t) = \int_x^t [\varphi(x+u) - \varphi(x)]du$ and therefore $\beta(x+t) = o(t)$ $t \to 0$ and $|\beta(t)| \leqq K_1 e^{M|t|}$.

To obtain the inversion result for the case $S_{2m}^{(2)} \geqq L(2)S_{2m}$ we use the estimation from Theorem 4.2; while for $|\mathrm{Re}\, a_k/a_k| > \eta$, $0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty$ we use Theorem 4.4, both are applicable to $G_{2m}(t + b_{2m})$.

REMARK 1. In case $a_{2k-1} = -a_{2k}$ (from some k onward) we can drop (5) and write instead

$$\int_0^h [\varphi(x \pm y) - \varphi(x \pm 0)]dy = o(h) \qquad h \to 0+$$

(if the numbers $\varphi(x \pm 0)$ exist) and then if we write $b_{2m} = 0$ instead of (4) and drop (6), we shall obtain

(5.2) $$\lim_{m\to\infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+0) + \varphi(x-0)]\,.$$

The proof is similar if we remember that $G_{2m}(t) = G_{2m}(-t)$ and therefore $\int_{-\infty}^{0} G_{2m}(t)dt = 1/2$.

REMARK 2. The condition (3) of Theorem 5.2 seems too strong since for the case where a_k are real the assumption could be dropped. We hope that at least for some classes of $\{a_k\}$ Theorem 5.2 could be proved without (3).

THEOREM 5.3. *Suppose*: (1) $G(t)$ *and* $E(s)$ *are defined by* (1.2) *and* $\{a_k\} \in B(2, \delta)$.

(2) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$

(3) *For some* M *and* K $|\varphi(t)| \leqq Ke^{M|t|}$ *where* $M = \min|\operatorname{Re} a_k|$.

(4) $\{(\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta})^{1/1+\delta}\}^{\beta} \leqq KQ_m^{1/4}$ *for some* $\beta \geqq 1$.

(5) $\varphi(x) - \varphi(t) = 0(|t-x|^{\beta-1})$ $t \to x$.

Then

(5.3) $$\lim_{m\to\infty} P_{2m}(D)f(x) = \varphi(x)\,.$$

Proof. We have

$$\begin{aligned}
|P_{2m}(D)f(x) - \varphi(x)| &= \left|\left\{\int_{|x-t|\geqq\delta} + \int_{|x-t|\leqq\delta}\right\}G_{2m}(x-t)[\varphi(t) - \varphi(x)dx\right| \\
&\leqq K_1\int_{|t-x|\geqq\delta} |G_{2m}(x-t)|\, e^{Mt}dt + \varepsilon\int_{-\infty}^{\infty} |G_{2m}(t)|\,|t|^{\beta-1}dt \\
&\leqq o(1) + \varepsilon K_2\int_{-\infty}^{\infty} Q_m^{-1/4}|t|^{\beta-1}\exp\left(-A\Big(\sum_{k=m+1}^{\infty}|a_k|^{-1-\delta}\Big)^{-1/1+\delta}|t|\right)dt \\
&\leqq o(1) + \varepsilon K_2 K\int_{-\infty}^{\infty}|u|^{\beta-1}\exp(-Au)du \leqq o(1) + \varepsilon K_2 K \\
&\qquad m \to \infty\,.
\end{aligned}$$

THEOREM 5.4. *Suppose*: (1) $G(t)$ *and* $E(s)$ *are defined by* (1.2) *and* $\{a_k\} \in C(2)$.

(2) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$

(3) *For some* K *and* M $|\varphi(X)| \leqq Ke^{M|t|}$ *where* $M = \min|\operatorname{Re} a_k|$.

(4) $S_{2m}^{\beta/2} \leqq KQ_m^{1/4}$ *for some* $\beta \geqq 1$.

(5) $\varphi(x) - \varphi(t) = o(|t-x|^{\beta-1})$ $t \to x$.

Then

$$(5.3) \qquad \lim_{m\to\infty} P_{2m}(D)f(x) = \varphi(x) .$$

Proof. Similar to that of Theorem 5.3.

REMARK. When β of condition (4) of Theorem 5.4 and Theorem 5.3 is equal to one, the condition on $\varphi(t)$ is mere continuity at a point $t = x$.

LEMMA 5.5. *If an integer r exists such that for all n $|a_{n+r}| > q|a_n|$ for $q > 1$, then*

$$\left(\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta}\right)^{1/1+\delta} \leqq KQ_m^{1/4} \qquad 0 \leqq \delta \leqq 1$$

(*for* $\delta = 1$ *we have* $S_{2m}^{1/2} \leqq KQ_m^{1/4}$).

Proof. Obvious.

COROLLARY 5.5. *If the kernel is defined by* $a_{2k} = 2^{k-1}(1+i)$ *and* $a_{2k-1} = -2^{k-1}(1+i)$ *then* (5.3) *is valid at any point of continuity.*

6. Examples, remarks and generalizations. In this section we shall show some examples of convolution transform by giving its related sequence $\{a_k\}$. When we say $G(t) \in A(2), B(2, \delta)$ or $C(2)$ we mean that there is an order for which $\{a_k\} \in A(2), B(2, \delta)$ and $C(2)$ respectively.

EXAMPLE 6.1. $\{a_k\}$ defined by $a_{2k-1} = k, a_{2k} = q^k e^{i\theta_k}$ for $q > 1$, $0 < \delta < \pi(1/2), |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta$. $\{a_k\} \in A(2)$. The kernel $G(t)$ related to $\{a_k\}$ is not necessarily one of those discussed in [6]; for instance in case $\theta_k = (2/5)\pi$ the result of Theorem 5.2 can be applied as $S_{2m}^{(j)} \geqq L(j)S_{2m}$ for all j ($j = 2$ is needed).

EXAMPLE 6.2. $G(t)$ defined by $a_{2k-1} = (2k-1)!$ $a_{2k} = (2k)!\ e^{i\theta_k}$ where $-\pi < \theta_k < \pi, |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta$ for some $0 < \delta < \pi/2$ where the a_k's are arranged in the order of $|a_k|$. Theorem 5.2 does not apply here as one can easily verify that $S_{2m}^{(j)} = o(S_{2m})$ $m \to \infty$ for all $j > 0$. We can apply Theorem 5.1 and get an inversion formula.

EXAMPLE 6.3. Let c_k be real, $\sum c_k^{-2} < \infty$ and

$$a_{2k-1} = c_k(\sin\theta_1)^{-1}e^{i\theta_2} , \qquad a_{2k} = c_k(\sin\theta_2)^{-1}e^{-i\theta_1}$$

where $0 < \theta_1, \theta_2 < \pi/2, 0 < \delta_1 < \theta_1 + \theta_2 < \pi/2 - \delta_2$. (1.5) is easily verified. (1.6) is valid also since $\sin^2\theta_1 + \sin^2\theta_2 - 4\sin^2\theta_1\sin^2\theta_2 \geqq \eta$ and $\cos^2\theta_1 > \sin^2\theta_2$ and therefore $\sin^2\theta_1 + \sin^2\theta_2 < 1$ implies

$$\left(1 - \frac{\eta}{2}\right)(\sin^2\theta_1 + \sin^2\theta_2) - 4\sin^2\theta_1\sin^2\theta_2 \geqq \frac{\eta}{2} .$$

Using $\sin^2\theta_1 < \cos^2\theta_2$ and $\sin^2\theta_2 < \cos^2\theta_1$ we get after some calculations that $\sin^2(\theta_1 + \theta_2)\sin^2(\theta_1 - \theta_2) < \sin^2\theta_1 + \sin^2\theta_2 - 4\sin^2\theta_1\sin^2\theta_2$ which implies (1.7). It should be noted that the class defined by $a_{2k-1} = a_{2k}$ and $\min(|\arg a_{2k}|\,|\arg - a_{2k}|) \leqq \pi/4 - \delta$ which includes Garder's class of transforms [5] as a very special case, is a special case of this example. Theorem 5.2 can be applied here.

EXAMPLE 6.4. Let c_k be real, $\sum c_k^{-2} < \infty$ and $a_{2k-1} = c_k(\sin\theta_1)^{-1}e^{i\theta_2}$. $a_{2k} = -c_k(\sin\theta_2)^{-1}e^{i\theta_1}$ where either $0<\theta_1,\ \theta_2<\pi/2,\ 0<\delta,\ <\theta_1 + \theta_2<\pi/2 - \delta_2$ or $-(\pi/2) < \theta_1,\ \theta_2 < 0,\ -(\pi/2) + \delta_2 < \theta_1 + \theta_2 < \delta_1 < 0$.

The inqualities used in Example 6.3 for the validity of $\{a_k\} \in A(2)$ can also be used here. It should be noted that the class of transforms defined by Dauns and Widder [1] is the case $\theta_1 = \theta_2$ here.

EXAMPLE 6.5. Let $a_{2n-1} = n^\gamma(1 + i)$, $a_{2n} = n^\gamma(1 - i)$, $\gamma > 1/2$. In this case $\{a_k\} \notin A(2)$ (since (1.6) is not satisfied) but clearly $\{a_k\} \in C(2)$. In this case β of Theorem 5.4 is easily computed as $S_{2m} = (1 + o(1))4\gamma m^{-2\gamma+1}$, $Q_m = (1 + o(1))4\gamma m^{-4\gamma-1}$ $m \to \infty$ and therefore

$$\left(-\gamma + \frac{1}{2}\right)\beta \leqq -\gamma + \frac{1}{4}$$

that is $\beta \geqq 1 + 1/2(2\gamma - 1)$. From this one can see easily that: (a) When $\gamma = 1$ it is enough to have at $t = x$ $\varphi(t) - \varphi(x) = o(|t - x|^{1/2})$ for Theorem 5.4.

(b) When $\gamma > 3/4$ it is enough to have $\varphi(t) - \varphi(x) = 0(t - x)$ $t \to x$ or it is enough for $\varphi(t)$ to have a left and right derivative at $t = x$.

EXAMPLE 6.6. $a_{2n-1} = n^\gamma(1 + i)$, $a_{2n} = -n^\gamma(1 + i)$. For $\gamma > 3/4$ $\{a_k\} \in B(2, 1/3)$. The following remarks will constitute generalizations of the Theorems of § 5 in various directions.

REMARK 6.1. In Theorem 5.1 $|\varphi(t)| \leqq Ke^{M|t|}$ can be replaced by $\left|\int_0^t \varphi(t)dt\right| \leqq Ke^{M|t|}$ if for every $\delta > 0$ if

$$(6.1) \qquad (S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2}\exp(-\delta S_m^{-1/2}) = o(1) \qquad m \to \infty .$$

This result can be achieved by a proper change of Theorem 4.2 that will yield now

$$(6.2) \qquad |G'_{2m}(t)| \leqq M(S_{2m}S^{(1)}_{2m}S^{(2)}_{2m})^{-1/2}\exp(-AS_{2m}^{-1/2}|t|)\,.$$

REMARK 6.2. In Theorems 5.3 and 5.4 condition (3) can be replaced by $\left|\int_0^t \varphi(t)dt\right| \leqq Ke^{M|t|}$ if either $Q_m^{(1)} \geqq LQ_m$ or if for all $\eta > 0$

$$(Q_m^{(1)}Q_m)^{-1/4}\exp\left(-\eta\sum_{m+1}^{\infty}|a_k|^{-1-\delta}\right)^{-1/(1+\delta)}\Bigg) = o(1) \qquad m \to \infty$$

for Theorem 5.3 and $(Q_m^{(1)}Q_m)^{-1/4}\exp(-\eta S_{2m}^{-1/2}) = o(1)$ $m \to \infty$ for Theorem 5.4. For the above generalization slight improvements of Theorems 4.6 and 4.7 are needed in case $Q_m^{(1)} \geqq LQ_m$ is not satisfied.

REMARK 6.3. If $S_{2m}^{-1/2} \leqq KQ_m^{1/4}$, then in Theorem 5.4 $\varphi(t) - \varphi(x) = 0(1)$ $t \to x$ can be replaced by

$$\int_x^{x+h}[\varphi(t) - \varphi(x)]dt = o(h) \qquad h \to 0\,.$$

REMARK 6.4. If in Theorem 5.3 (5) is replaced by

$$\varphi(t) - \varphi(x+) = o(|t - x|^{\beta-1}) \qquad t \to x+$$

and

$$\varphi(t) - \varphi(x-) = o(|t - x|^{\beta-1}) \qquad t \to x-\,,$$

then

$$\lim_{m\to\infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+) + \varphi(x-)]\,.$$

REMARK 6.5. If in Theorem 5.3 we have

$$\left(\sum_{m+1}|a_k|^{-1-\delta}\right)^{1/1+\delta} \leqq K_1Q_m\,,$$

then $\varphi(t) - \varphi(x) = o(1)$ $t \to x$ can be replaced by

$$\int_x^{x+h}[\varphi(x \pm t) - \varphi(x \pm 0)dt = o(h) \qquad h \to 0+$$

and then

$$\lim_{m\to\infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x + 0) + \varphi(x - 0)]\,.$$

REFERENCES

1. J. Dauns and D. V. Widder, *Convolution transforms whose inversion function have complex roots*, Pacific J. Math. (2) **15** (1965), 427-442.
2. Z. Ditzian, *Inversion formulae and various properties of integral transforms*, Ph. D. thesis, 1965, Hebrew University, Jerusalem, Israel (Hebrew, English summary).
3. ———, *On asymptotic estimates for kernels of convolution transforms*, Pacific J. Math. **21** (1967), 249-254.
4. Z. Ditzian and A. Jakimovski, *Inversion and jump formulae for variation diminishing transforms*. (to appear)
5. A. O. Garder, *The inversion of a special class of convolution transforms* M. A. thesis, 1950, Washington University, St. Louis Missouri.
6. I. I. Hirschman and D. V. Widder, *Convolution transforms with complex kernels*, Pacific J. Math. **1** (1951).
7. ———, *The Convolution Transform*, Princeton Press, 1955.
8. R. M. Mcleod, *Mean value theorems for vector valued functions*, Proc. Edin. Math. Soc. (11), **14** (1965) Part 3, 197-211.
9. D. V. Widder, *The Laplace Transform*, 2nd ed., Princeton Press. 1946.

Received December 20, 1966.

THE UNIVERSITY OF ALBERTA
EDMONTON, CANADA

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 25, No. 1 September, 1968