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**ON A CLASS OF CONVOLUTION TRANSFORMS**

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**In this paper the convolution transform**

$$(1.1) \quad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt \equiv (G^*\varphi)(x)$$

**whose kernel  $G(t)$  is the Fourier transform of  $[E(iy)]^{-1}$  where  $E(s)$  is defined by**

$$(1.2) \quad E(s) = e^{bs} \prod_{k=1}^{\infty} (1 - s/a_k) \exp(s \operatorname{Re} a_k^{-1}),$$

$$\operatorname{Re} b = b \text{ and } \sum |a_k|^{-2} < \infty$$

**will be studied. An inversion theory similar to that achieved when  $a_k$  of (1.2) are real will be obtained. The results will show that under certain rather weak conditions, an infinite subsequence  $a_{k(i)}$  of  $a_k$  can satisfy**

$$\min \{ |\arg a_{k(i)}|, |\arg -a_{k(i)}| \} \geq \frac{\pi}{4}.$$

**Classes of transforms will be introduced that allow the occurrence of  $\min \{ |\arg a_k|, |\arg -a_k| \} \geq \pi/4$  for all  $k$ .**

We hope this will partly answer a problem set by Dauns and Widder [1] in Remark 1, page 441.

The inversion operator  $P_m(D)$  is defined by

$$(1.3) \quad P_m(D) = \exp((b - b_m)D) \prod_{k=1}^m \left(1 - \frac{D}{a_k}\right) \exp\left(\left(\operatorname{Re} \frac{1}{a_k}\right)D\right)$$

where  $D \equiv d/dx$ ,  $\exp(kD)f(x) = f(x+k)$  and  $\lim_{m \rightarrow \infty} b_m = 0$ .

The inversion formula will be

$$(1.4) \quad \lim_{i \rightarrow \infty} P_{m(i)}(D)f(x) = \varphi(x).$$

This inversion formula was achieved under general conditions on  $\varphi(x)$  in the case  $a_k$  were real by I. I. Hirschman and D. V. Widder in a series of papers and in their book, "The convolution transform" [7]. Hirschman and Widder [6] also found a slightly changed version of (1.4) when  $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / \operatorname{Re} a_k)^2 < \infty$ . A. O. Garder [5] showed that if  $a_{2k-1} = \bar{a}_{2k}$  then  $\arg a_{2k}$  can tend to 0 or  $\pi$  slower than is required in [6]. Dauns and Widder [1] showed that if  $a_{2k-1} = -a_{2k}$ ,  $0 \leq \operatorname{Re} a_{2k-1} \in \uparrow$  and  $|\arg a_{2k-1}| < (\pi/4) - \eta$ , where  $\eta$  is independent of  $k$ , then (1.4) can be achieved.

It will be noted that in [1] and [5] the  $a_k$ 's were in a special order. The order of the  $a_k$ 's, though having no influence on  $E(s)$ ,

may be quite important when treating (1.4) as discussed with some examples in [2] and [4].

We shall define class  $A(2)$  (that will depend also on the order of the  $a_k$ 's). The sequence  $\{a_k\}$  belongs to class  $A(2)$  if  $\operatorname{Re} a_k \neq 0$ ,

$$(1.5) \quad \sum_{k=1}^{\infty} (\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 / (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \infty ,$$

$$(1.6) \quad (1 - \theta)(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1} > 0$$

for  $k > k_0$  for some  $\theta$ ,  $0 < \theta < 1$  where  $\theta$  is independent of  $k$ , and

$$(1.7) \quad \frac{(\operatorname{Im} \{ \overline{(a_{2k-1}^{-1} + a_{2k}^{-1})} a_{2k-1}^{-1} a_{2k}^{-1} \})^2 |a_{2k-1} a_{2k}|^2}{|a_{2k-1}|^2 + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}} < 1 - \eta$$

for  $k \geq k_1$  for some  $\eta$ ,  $0 < \eta < 1$  where  $\eta$  is independent of  $k$ .

A transform belongs to  $A(2)$  if there is an order under which  $\{a_n\} \in A(2)$ . Class  $A(2)$  includes the transforms of [1], [5] and [6].

**LEMMA 1.1.**  $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / \operatorname{Re} a_k)^2 < \infty$  implies  $\{a_k\} \in A(2)$  (and the order does not matter).

*Proof.*  $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / \operatorname{Re} a_k)^2 < \infty$  implies  $\sum_{k=1}^{\infty} (\operatorname{Im} a_k / |a_k|)^2 < \infty$  which implies  $\sum_{k=1}^{\infty} (\operatorname{Im} a_k^{-1})^2 / |a_k|^{-2} < \infty$  which implies (1.5). To prove that  $a_k$  satisfies (1.6) and (1.7) is not difficult.

**REMARK.** The inversion operator introduced by Hirschman and Widder [6] was slightly different from (1.4) but since

$$\sum_{k=1}^{\infty} \{(\operatorname{Re} a_k)^{-1} - \operatorname{Re} a_k^{-1}\} = \sum_{k=1}^{\infty} \frac{(\operatorname{Im} a_k)^2}{|a_k|^2 \operatorname{Re} a_k} < \infty ,$$

the difference is a change in  $b$  and  $b_m$  without changing  $\lim_{m \rightarrow \infty} b_m = 0$ .

**LEMMA 1.2.** Let  $a_{2k-1} = -a_{2k}$ , let  $\operatorname{Re} a_{2k} > 0$  and  $|\arg a_{2k}| < (\pi/4) - \eta_1$  for  $k > k_2$ , where  $\eta_1$  satisfies  $0 < \eta_1 < \pi/4$  and  $\eta_1$  is independent of  $k$ , then  $\{a_k\} \in \text{class } A(2)$ .

*Proof.* It is easy to see that the sum in (1.5) is equal to zero and the right side of (1.7) is equal to zero.  $|\arg a_{2k}| < (\pi/4) - \eta_1$  implies (1.6), with  $\theta = 1 - 2(\sin((\pi/4) - \eta_1))^2$ , for  $k > k_2$ .

This shows that the transforms treated in [1] are included in class  $A(2)$ .

**LEMMA 1.3.** Let  $a_{2k-1} = \overline{a_{2k}}$  and let  $\min\{|\arg a_{2k}|, |\arg -a_{2k}|\} < (\pi/4) - \eta_2$  for  $k \geq k_2$  where  $\eta_2$ ,  $0 < \eta_2 < \pi/4$ , is independent of  $k$ , then  $\{a_k\} \in A(2)$ .

*Proof.* It is easy to see that the sum in (1.5) and the right side of (1.7) are equal to zero. One can show that  $\min\{|\arg a_{2k}|, |\arg -a_{2k}|\} < (\pi/4) - \eta_2$  implies (1.6) with  $\theta = 1 - 2(\sin((\pi/4) - \eta_2))^2$  for  $k \geq k_2$ .

Lemma 1.3 shows that the transforms treated by A. O. Garder [5] belong to class  $A(2)$ . Some cases which do not belong to class  $A(2)$  will be treated, among them will be the case when  $a_{2k-1} = -a_{2k}$  and  $\min\{|\arg a_{2k}|, |\arg -a_{2k}|\} = \pi/4$  (see Remark 2, [1], p. 442) where estimates different from those achieved for class  $A(2)$  will be obtained.

For the definition of  $G(t)$

$$(1.8) \quad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st} dt$$

we have to assume that the integral on the right converges.

For the convergence of (1.8) we shall have to estimate  $E(iy)$  and to these estimates the various classes correspond.

2. Estimates for  $E_{2m}(s)$  when  $\{a_k\} \in \text{class } A(2)$ . In previous papers (see [1] and [6] for example) it was found useful and important to estimate  $E_m(s)$  which is defined by

$$(2.1) \quad E_m(s) = e^{b_m s} \prod_{k=m+1}^{\infty} (1 - s/a_k) \exp(s \operatorname{Re} a_k^{-1}).$$

In order to estimate  $E_m(s)$  we shall estimate one term first.

LEMMA 2.1. *Let  $\{a_k\} \in \text{class } A(2)$  then for  $k \geq K$*

$$(2.2) \quad \begin{aligned} & |(1 - iy/a_{2k-1})(1 - iy/a_{2k})|^2 \\ & \geq (1 + \alpha y^2/|a_{2k-1}|^2)(1 + \alpha y^2/|a_{2k}|^2)(1 - \alpha^{-1}[(\operatorname{Im}(a_{2k-1}^{-1} \\ & \quad + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})]). \end{aligned}$$

where  $0 < \alpha < 1$  and  $\alpha$  is independent of  $k$ . ( $\alpha$  does depend on  $\theta$  and  $\eta$  of the definition of class  $A(2)$ ).

*Proof.* By a simple calculation we get

$$\begin{aligned} I_k & \equiv |(1 - iy/a_{2k-1})(1 - iy/a_{2k})|^2 = 1 + 2y \operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}) \\ & \quad + y^2\{|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}\} \\ & \quad + 2y^3 \operatorname{Im}\{a_{2k-1}^{-1} a_{2k}^{-1} (\overline{a_{2k-1}^{-1}} + \overline{a_{2k}^{-1}})\} + y^4 |a_{2k-1}|^{-2} |a_{2k}|^{-2}. \end{aligned}$$

We assume  $K \geq k_1$  and therefore by (1.7) we get

$$\frac{(\operatorname{Im} \{(\overline{a_{2k-1}^{-1}} + \overline{a_{2k}^{-1}})(a_{2k-1}^{-1} \cdot a_{2k}^{-1})\})^2}{\left[ \left(1 - \frac{\eta}{2}\right)(|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}) \right] \cdot \left[ \left(1 - \frac{\eta}{2}\right) |a_{2k-1} \cdot a_{2k}|^{-2} \right]} \\ < (1 - \eta) / \left(1 - \eta + \frac{\eta^2}{4}\right) < 1 - \frac{\eta^2}{4}.$$

It is easy to see that  $y^2(A + 2By + Cy^2) \geq 0$  whenever  $A > 0$ ,  $C > 0$  and  $B^2 < AC$ . We substitute

$$\begin{aligned} A &= \left(1 - \frac{\eta}{2}\right) \{|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}\}, \\ B &= \operatorname{Im} \{(\overline{a_{2k-1}^{-1}} + \overline{a_{2k}^{-1}})a_{2k-1}^{-1}a_{2k}^{-1}\} \quad \text{and} \\ C &= \left(1 - \frac{\eta}{2}\right) |a_{2k-1}a_{2k}|^{-2}. \end{aligned}$$

We use (1.6), (1.7) and the above calculation to show that, for  $k > \max(k_0, k_1)$ ,  $A > 0$ ,  $C > 0$  and  $B^2 > AC$ . By omitting  $y^2(A + 2By + Cy^2)$  from the right side of the equation defining  $I_k$  we obtain

$$(2.3) \quad \begin{aligned} I_k &\geq 1 + 2y \operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}) \\ &\quad + \frac{\eta\theta}{2} y^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) + \frac{\eta y^4}{2} |a_{2k-1}a_{2k}|^{-2} \end{aligned}$$

by minimum consideration

$$(2.4) \quad \begin{aligned} &1 + 2y \operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}) \\ &\quad + \frac{\eta\theta}{4} y^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \geq 1 - \frac{\frac{4}{\eta\theta} (\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})} \end{aligned}$$

the last term tends to 1 for large  $k$  because of (1.5). Using (2.3), (2.4) and letting the coefficients of  $y^2$  and  $y^4$  be smaller, we obtain (2.2) with  $\alpha = \eta\theta/4$ .

**LEMMA 2.2.** *Suppose  $\{a_k\} \in \text{class } A(2)$ . Then for  $k > K$  there exist  $A$  and  $B$ ,  $0 < A < B < 1$  independent of  $k$  (but they depend on  $\eta$  and  $\theta$ ) so that for any  $r$ ,  $r < \min(|a_{2k-1}|, |a_{2k}|)$ , we shall have:*

(a) *For  $|\sigma| \leq Ar$  and  $|y| \leq Br$*

$$\begin{aligned} H_k(\sigma) &\equiv |(1 - (\sigma + iy)/a_{2k-1})(1 - (\sigma + iy)/a_{2k})|^2 \exp(2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1})) \\ &\geq 1 - 2\alpha^{-1} \frac{\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1})}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}} - \frac{r^2}{4} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \\ &\quad - 4\sigma^2 (\operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2. \end{aligned}$$

(b) *For  $|\sigma| \leq Ar$  and  $|y| \geq Br$*

$$H_k(\sigma) \geq \left(1 + \frac{\alpha}{4} y^2 |a_{2k-1}|^{-2}\right) \left(1 + \frac{\alpha}{4} y^2 |a_{2k}|^{-2}\right) \\ \times \left(1 - \frac{2}{\alpha} \frac{(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}}\right),$$

where  $\alpha$  is that of Lemma 2.1.

*Proof.* By a simple calculation

$$\left| \left(1 - \frac{\sigma + iy}{a_{2k-1}}\right) \left(1 - \frac{\sigma + iy}{a_{2k}}\right) \right|^2 = \left| \left(1 - \frac{iy}{a_{2k-1}}\right) \left(1 - \frac{iy}{a_{2k}}\right) \right|^2 \\ - 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) + [\sigma^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1}) \\ + \sigma^4 |a_{2k-1} a_{2k}|^{-2} + 2\sigma^2 y^2 |a_{2k-1} a_{2k}|^{-2} - 4\sigma y \operatorname{Im}(a_{2k-1}^{-1} a_{2k}^{-1}) \\ - 2(\sigma^2 + y^2) \sigma \operatorname{Re} \{(a_{2k-1}^{-1} + a_{2k}^{-1}) a_{2k-1}^{-1} a_{2k}^{-1}\} \\ + 2\sigma^2 y \operatorname{Im} \{(\overline{a_{2k-1}^{-1} + a_{2k}^{-1}}) a_{2k-1}^{-1} a_{2k}^{-1}\}] \equiv I_k - 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) + J_k.$$

For the estimation of  $J_k$  we shall recall that

$$(2.5) \quad |(a_{2k-1}^{-1} + a_{2k}^{-1}) a_{2k-1}^{-1} a_{2k}^{-1}| \leq 2(|a_{2k-1}|^{-3} + |a_{2k}|^{-3})$$

and

$$(2.6) \quad |a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1} \geq -2 |\operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1}| \\ \geq -(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}).$$

To prove (a) assume  $|\sigma| \leq Ar$ ,  $|y| \leq Br$ . Using (2.5) and (2.6) and dropping positive terms we obtain for  $A < B$

$$J_k \geq (-A^2 - |2AB)r^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \\ + (-4(A^2 + B^2)A - 4A^2 B)r^3(|a_{2k-1}|^{-3} + |a_{2k}|^{-3}) \\ \geq (-3B^2 - 12B^3)r^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}).$$

Choosing  $A < B$  and (for instance)  $B = 3^{-2}$  and using Lemma 2.1 with  $y = 0$  we obtain

$$H_k(\sigma) \geq \left(1 - \frac{\alpha^{-1}(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|a_{2k-1}|^{-2} + |a_{2k}|^{-2}} - \frac{1}{9} r^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})\right) \\ - 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) \exp(2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1})) \\ \geq 1 - 2\alpha^{-1}(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2[|a_{2k-1}|^{-2} + |a_{2k}|^{-2}]^{-1} \\ - \frac{1}{4} r^2(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) - 4\sigma^2(\operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2.$$

(The coefficients in the above estimation are not the best but they are convenient). To prove (b) (for which we are free to choose  $A, A < B$ ) we recall that for  $A \leq \beta B$ ,  $0 < \beta < 1$  and  $|\sigma| < Ar$  we

have

$$\begin{aligned}
& |2\sigma^2 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1}| \leq \beta^2 B^2 r^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) , \\
& |2\sigma^2 y \operatorname{Im} \{(\overline{a_{2k-1}^{-1} + a_{2k}^{-1}}) a_{2k-1}^{-1} \cdot a_{2k}^{-1}\}| \\
& \quad \leq \sigma^2 y^2 |a_{2k-1} a_{2k}|^{-2} + 2\sigma^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) , \\
& |2y^2 \sigma \operatorname{Re} \{(\overline{a_{2k-1}^{-1} + a_{2k}^{-1}}) a_{2k-1}^{-1} a_{2k}^{-1}\}| \\
& \quad \leq \beta y^4 |a_{2k-1} a_{2k}|^{-2} + 2\beta y^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2})
\end{aligned}$$

and

$$\begin{aligned}
& |2\sigma^3 \operatorname{Re} \{(\overline{a_{2k-1}^{-1} + a_{2k}^{-1}}) a_{2k-1}^{-1} a_{2k}^{-1}\}| \\
& \quad \leq \sigma^4 |a_{2k-1} a_{2k}|^{-2} + 2\sigma^2 (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) .
\end{aligned}$$

Choosing  $\beta$  so that  $5\beta^2 + 4\beta < \alpha/4$  and  $K$  so that

$$\alpha^{-1} (\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 / (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \leq \frac{1}{4}$$

for  $k \geq K$  we obtain by the above estimations

$$\begin{aligned}
H_k(\sigma) & \leq \left(1 + \frac{\alpha}{2} y^2 |a_{2k-1}|^{-2}\right) \left(1 + \frac{\alpha}{2} |a_{2k}|^{-2}\right) \left(1 - \frac{1}{\alpha} (\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2\right) , \\
& \times [|a_{2k-1}|^{-2} + |a_{2k}|^{-2}]^{-1} - 2\sigma \operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}) \cdot \exp(2\sigma \operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1})) \\
& \leq \left(1 + \frac{3\alpha}{8} y^2 |a_{2k-1}|^{-2}\right) \left(1 + \frac{3\alpha}{8} y^2 |a_{2k}|^{-2}\right) \left(1 - \frac{2}{\alpha} (\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2\right) , \\
& \times [|a_{2k-1}|^{-2} + |a_{2k}|^{-2}]^{-1} \cdot (1 - 4\sigma^2 (\operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2) .
\end{aligned}$$

Since  $\beta < \alpha \cdot 4^{-2}$ ,  $\beta^2 < \alpha \cdot 4^{-4}$  we obtain (b) easily.

Define  $S_m$  and  $S_m^{(l)}$  (see [7], [2] and [4]) by

$$(2.7) \quad S_m = \sum_{k=m+1}^{\infty} |a_k|^{-2}$$

$$(2.8) \quad S_m^{(l)} = S_m - \max_{k(1) < \dots < k(l)} \sum_{i=1}^l |a_{k(i)}|^{-2} .$$

Define also  $r_m$  by

$$(2.9) \quad r_m = \min_{k > m} |a_k|$$

One can easily see that  $S_m^{(0)} = S_m$  and  $S_m^{(1)} = S_m - r_m^{-2}$ .

**THEOREM 2.3.** *Let  $\{a_k\} \in \text{class } A(2)$ , then for  $m \geq K$ ,  $|\sigma| \leq AS_{2m}^{-1/2}$ , and  $b_{2m} = 0$  we have*

$$(2.10) \quad |E_{2m}(\sigma + iy)| \geq \sqrt{2}/2 .$$

( $A$  being that of Lemma 2.2.)

*Proof.* To prove (2.10) we use Lemma 2.2(a) whose conditions are satisfied since  $S_{2m} > r_{2m}^{-2}$ ,  $S_{2m}^{-1/2} < r_{2m} = \min_{k>2m} |a_k|$ . We also recall that for  $A_n > 0$  and  $\sum_{n=m+1}^{\infty} A_n < 1/2$  we have

$$\prod_{n=m+1}^{\infty} (1 - A_n) \geq 1 - \sum_{n=m+1}^{\infty} A_n \geq \frac{1}{2}.$$

Remembering that for large  $m$

$$2\alpha^{-1} \sum_{k=m+1}^{\infty} (\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 / (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \frac{1}{8}$$

and

$$\begin{aligned} 4\sigma^2 \sum_{k=m+1}^{\infty} (\operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 &\leq 8A^2 S_{2m} \sum_{k=m+1}^{\infty} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \\ &\leq 8A^2 < 8^{-1} \end{aligned}$$

and using Lemma 2.2(a) we conclude the proof of (2.10) in the case where  $|\sigma| \leq S_{2m}^{-1/2}$  and  $|y| \leq BS_{2m}^{-1/2}$ . Using Lemma 2.2(b), (2.10) in the case where  $|\sigma| \leq AS_{2m}^{-1/2}$ ,  $|y| \geq BS_{2m}^{-1/2}$  follows by an argumentation similar to that used in the first part. Then:

**THEOREM 2.4.** *Let  $\{a_k\} \in A(2)$ ,  $b_{2m} = 0$ , then for  $m \geq k$ ,  $|\sigma| \leq AS_{2m}^{-1/2}$  and  $|y| \geq BS_{2m}^{-1/2}$  we have*

$$\begin{aligned} (2.11) \quad |E_{2m}(\sigma + iy)| &\geq \frac{3}{4} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \cdot \left( \frac{\alpha}{4} \right)^n \cdot \prod_{l=0}^{n-1} S_{2m}^{(l)} \right)^{1/2} \\ &\geq \frac{3}{4} \left( 1 + \frac{1}{n!} y^{2n} \left( \frac{\alpha}{4} \right)^n \prod_{l=0}^{n-1} S_{2m}^{(l)} \right)^{1/2}. \end{aligned}$$

*Proof.* Using (1.5) we can choose, by the method in the proof of Theorem 2.3,  $m$  so that

$$(2.12) \quad \sum_{k=m+1}^{\infty} \left( 1 - \frac{2}{\alpha} [(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 / (|a_{2k-1}|^{-2} + |a_{2k}|^{-2})] \right) \geq \frac{9}{16}$$

(9/16 can be replaced of course by any  $1 - \varepsilon$ ).

$$\sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \left( \frac{\alpha}{2} \right)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}$$

converges for all  $y$  since  $S_{2m} = S_{2m}^{(0)} > S_{2m}^{(1)} > \dots > S_{2m}^{(L)}$ . By Lemma 2.2 and (2.12) we have

$$|E_{2m}(\sigma + iy)| \geq \frac{3}{4} \left( 1 + \sum_{n=1}^{\infty} y^{2n} \left( \frac{\alpha}{4} \right)^n \sum_{\substack{k(1) > 2m \\ k(i) < k(i+1)}} |a_{k(1)} \cdots a_{k(n)}|^{-2} \right)^{1/2}.$$

But we have



$$\begin{aligned}
I(n, m) &\equiv \sum_{\substack{2m < k(1) \\ k(i) < k(i+1)}} |a_{k(1)} \cdots a_{k(n)}|^{-2} = \frac{1}{n!} \sum_{\substack{k(i) > 2m \\ k(i) \neq k(j), j \neq i}} |a_{k(1)} \cdots a_{k(n)}|^{-2} \\
&\geq \frac{1}{n!} \sum_{\substack{k(i) > 2m \\ k(i) \neq k(j), j \neq i}} \left( S_{2m} - \sum_{r=1}^{n-1} |a_{k(r)}|^{-2} \right) |a_{k(1)} \cdots a_{k(n-1)}|^{-2} \\
&\geq \frac{1}{n!} S_{2m}^{(n-1)} \sum_{\substack{k(i) > 2m \\ k(i) \neq k(j), j \neq i}} |a_{k(1)} \cdots a_{k(n-1)}|^{-2}.
\end{aligned}$$

Since

$$\sum_{k(i) > 2m} |a_{k(i)}|^{-2} = S_{2m} = S_{2m}^{(0)},$$

by induction  $I(n, m) \geq 1/n!$ .  $\prod_{l=0}^{n-1} S_{2m}^{(l)}$ , which concludes the proof of the theorem.

**THEOREM 2.5.** *Let  $\{a_k\} \in A(2)$ ,  $b_{2m} = 0$ , and  $\sigma$  satisfies  $\operatorname{Re} a_k \neq \sigma$  for all  $k > n$ , then for  $p, n = 0, 1, 2, \dots$  there exist  $k_1(p, \sigma, n)$  and  $k_2(p, \sigma, n)$  so that*

$$(2.13) \quad |E_{2n}(\sigma + i\tau)|^2 \geq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p}.$$

*Proof.* Since  $S_{2m} = o(1)m \rightarrow \infty$  we can choose  $m$  so that  $AS_{2m}^{-1/2} \geq \sigma$  (for  $A$  of Theorems 2.3 and 2.4). Combining Theorems 2.3, 2.4 and the fact that  $|\prod_{k=2n+1}^{2m} (1 - \sigma + i\tau/a_k)| e^{\sigma \operatorname{Re} a_k^{-1}} \geq \delta$  whenever  $\operatorname{Re} a_k \neq \sigma$ , we obtain (2.13).

**3. Estimates for  $E_{m(i)}(s)$  in special cases when  $\{a_k\} \in A(2)$ .** In this section we shall estimate  $E_{2m}(s)$  in case  $\{a_k\}$  does not necessarily belong to  $A(2)$  but  $a_{2k-1} = -a_{2k}$  or  $a_{2k-1} = \bar{a}_{2k}$  and some other conditions are satisfied.

First we prove some lemmas concerning the above mentioned cases.

**LEMMA 3.1.** *Let  $a$  be a complex number  $\operatorname{Re} a \neq 0$ , then for all real  $y$  and  $q \geq 1$*

$$\begin{aligned}
I(a) &= \left| \left(1 - \frac{iy}{a}\right) \left(1 + \frac{iy}{a}\right) \right|^2 = \left| \left(1 - \frac{iy}{a}\right) \left(1 - \frac{iy}{\bar{a}}\right) \right|^2 \\
(3.1) \quad &\geq \begin{cases} 1 - q \left( \frac{\operatorname{Re} a^2}{|a|^2} \right)^2 + \left(1 - \frac{1}{q}\right) y^4 |a|^{-4} \\ 1 + y^4 |a|^{-4} & \operatorname{Re} a^2 \geq 0. \end{cases}
\end{aligned}$$

*Proof.* Simple calculation yields

$$\begin{aligned}
\left| \left(1 - \frac{iy}{a}\right) \left(1 - \frac{iy}{\bar{a}}\right) \right|^2 &= 1 - q \left( \frac{\operatorname{Re} a^2}{|a|^2} \right)^2 \\
&+ \left( \sqrt{q} \frac{\operatorname{Re} a^2}{|a|^2} + \frac{1}{\sqrt{q}} y^2 |a|^{-2} \right)^2 + \left(1 - \frac{1}{q}\right) y^4 |a|^{-4}
\end{aligned}$$

from which (3.1) is immediate.

**LEMMA 3.2.** *Let  $a$  be complex number,  $\operatorname{Re} a \neq 0$ , then*

$$(3.2) \quad \begin{aligned} & \left| \left( 1 - \frac{\sigma + iy}{a} \right) \left( 1 + \frac{\sigma + iy}{a} \right) \right|^2 \\ &= I(a) + 2\sigma^2(|a|^{-2} - 2(\operatorname{Re} a)^2 |a|^{-4}) + \sigma^4 |a|^{-4} \\ & \quad + 2\sigma^2 y^2 |a|^{-4} + 4\sigma y(\operatorname{Im} a^2) |a|^{-4}, \end{aligned}$$

$$(3.3) \quad \begin{aligned} & \left| \left( 1 - \frac{\sigma + iy}{a} \right) \left( 1 - \frac{\sigma + iy}{\bar{a}} \right) \right|^2 \\ &= I(a) - 4\sigma \operatorname{Re} a |a|^{-2} + \sigma^2(2|a|^{-2} + 4(\operatorname{Re} a)^2 |a|^{-4}) \\ & \quad + \sigma^4 |a|^{-4} + 2\sigma^2 y^2 |a|^{-4} - 4(\sigma^2 + y^2)\sigma |a|^{-4} \operatorname{Re} a, \end{aligned}$$

where  $I(a)$  is defined in Lemma 3.1.

*Proof.* The proof is a corollary of the proof of Lemma 2.2 combined with Lemma 3.1.

**LEMMA 3.3.** *Let  $\operatorname{Re} a \neq 0$ , then for  $K > 1$  there exists  $A$  and  $B$ , independent of  $a$ ,  $0 < A < B < 1$  such that for  $r < |\operatorname{Re} a|$  we have:*

(a) *For  $|\sigma| \leq Ar$  and  $|y| \leq Br$*

$$(3.4) \quad \begin{aligned} & \left| \left( 1 - \frac{\sigma + iy}{a} \right) \left( 1 + \frac{\sigma + iy}{a} \right) \right|^2 \\ & \geq 1 - K^{-1} r^2 |a|^{-2} - (\min(0, (\operatorname{Re} a^2) \cdot |a|^{-2}))^2. \end{aligned}$$

(b) *For  $|\sigma| \leq Ar_1 \leq Ar$ ,  $|y| \geq Br$  and  $\delta > 0$*

$$(3.5) \quad \begin{aligned} & \left| \left( 1 - \frac{\sigma + iy}{a} \right) \left( 1 + \frac{\sigma + iy}{a} \right) \right|^2 \\ & \geq \left( 1 + \frac{1}{4} y^4 |a|^{-4} \right) (1 - 2(\min(0, \operatorname{Re} a^2 / |a|^2))^2 \\ & \quad - K^{-1}(r^2 |a|^{-2} + r_1^2 |a|^{-1-\delta} + |a|^{-2+2\delta})). \end{aligned}$$

*Proof.* To prove (3.4) we use (3.2) and (3.1) with  $q = 1$  and obtain the result by choosing  $B$  so that  $6B^2 < K^{-1}$ , and dropping some positive terms.

To prove (3.5) we use

$$4\sigma y(\operatorname{Im} a^2) |a|^{-4} \geq -4|\sigma y| |a|^{-2} \geq -\left( \frac{1}{\beta^2} y^2 |a|^{-(3-\delta)} + 2\beta^2 \sigma^2 |a|^{-(1+\delta)} \right)$$

and

$$-\frac{1}{\beta^2} y^2 |a|^{-(3-\delta)} + \frac{1}{4} y^4 |a|^{-4} \geq -\frac{4}{\beta^4} |a|^{-(2-2\delta)}.$$

Choosing  $4/\beta^4 \leq 1/K$  or  $\beta \geq \sqrt[4]{4K}$  and  $A$  so that  $2\beta^2 A^2 < K^{-1}$  or  $A^2 < 1/4K\sqrt{K}$  or  $A < 1/2K$  one can conclude the proof by using Lemma 3.1 (choosing there  $q = 2$  in case  $\operatorname{Re} a^2 < 0$ ) and dropping some positive terms.

**LEMMA 3.4.** *Let  $\operatorname{Re} a \neq 0$ , then for  $K > 1$  there exist  $A$  independent of  $a$ ,  $0 < A < 1$ , such that for  $r < |\operatorname{Re} a|$  and  $|\sigma| \leq Ar$  we have*

$$(3.6) \quad \left| \left( 1 - \frac{\sigma + iy}{a} \right) \left( 1 - \frac{\sigma + iy}{\bar{a}} \right) \right|^2 \exp(4\sigma \operatorname{Re} a / |a|^2) \\ \geq \left( 1 + \frac{1}{4} y^4 |a|^{-4} \right) (1 - 2(\min(0, \operatorname{Re} a^2 / |a|^2))^2 - K^{-1} r^2 |a|^{-2}).$$

*Proof.* Using (3.3) of Lemma 3.2, Lemma 3.1 with  $q = 3/2$ , the estimations

$$-4\sigma^3 |a|^{-4} \operatorname{Re} a \geq -\sigma^4 |a|^{-4} - 4\sigma^2 (\operatorname{Re} a)^2 |a|^{-4}, \\ -4y^2 \sigma |a|^{-4} \operatorname{Re} a \geq -\frac{1}{2^5} y^4 |a|^{-4} - 4^3 \sigma^2 |a|^{-2}$$

and dropping some positive terms we obtain

$$\left| \left( 1 - \frac{\sigma + iy}{a} \right) \left( 1 - \frac{\sigma + iy}{\bar{a}} \right) \right|^2 \geq 1 + \left( \frac{1}{3} - \frac{1}{2^4} \right) y^4 |a|^{-4} \\ - \frac{3}{2} (\min(0, \operatorname{Re} a^2 / |a|^2))^2 - 4^3 A^2 r^2 |a|^{-2} - 4\sigma (\operatorname{Re} a) |a|^{-2}.$$

Choosing  $A$  so that  $4^3 A^2 < 1/4K$ , which implies

$$-4^3 A^2 r^2 |a|^{-2} > -\frac{1}{4K} r^2 |a|^{-2}, \quad 4|\sigma| |a|^{-1} < \frac{1}{4},$$

and

$$\exp(4\sigma (\operatorname{Re} a) |a|^{-2}) \geq 1 + 4\sigma (\operatorname{Re} a) |a|^{-2} - 4^2 \sigma^2 |a|^{-2}$$

from which (3.6) follows.

We shall define now two classes of convolution transforms by the function  $E(s)$  and the sequence  $\{a_k\}$ .

**DEFINITION 3.1.**  $\{a_k\} \in \text{class } B(2, \delta)$  if

$$(3.7) \quad E(s) = \prod_{k=1}^{\infty} (1 - s^2 a_k^{-2}) ,$$

$$(3.8) \quad \sum_{\operatorname{Re} a_k^2 < 0} |a_k|^{-4} (\operatorname{Re} a_k^2)^2 < \infty$$

and

$$(3.9) \quad \sum_{k=1}^{\infty} |a_k|^{-1-\delta} < \infty , \quad \sum_{k=1}^{\infty} |a_k|^{-2+\delta} < \infty \quad \text{for some } \delta > 0 .$$

DEFINITION 3.2.  $\{a_k\} \in B(2)$  if there is  $\delta > 0$  so that  $\{a_k\} \in B(2, \delta)$ .

DEFINITION 3.3.  $\{a_k\} \in \text{class } C(2)$  if

$$(3.10) \quad E(s) = \prod_{k=1}^{\infty} (1 - s a_k^{-1})(1 - s \cdot \bar{a}_k^{-1}) ,$$

if condition (3.8) is satisfied and  $\sum |a_k|^{-2} < \infty$ .

REMARK.  $S_{2m} = 2 \sum_{k=m+1}^{\infty} |a_k|^{-2}$  in case of class  $B(2)$  and  $C(2)$ . We have to introduce some more notations before being able to prove the estimation on  $E(s)$  for transforms of classes  $B(2)$  and  $C(2)$ .

$$(3.11) \quad Q_m = \sum_{k=m+1}^{\infty} |a_k|^{-4} .$$

$$(3.12) \quad Q_m^{(j)} = Q_m - \max_{m < k(1) < \dots < k(j)} \left\{ \sum_{r=1}^j |a_{k(r)}|^{-4} \right\} .$$

We shall state the estimations for classes  $B(2)$  and  $C(2)$  together and then outline the proofs.

THEOREM 3.5. *If  $\{a_k\} \in B(2, \delta)$ , then for  $m \geq M$  and some  $A$  and  $B$  we have:*

$$(a) \quad |\sigma| \leq A S_{2m}^{-1/2}, |y| \leq B S_{2m}^{-1/2} \text{ imply}$$

$$(3.13) \quad |E_{2m}(s)| \geq 3/4 .$$

$$(b) \quad |\sigma| \leq A (\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta})^{-1/1+\delta} \text{ and } |y| \geq B S_{2m}^{-1/2} \text{ imply}$$

$$(3.14) \quad |E_{2m}(s)| \geq \frac{3}{4} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{4n} \prod_{l=0}^{n-1} Q_m^{(l)} \right)^{1/2} .$$

THEOREM 3.6. *If  $\{a_k\} \in C(2)$  then for  $m \geq M$  there exists an  $A$  so that for  $|\sigma| \leq A S_{2m}^{-1/2}$  (3.14) is valid.*

*Proof of Theorems 3.5 and 3.6.* The proof follows the proof of Theorems 2.3 and 2.4 Using Lemmas 3.3 and 3.4 we have to choose

$r = S_{2m}^{-1/2}$  and  $r_1 = (2 \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta})^{-1/(1+\delta)}$  ( $r_1$  necessary only in proving Theorem 3.5 from Lemma 3.3). Obviously  $r_1 < \min_{k>m} |a_k|$ ,  $r \leq \min_{k>m} |a_k|$ . Also we have

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} &\geq \left( \sum_{k=m+1}^{\infty} |a_k|^{-2} \right) \left( \min_{k>m} |a_k| \right)^{1-\delta} \\ &\geq \left( \sum_{k=m+1}^{\infty} |a_k|^{-2} \right)^{(1+\delta)/2} \cdot \left( \sum_{k=m+1}^{\infty} |a_k|^{-2} \right)^{(1-\delta)/2} \left( \min_{k>m} |a_k| \right)^{1-\delta} \\ &\geq \left( \sum_{k=m+1}^{\infty} |a_k|^{-2} \right)^{(1+\delta)/2}. \end{aligned}$$

This implies

$$r_1 = \left( 2 \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} \leq (S_{2m})^{-1/2} \leq r.$$

Choose  $m$  and  $K$  so that  $\sum_{k>m} (\min(0, \operatorname{Re} a_k^2 / |a_k|^2))^2 < \varepsilon_1$ ,  $1/K < \varepsilon_1$  ( $K$  of Lemmas 3.3 and 3.4) and, for proving Theorem 3.5,  $\sum_{k=m+1}^{\infty} |a_k|^{-2+2\delta} < \varepsilon_1$ . The choice  $\varepsilon_1 \leq 1/16$  will yield the number  $3/4$  in (3.13) (every  $1 - \eta$  could be achieved by  $\varepsilon_1$  small enough) and the coefficient  $3/4$  in (3.14).

To complete the proof we have to show

$$\prod_{k=m+1}^{\infty} \left( 1 + \frac{1}{4} y^4 |a_k|^{-4} \right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{4n} \prod_{l=0}^{n-1} Q_m^{(l)},$$

the proof of which follows stepwise that of Theorem 2.4.

The classes in this section are not included in  $A(2)$  since (1.6) may fail to be valid. The estimates in this section are weaker in the case where the transforms are also  $A(2)$ .

**THEOREM 3.7.** *Let  $\{a_k\} \in B(2)$  or  $C(2)$ . Then for  $\sigma$  satisfying  $\operatorname{Re} a_k \neq 0$  for all  $k > n$ , and for  $p, n = 0, 1, 2 \dots$  there exist  $k_1(p, \sigma, n)$  and  $k_2(p, \sigma, n)$  such that when  $\sigma \neq \operatorname{Re} a_k$*

$$(3.15) \quad |E_{2n}(\sigma + i\tau)|^2 \geq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p}.$$

*Proof.* Deduced from Theorems 3.5 and 3.6 as Theorem 2.5 is deduced from Theorem 2.4 and 2.3.

4. Estimates for  $G_m(t)$ . We define  $G_m(t)$ , in the usual manner, by

$$(4.1) \quad G_m(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E_m(s)]^{-1} e^{st} ds, \quad G_0(t) = G(t).$$

We define also:

$$(4.2) \quad \alpha(m) = \max \{ \operatorname{Re} a_k, -\infty \mid \operatorname{Re} a_k < 0 \text{ and } k > m \} .$$

$$(4.3) \quad \beta(m) = \min \{ \operatorname{Re} a_k, \infty \mid \operatorname{Re} a_k > 0 \text{ and } k > m \} .$$

We recall that in the cases  $\{a_k\} \in A(2)$ ,  $\{a_k\} \in B(2)$  and  $\{a_k\} \in C(2)$  we have

$$(4.4) \quad |E_{2n}(\sigma + i\tau)|^2 \geq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} ,$$

for  $n, p = 0, 1, 2 \dots$  and  $\alpha(2n) < \sigma < \beta(2n)$ .

**THEOREM 4.1.** *Let  $E_n(s)$ ,  $P_n(D)$  and  $G_n(t)$  be defined by (2.1), (1.3) and (4.1); let (4.4) be satisfied for  $m(l)$ , a subsequence of  $m$ , then:*

A. *For any  $\sigma$  satisfying  $\sigma(m(l)) < \sigma < \beta(m(l))$  we have*

$$(4.5) \quad G_{m(l)}(t) = P_{m(l)}(D)G(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} [E_{m(l)}(s)]^{-1} e^{st} ds .$$

B. *Suppose in case  $\alpha(m(l)) \neq -\infty$  that  $a_{k(1,1)} = \dots = a_{k(1, m_1+1)}$ ,  $a_{k(2,1)} = \dots = a_{k(2, m_2+1)}$ ,  $\dots$ ,  $a_{k(r,1)} = \dots = a_{k(r, m_r+1)}$  are all with indices greater than  $m(l)$  and  $\alpha(m(l)) = \operatorname{Re} a_{k(1,1)} = \operatorname{Re} a_{k(2,1)} = \dots = \operatorname{Re} a_{k(r,1)}$ , then*

$$(4.6) \quad \frac{d^v}{dt^v} G_{m(l)}(t) = \sum_{i=1}^r \frac{d^v}{dt^v} \{P_i(t) e^{t a_{k(i,1)}}\} + 0(e^{k(t)}) \quad t \rightarrow \infty$$

where  $p_i(t)$  are polynomials of order  $m_i$  and  $k$  is any real number satisfying

$$\max \{ \operatorname{Re} a_k, -\infty \mid k > m(l), \operatorname{Re} a_k < \alpha(m(l)) \} < k < \alpha(m(l)) .$$

C. *Suppose  $\alpha(m(l)) = -\infty$ , then*

$$(4.7) \quad \frac{d^v}{dt^v} G_{m(l)}(t) = 0(e^{kt}) \quad t \rightarrow \infty \text{ for any real } k, k < 0 .$$

D. *Suppose in case  $\beta(m(l)) \neq \infty$  that  $a_{r(1,1)} = \dots = a_{r(1, m_1+1)}$ ,  $\dots$ ,  $a_{r(j,1)} = \dots = a_{r(j, m_j+1)}$  are all with indices greater than  $m(l)$  and  $\beta(m(l)) = \operatorname{Re} a_{r(1,1)} = \dots = \operatorname{Re} a_{r(j,1)}$ , then*

$$(4.8) \quad \frac{d^v}{dt^v} G_{m(l)}(t) = \sum_{i=1}^j \frac{d^v}{dt^v} \{q_i(t) e^{t a_{r(i,1)}}\} + 0(e^{kt}) \quad t \rightarrow -\infty$$

where  $q_i(t)$  are polynomials of order  $m_i$  and  $k$  is a real number satisfying  $\beta(m(l)) < k < \min \{ \operatorname{Re} a_k, \infty \mid k > m(l), \operatorname{Re} a_k > \beta(m(l)) \}$ .

E. *Suppose  $\beta(m(l)) = \infty$ , then*

$$(4.9) \quad \frac{d^v}{dt^v} G_{m(l)}(t) = 0(e^{kt}) \quad t \rightarrow -\infty$$

where  $k$  is any real positive number.

F. For  $\alpha(m(l)) < \operatorname{Re} s < \beta(m(l))$  we have

$$(4.10) \quad \frac{1}{E_{m(l)}(s)} = \int_{-\infty}^{\infty} e^{st} G_{m(l)}(t) dt$$

which implies

$$(4.11) \quad 1 = \int_{-\infty}^{\infty} G_{m(l)}(t) dt .$$

*Proof.* The proof follows the method used in Hirschman and Widder's book "The convolution transform" [6, p. 108]. Formula (4.4), that was proved for class  $A(2)$ ,  $B(2)$  and  $C(2)$ , is used here instead of the theorems on  $E_m(s)$  in [6].

The following result will estimate  $G_{2m}(t)$  in the case when  $m$  is large near the point  $t = 0$  as well as when  $|t| \rightarrow \infty$ .

**THEOREM 4.2.** Let  $\{a_k\} \in A(2)$  and suppose that for some  $n$   $S_{2m}^{(n+1)} \geq L_n S_{2m}$  where  $L_n > 0$  is independent of  $m$ , then there exist  $M(n) > 0$  and  $A > 0$  such that

$$(4.12) \quad |G_{2m}^{(n)}(t)| \leq M(n) S_{2m}^{-(n+1)/2} \exp(-A \cdot S_{2m}^{-1/2} |t|) .$$

*Proof.* By Theorem 4.1.A we have

$$G_{2m}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-(\sigma + iy)t}}{E_{2m}(\sigma + iy)} dy$$

and therefore

$$G_{2m}^{(n)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(\sigma + iy)^n e^{-(\sigma + iy)t}}{E_{2m}(\sigma + iy)} dy .$$

Remembering that  $S_{2m}^{(n+1)} \geq L_n S_{2m}$  implies  $S_{2m}^{(k)} \geq L_n S_{2m}$  for  $k \leq n+1$ , and using Theorems 2.3 and 2.4 we obtain, choosing  $\sigma = A S_{2m}^{-1/2}$  for the case  $t > 0$ ,

$$\begin{aligned} |G_{2m}^{(n)}(t)| &\leq \frac{1}{2\pi} \exp(-A S_{2m}^{-1/2} t) \left\{ \int_{-B S_{2m}^{-1/2}}^{B S_{2m}^{-1/2}} \frac{(|\sigma| + |y|)^n}{|E_{2m}(\sigma + iy)|} dy \right. \\ &+ \left. \int_{|y| \geq B S_{2m}^{-1/2}} \frac{(|\sigma| + |y|)^n dy}{|E_{2m}(\sigma + iy)|} \right\} \leq \exp(-A S_{2m}^{-1/2} t) \left\{ \frac{\sqrt{2}}{2\pi} (A + B)^n 2 B S_{2m}^{-(n+1)/2} \right. \\ &+ \left. 2 \sum_{k=0}^n \binom{n}{k} S_{2m}^{-k/2} \frac{2}{3\pi} \int_{B S_{2m}^{-1/2}}^{\infty} \frac{y^{n-k} dy}{\left(1 + y^{2(n+2)} \frac{1}{(n+2)!} (L_{(n)})^{n+1} S_{2m}^{n+2} \left(\frac{\alpha}{4}\right)^{n+2}\right)^{1/2}} \right\} \\ &\leq M(n) S_{2m}^{-(n+1)/2} \exp(-A S_{2m}^{-1/2} t) . \end{aligned}$$

The result for  $t < 0$  is achieved choosing  $\sigma = -AS_{2m}^{-1/2}$ .

**REMARK.** When  $a_{2k-1} = -a_{2k}$  we have  $S_{2m}^{(1)} \geq (1/2)S_{2m}$  and therefore Theorem 4.2 for  $n = 0$  includes Lemma 2.4 of [1, p. 432]. Whenever the connection between pair is  $0 < \theta_1 \leq |a_{2k-1}/a_{2k}| \leq \theta_2 < \infty$ , where  $\theta_1, \theta_2$  are fixed for all  $m$ , we have  $S_{2m}^{(1)} \geq L_1 S_{2m} L_1 > 0$ . But in case of  $n = 0$  the restriction  $S_{2m}^{(1)} \geq L_1 S_{2m}$  is not necessary as is proved by the following.

**THEOREM 4.3.** *Let  $\{a_k\} \in A(2)$ , then for some  $A > 0$  we have*

$$(4.12) \quad |G_{2m}(t)| \leq MS_{2m}^{-1/2} \exp(-AS_{2m}^{-1/2} |t|).$$

*Proof.* Following the proof of Theorem 4.2 and using Theorem 2.4 we have for  $t > 0$  ( $t < 0$  can be treated similarly)

$$\begin{aligned} |G_{2m}(t)| &\leq \exp(-AS_{2m}^{-1/2}t) \left\{ \frac{\sqrt{2}}{\pi} BS_{2m}^{-1/2} \right. \\ &\quad \left. + \frac{4}{3\pi} \int_{BS_{2m}^{-1/2}}^{\infty} \frac{dy}{\left(1 + \frac{1}{2}y^4 S_{2m} S_{2m}^{(1)} \left(\frac{\alpha}{4}\right)^2\right)^{1/2}} \right\}. \\ \int_{BS_{2m}^{-1/2}}^{\infty} \frac{dy}{(1 + Ly^4 S_{2m} S_{2m}^{(1)})^{1/2}} &= S_{2m}^{-1/2} (S_{2m}/S_{2m}^{(1)})^{1/4} \int_{(B/L^{1/4})(S_{2m}/S_{2m}^{(1)})^{1/4}}^{\infty} \frac{1}{(1 + y^4)^{1/2}} dy \\ &\leq 2^d S_{2m}^{-1/2} (S_{2m}/S_{2m}^{(1)})^{1/4} \int_{B_1(S_{2m}/S_{2m}^{(1)})^{1/4}}^{\infty} \frac{dy}{1 + y^2} \\ &\leq 2S_{2m}^{-1/2} (S_{2m}/S_{2m}^{(1)})^{1/4} \lim_{\zeta \rightarrow \infty} (\arctan \zeta - \arctan B_1(S_{2m}/S_{2m}^{(1)})^{1/4}) \\ &\leq 2S_{2m}^{-1/2} (S_{2m}/S_{2m}^{(1)})^{1/4} \lim_{\zeta \rightarrow \infty} \arctan \zeta \left\{ \frac{1 - B_1(S_{2m}/S_{2m}^{(1)})^{1/4} \zeta}{\zeta^{-1} + B_1(S_{2m}/S_{2m}^{(1)})^{1/4}} \right\} \leq B_2 S_{2m}^{-1/2}. \end{aligned}$$

From this the proof can be easily concluded.

Lemma 3.2A of [1, p. 434] is generalized by Theorem 4.2 in case  $S_{2m}^{(2)} \geq L_2 S_{2m}$  for some  $L_2$ . Case B is covered only in part. The following theorem generalizes Lemma 3.2B [1, p. 434].

**THEOREM 4.4.** *Let  $a_k \in A(2)$ ,  $b_{2m} = 0$  and suppose  $0 < \theta_1 < |a_{2k}/a_{2k-1}| < \theta_2 < \infty$  where  $\theta_1, \theta_2$  are independent of  $k$  and  $|\operatorname{Re} a_k|/|a_k| > \eta$ , then for some  $A_1 > 0$  and  $M_1$  we have:*

$$(4.13) \quad |G'_{2m}(t)| \leq M_1 S_{2m}^{-1} \exp(-A_1 S_{2m}^{-1/2} |t|).$$

*Proof.* Let us split the proof into two cases



$$(a) \quad S_{2m} - \max_{k > m} (|a_{k-1}|^{-2} + |a_{2k}|^{-2}) \geq \frac{1}{K} S_{2m}$$

and

$$(b) \quad S_{2m} - \max_{k > m} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \frac{1}{K} S_{2m} .$$

In case (a) (4.13) was proved by Theorem 4.2 for any arbitrary  $K$ . We shall choose  $K > 2$ . To prove (4.13) in case (b) we define  $k_0$  by

$$\max_{k > m} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) = |a_{2k_0-1}|^{-2} + |a_{2k_0}|^{-2} .$$

(In case (b) the choice of  $k_0$  is unique.) Define  $g_{k_0}^*(t)$  and  $G_{2m+2}(t)$  by:

$$(4.14) \quad g_{k_0}^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0})]^{-1} e^{-iyt} dy .$$

$$(4.15) \quad G_{2m+2}^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0})] (E_{2m}(iy))^{-1} e^{-iyt} dy .$$

By [9, p. 255] we have

$$G_{2m}(t) = g_{k_0}^*(t) * G_{2m+2}^*(t) .$$

One can calculate  $g_{k_0}^*(t)$ :

$$g_{k_0}^*(t) = \frac{a_{2k_0-1} a_{2k_0}}{a_{2k_0-1} - a_{2k_0}} \begin{cases} e^{a_{2k_0} t} & t \geq 0 \\ e^{a_{2k_0-1} t} & t < 0 \end{cases}$$

when  $\operatorname{Re} a_{2k_0-1} > 0, \operatorname{Re} a_{2k_0} < 0$ .

$$g_{k_0}^*(t) = \begin{cases} \frac{a_{2k_0-1} a_{2k_0}}{a_{2k_0} - a_{2k_0-1}} [e^{a_{2k_0-1} t} - e^{a_{2k_0} t}] & t < 0 \\ 0 & t > 0 \end{cases}$$

when  $\operatorname{Re} a_{2k_0-1} > 0, \operatorname{Re} a_{2k_0} > 0, a_{2k_0} \neq a_{2k_0-1}$ .

$$g_{k_0}^*(t) = \begin{cases} -a_{2k_0}^2 t e^{a_{2k_0} t} & t < 0 \\ 0 & t > 0 \end{cases}$$

when  $a_{2k_0} = a_{2k_0-1}, \operatorname{Re} a_{2k_0} > 0$ .

Either  $g_{k_0}^*(t)$  or  $g_{k_0}^*(-t)$  is of the above form.

$$G_{2m+2}^*(t - \operatorname{Re}(a_{2k_0+1}^{-1} + a_{2k_0}^{-1}))$$

satisfies the assumptions of Theorem 4.3 with  $S_{2m+2}^* = S_{2m} - |a_{2k_0-1}|^2 - |a_{2k_0}|^2$  and therefore

$$|G_{2m+2}^*(t)| \leq M(S_{2m+2}^*)^{1/2} \exp(-AS_{2m+2}^{*-1/2} |t + \operatorname{Re}(a_{2k_0+1}^{-1} + a_{2k_0}^{-1})|).$$

Integrating by parts

$$\begin{aligned} G_{2m}'(t) &= \int_{-\infty}^{\infty} g_{k_0}^*(u) \frac{d}{dt} G_{2m+2}^*(t-u) du \\ &= \left\{ \int_{-\infty}^0 + \int_0^{\infty} \right\} \left( \frac{d}{du} g_{k_0}^*(u) \right) G_{2m+2}^*(t-u) du. \end{aligned}$$

Since

$$\theta_1^2 |a_{2k_0-1}|^2 \leq |a_{2k_0}|^2 \quad \text{and} \quad |a_{2k_0}|^2 \leq \theta_2^2 |a_{2k_0-1}|^2$$

we have

$$(\theta_1^{-2} + 1) |a_{2k_0}|^{-2} \geq \frac{1}{2} S_{2m} \quad \text{and} \quad (\theta_2^2 + 1) |a_{2k_0-1}|^{-2} \geq \frac{1}{2} S_{2m};$$

therefore

$$\max(|a_{2k_0}|, |a_{2k_0-1}|) \leq [(2\theta_1^{-2} + 2)^{1/2} + (2\theta_2^2 + 2)^{1/2}] S_{2m}^{-1/2} = R_1 S_{2m}^{-1/2}.$$

By the same method  $(\theta_2^{-2} + 1) |a_{2k_0}|^{-2} \leq S_{2m}$  and  $(\theta_1^2 + 1) |a_{2k_0-1}|^2 \leq S_{2m}$ , from which we deduce

$$\begin{aligned} |\operatorname{Re} a_{2k_0}| &\geq \gamma |a_{2k_0}| \geq \gamma (\theta_2^{-2} + 1)^{-1/2} S_{2m}^{-1/2}, \\ |\operatorname{Re} a_{2k_0-1}| &\geq \gamma (\theta_1^2 + 1)^{-1/2} S_{2m}^{-1/2} \end{aligned}$$

and

$$\min(|\operatorname{Re} a_{2k_0}|, |\operatorname{Re} a_{2k_0-1}|) \geq R_2 S_{2m}^{-1/2} > 0$$

where

$$R_2 = \gamma \cdot \min((\theta_2^{-2} + 1)^{-1/2}, (\theta_1^2 + 1)^{-1/2}).$$

One has to estimate  $G_{2m}(t)$  for different cases of  $g_{k_0}^*(t)$  of which the case where  $\operatorname{Re} a_{2k_0} > 0$ ,  $\operatorname{Re} a_{2k_0-1} > 0$  and  $a_{2k_0} \neq a_{2k_0-1}$  will be done here. The other cases are similar and simpler.

$$\frac{dg^*(u)}{du} = \begin{cases} a_{2k_0-1} a_{2k_0} \frac{(a_{2k_0-1} \exp(a_{2k_0-1} u) - a_{2k_0} \exp(a_{2k_0} u))}{a_{2k_0} - a_{2k_0-1}} & u < 0 \\ 0 & u > 0. \end{cases}$$

Let us recall from [8, p. 203] that if  $f'(t)$  is continuous and  $f(t)$  is complex valued, then

$$\frac{f(a) - f(b)}{a - b} = \lambda f'(t_1) + (1 - \lambda) f'(t_2) \quad t_1, t_2 \in (a, b) \quad 0 < \lambda < 1$$

from which it is obvious that

$$\frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2} = \lambda f'(\zeta_3) + (1 - \lambda)f'(\zeta_4) \quad 0 < \lambda < 1$$

where  $\zeta_i = \alpha_i \zeta_1 + (1 - \alpha_i)\zeta_2$ ,  $0 \leq \alpha_i \leq 1$  and  $i = 3, 4$ . Substituting  $f(\zeta) = \zeta e^\zeta$ ,  $f'(\zeta) = e^\zeta + \zeta u e^\zeta$ , we obtain the following estimate for  $(d/du)g^*(u)$  when  $u < 0$ :

$$\left| \frac{dg^*(u)}{du} \right| \leq |a_{2k_0-1}a_{2k_0}| (\exp(R_{k_0}u) + \max(|a_{2k_0-1}|, |a_{2k_0}|) |u| \exp(R_{k_0}u))$$

where  $R_{k_0} = \min(\operatorname{Re} a_{2k_0-1}, \operatorname{Re} a_{2k_0})$ . Therefore we obtain

$$\begin{aligned} |G'_{2m}(t)| &= |a_{2k_0-1}a_{2k_0}| \int_{-\infty}^0 \exp(R_{k_0}u) \{1 + |u| \max(|a_{2k_0-1}|, |a_{2k_0}|)\} \\ &\quad \cdot S_{2m+2}^{*-1/2} \exp(-AS_{2m+2}^{*-1/2} |t - u + \operatorname{Re}(a_{2k_0-1}^{-1} + a_{2k_0}^{-1})|) du. \end{aligned}$$

Using relations among  $S_{2m+2}^*$ ,  $S_{2m}$ ,  $a_{2k_0}$  and  $a_{2k_0-1}$  one obtains

$$\exp(-AS_{2m+2}^{*-1/2} |t - u + \operatorname{Re}(a_{2k_0-1}^{-1} + a_{2k_0}^{-1})|) \leq M_2 \exp(-AS_{2m+2}^{-1/2} |t - u|).$$

Using this and the definition of  $R_1$  and  $R_2$  one derives

$$\begin{aligned} |G'_{2m}(t)| &\leq M_2 R_1^2 S_{2m}^{-1} \int_{-\infty}^0 \exp(R_2 S_{2m}^{-1/2} u) \{1 + |u| R_1 S_{2m}^{-1/2}\} S_{2m+2}^{*-1/2} \\ &\quad \cdot \exp(-AS_{2m+2}^{*-1/2} |t - u|) du. \end{aligned}$$

We have to distinguish two cases  $t < 0$  and  $t \geq 0$ . Let us prove first the theorem in case  $t < 0$ :

$$\begin{aligned} |G'_{2m}(t)| &\leq M_2 R_1^2 S_{2m}^{-1} \exp(-At S_{2m+2}^{*-1/2}) \int_{-\infty}^t \{1 - u R_1 S_{2m}^{-1/2}\} S_{2m+2}^{*-1/2} \\ &\quad \cdot \exp\{(R_2 S_{2m}^{-1/2} + AS_{2m+2}^{*-1/2})u\} du \\ &\quad + M_2 R_1^2 S_{2m}^{-1} \exp(At S_{2m+2}^{*-1/2}) \int_t^0 \{1 - u R_1 S_{2m}^{-1/2}\} S_{2m+2}^{*-1/2} \\ &\quad \cdot \exp\{(R_2 S_{2m}^{-1/2} - AS_{2m+2}^{*-1/2})u\} du. \end{aligned}$$

Choosing  $K$  so that  $AS_{2m+2}^{*-1/2} > 2R_2 S_{2m}^{-1/2}$  we have

$$|G'_{2m}(t)| \leq M_1 S_{2m}^{-1} \exp(R_2 t S_{2m}^{-1/2}).$$

For  $t > 0$

$$\begin{aligned} |G'_{2m}(t)| &\leq M_2 R_1^2 S_{2m}^{-1} \exp(-At S_{2m+2}^{*-1/2}) \cdot \int_{-\infty}^0 \{1 - u R_1 S_{2m}^{-1/2}\} \cdot S_{2m+2}^{*-1/2} \\ &\quad \cdot \exp\{(R_2 S_{2m}^{-1/2} + AS_{2m+2}^{*-1/2})u\} du \leq M_1 S_{2m}^{-1} \exp(-At S_{2m+2}^{*-1/2}) \\ &\leq M_1 S_{2m}^{-1} \exp(-R_2 S_{2m}^{-1/2} t). \end{aligned}$$

Estimations similar to those achieved in Theorem 4.2 for  $\{a_k\} \in B(2, \delta)$  and  $\{a_k\} \in C(2)$  are developed in the following theorems.

**THEOREM 4.6.** *Let  $\{a_k\} \in B(2, \delta)$  and  $Q_m^{(j)} \geq L(j)Q_m$  for some  $j$ , then there exist  $A > 0$  and  $M > 0$  (independent of  $m$ ) so that for  $k \leq 2j$ :*

$$(4.14) \quad |G_{2m}^{(k)}(t)| \leq MQ_m^{-k/4} \exp \left( -A \left( 2 \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} |t| \right).$$

**THEOREM 4.7.** *Let  $\{a_k\} \in C(2)$  and  $Q_m^{(j)} \geq L(j)Q_m$  for some  $j$ , then there exist  $A > 0$  and  $M > 0$  (independent of  $m$ ) so that for  $k \leq 2j$ :*

$$(4.15) \quad G_{2m}^{(k)}(t) \leq MQ_m^{-k/4} \exp (-AS_{2m}^{-1/2} |t|).$$

One can note that in case  $k = 0$  no condition of the form  $Q_m^{(j)} \geq L(j)Q_m$  is needed.

*Proof of Theorems 4.6 and 4.7.* Using Theorems 3.5 and 3.6 (for Theorems 4.6 and 4.7 respectively) we obtain by Theorem 4.1

$$|G_{2m}^{(k)}(t)| \leq \left| e^{-\sigma t} \int_{\sigma+iy}^{\sigma-i\infty} \frac{(\sigma+iy)^k e^{-iyt}}{E_{2m}(\sigma+iy)} dy \right|, \quad \beta(2m) < \sigma < \alpha(2m).$$

Using the fact that  $Q_m^{-1/4} < ((1/2)S_{2m})^{-1/2}$ , as  $S_m^2 > Q_m$  (which is achieved by dropping many positive terms) and recalling that

$$S_{2m}^{-1/2} < \left( 2 \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/1+\delta},$$

we obtain

$$Q_m^{-1/4} < \left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/1+\delta}.$$

The completion of the proof is similar to the proof of Theorem 4.2.

**5. Some inversion theorems.** In this section we shall show that inversion formulae can be given for  $\{a_k\} \in A(2)$ ,  $\{a_k\} \in B(2, \delta)$  and  $\{a_k\} \in C(2)$ .

**THEOREM 5.1.** *Suppose: (1)  $G(t)$  and  $E(s)$  are defined by (1.2) and  $\{a_k\} \in A(2)$ .*

$$(2) \quad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$$

$$(3) \quad \text{For some } M \text{ and } K, |\varphi(t)| \leq Ke^{M|t|}, \text{ where } M < \min |\operatorname{Re} a_n|.$$

$$(4) \quad b_{2m} = o(1) \quad m \uparrow \infty.$$

*Then*

$$(5.1) \quad \lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \lim_{m \rightarrow \infty} \exp((b - b_{2m})D) \prod_{k=1}^m \left(1 - \frac{D}{\alpha_{2k-1}}\right) \left(1 - \frac{D}{\alpha_{2k}}\right).$$

$\exp((\operatorname{Re} a_{2k-1}^{-1} + a_{2k}^{-1})D)f(x) = \varphi(x)$  at any point of continuity of  $\varphi(t)$ .

*Proof.* By steps following those of [1; p. 433]

$$|P_{2m}(D)f(x) - \varphi(x)| \leq \sup_{|t| < \delta} |\varphi(x-t) - \varphi(x)| \int_{-\infty}^{\infty} |G_{2m}(t)| dt + M_0 \int_{|t| > \delta} |G_{2m}(t)| e^{M|t|} dt.$$

Using Theorem 4.3, the conditions of which are satisfied by the kernel  $G_{2m}(t + b_m)$ , choosing  $m$  so big that  $|b_m| < \delta/2$  and  $AS_{2m}^{-1/2} > 4M$ , we conclude the proof of the theorem.

**THEOREM 5.2.** *Suppose: Assumptions (1) and (2) of Theorem 5.1 are satisfied*

(3) For  $\alpha(t) = \int_0^t \varphi(u) du$  there exist positive  $M$  and  $K$  so that  $|\alpha(t)| \leq ke^{M|t|}$  where  $M < \min |\operatorname{Re} a_k|$ .

(4)  $b_{2m} = o(S_{2m}^{1/2})$   $m \rightarrow \infty$ .

(5)  $\int_0^h [\varphi(x+y) - \varphi(x)] dy = o(h)$   $h \rightarrow 0$ .

(6) Either  $S_{2m}^{(2)} \geq L(2)S_{2m}$  or  $0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty$  and  $|\operatorname{Re} a_k|/|a_k| > \eta$ .

Then  $\lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \varphi(x)$ .

*Proof.* Integrating by parts and since  $\int_{-\infty}^{\infty} G_{2m}(t) dt = 1$  we obtain

$$|P_{2m}(D)f(x) - \varphi(x)| > \int_{|x-t| \leq \delta} |\beta(t)| |G'_{2m}(x-t)| dt + \int_{|x-t| \geq \delta} |G'_{2m}(x-t)| |\beta(t)| dt,$$

where  $\beta(t) = \int_x^t [\varphi(x+u) - \varphi(x)] du$  and therefore  $\beta(x+t) = o(t)$   $t \rightarrow 0$  and  $|\beta(t)| \leq K_1 e^{M|t|}$ .

To obtain the inversion result for the case  $S_{2m}^{(2)} \geq L(2)S_{2m}$  we use the estimation from Theorem 4.2; while for  $|\operatorname{Re} a_k/a_k| > \eta$ ,  $0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty$  we use Theorem 4.4, both are applicable to  $G_{2m}(t + b_{2m})$ .

**REMARK 1.** In case  $a_{2k-1} = -a_{2k}$  (from some  $k$  onward) we can drop (5) and write instead

$$\int_0^h [\varphi(x \pm y) - \varphi(x \pm 0)] dy = o(h) \quad h \rightarrow 0+$$

(if the numbers  $\varphi(x \pm 0)$  exist) and then if we write  $b_{2m} = 0$  instead of (4) and drop (6), we shall obtain

$$(5.2) \quad \lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+0) + \varphi(x-0)] .$$

The proof is similar if we remember that  $G_{2m}(t) = G_{2m}(-t)$  and therefore  $\int_{-\infty}^0 G_{2m}(t)dt = 1/2$ .

REMARK 2. The condition (3) of Theorem 5.2 seems too strong since for the case where  $a_k$  are real the assumption could be dropped. We hope that at least for some classes of  $\{a_k\}$  Theorem 5.2 could be proved without (3).

THEOREM 5.3. Suppose: (1)  $G(t)$  and  $E(s)$  are defined by (1.2) and  $\{a_k\} \in B(2, \delta)$ .

$$(2) \quad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$$

$$(3) \quad \text{For some } M \text{ and } K \quad |\varphi(t)| \leq Ke^{M|t|} \text{ where } M = \min |\operatorname{Re} a_k| .$$

$$(4) \quad \{(\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta})^{1/(1+\delta)}\}^{\beta} \leq KQ_m^{1/4} \text{ for some } \beta \geq 1.$$

$$(5) \quad \varphi(x) - \varphi(t) = o(|t-x|^{\beta-1}) \quad t \rightarrow x.$$

Then

$$(5.3) \quad \lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \varphi(x) .$$

*Proof.* We have

$$\begin{aligned} |P_{2m}(D)f(x) - \varphi(x)| &= \left| \left\{ \int_{|x-t| \geq \delta} + \int_{|x-t| \leq \delta} \right\} G_{2m}(x-t)[\varphi(t) - \varphi(x)]dx \right| \\ &\leq K_1 \int_{|t-x| \geq \delta} |G_{2m}(x-t)| e^{Mt} dt + \varepsilon \int_{-\infty}^{\infty} |G_{2m}(t)| |t|^{\beta-1} dt \\ &\leq o(1) + \varepsilon K_2 \int_{-\infty}^{\infty} Q_m^{-1/4} |t|^{\beta-1} \exp \left( -A \left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{-1/(1+\delta)} |t| \right) dt \\ &\leq o(1) + \varepsilon K_2 K \int_{-\infty}^{\infty} |u|^{\beta-1} \exp(-Au) du \leq o(1) + \varepsilon K_2 K \end{aligned}$$

$m \rightarrow \infty$  .

THEOREM 5.4. Suppose: (1)  $G(t)$  and  $E(s)$  are defined by (1.2) and  $\{a_k\} \in C(2)$ .

$$(2) \quad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$$

$$(3) \quad \text{For some } K \text{ and } M \quad |\varphi(X)| \leq Ke^{M|t|} \text{ where } M = \min |\operatorname{Re} a_k| .$$

$$(4) \quad S_{2m}^{\beta/2} \leq KQ_m^{1/4} \text{ for some } \beta \geq 1.$$

$$(5) \quad \varphi(x) - \varphi(t) = o(|t-x|^{\beta-1}) \quad t \rightarrow x.$$

Then

$$(5.3) \quad \lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \varphi(x) .$$

*Proof.* Similar to that of Theorem 5.3.

REMARK. When  $\beta$  of condition (4) of Theorem 5.4 and Theorem 5.3 is equal to one, the condition on  $\varphi(t)$  is mere continuity at a point  $t = x$ .

LEMMA 5.5. *If an integer  $r$  exists such that for all  $n$   $|a_{n+r}| > q|a_n|$  for  $q > 1$ , then*

$$\left( \sum_{k=m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/(1+\delta)} \leq KQ_m^{1/4} \quad 0 \leq \delta \leq 1$$

(for  $\delta = 1$  we have  $S_{2m}^{1/2} \leq KQ_m^{1/4}$ ).

*Proof.* Obvious.

COROLLARY 5.5. *If the kernel is defined by  $a_{2k} = 2^{k-1}(1+i)$  and  $a_{2k-1} = -2^{k-1}(1+i)$  then (5.3) is valid at any point of continuity.*

6. Examples, remarks and generalizations. In this section we shall show some examples of convolution transform by giving its related sequence  $\{a_k\}$ . When we say  $G(t) \in A(2), B(2, \delta)$  or  $C(2)$  we mean that there is an order for which  $\{a_k\} \in A(2), B(2, \delta)$  and  $C(2)$  respectively.

EXAMPLE 6.1.  $\{a_k\}$  defined by  $a_{2k-1} = k, a_{2k} = q^k e^{i\theta_k}$  for  $q > 1, 0 < \delta < \pi(1/2), |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta$ .  $\{a_k\} \in A(2)$ . The kernel  $G(t)$  related to  $\{a_k\}$  is not necessarily one of those discussed in [6]; for instance in case  $\theta_k = (2/5)\pi$  the result of Theorem 5.2 can be applied as  $S_{2m}^{(j)} \geq L(j)S_{2m}$  for all  $j$  ( $j = 2$  is needed).

EXAMPLE 6.2.  $G(t)$  defined by  $a_{2k-1} = (2k-1)!, a_{2k} = (2k)! e^{i\theta_k}$  where  $-\pi < \theta_k < \pi, |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta$  for some  $0 < \delta < \pi/2$  where the  $a_k$ 's are arranged in the order of  $|a_k|$ . Theorem 5.2 does not apply here as one can easily verify that  $S_{2m}^{(j)} = o(S_{2m})$   $m \rightarrow \infty$  for all  $j > 0$ . We can apply Theorem 5.1 and get an inversion formula.

EXAMPLE 6.3. Let  $c_k$  be real,  $\sum c_k^{-2} < \infty$  and

$$a_{2k-1} = c_k(\sin \theta_1)^{-1} e^{i\theta_2}, \quad a_{2k} = c_k(\sin \theta_2)^{-1} e^{-i\theta_1}$$

where  $0 < \theta_1, \theta_2 < \pi/2, 0 < \delta_1 < \theta_1 + \theta_2 < \pi/2 - \delta_2$ . (1.5) is easily veri-

fied. (1.6) is valid also since  $\sin^2 \theta_1 + \sin^2 \theta_2 - 4 \sin^2 \theta_1 \sin^2 \theta_2 \geq \eta$  and  $\cos^2 \theta_1 > \sin^2 \theta_2$  and therefore  $\sin^2 \theta_1 + \sin^2 \theta_2 < 1$  implies

$$\left(1 - \frac{\eta}{2}\right)(\sin^2 \theta_1 + \sin^2 \theta_2) - 4 \sin^2 \theta_1 \sin^2 \theta_2 \geq \frac{\eta}{2}.$$

Using  $\sin^2 \theta_1 < \cos^2 \theta_2$  and  $\sin^2 \theta_2 < \cos^2 \theta_1$  we get after some calculations that  $\sin^2(\theta_1 + \theta_2) \sin^2(\theta_1 - \theta_2) < \sin^2 \theta_1 + \sin^2 \theta_2 - 4 \sin^2 \theta_1 \sin^2 \theta_2$  which implies (1.7). It should be noted that the class defined by  $a_{2k-1} = a_{2k}$  and  $\min(|\arg a_{2k}|, |\arg -a_{2k}|) \leq \pi/4 - \delta$  which includes Garder's class of transforms [5] as a very special case, is a special case of this example. Theorem 5.2 can be applied here.

EXAMPLE 6.4. Let  $c_k$  be real,  $\sum c_k^{-2} < \infty$  and  $a_{2k-1} = c_k(\sin \theta_1)^{-1}e^{i\theta_2}$ ,  $a_{2k} = -c_k(\sin \theta_2)^{-1}e^{i\theta_1}$  where either  $0 < \theta_1, \theta_2 < \pi/2, 0 < \delta, < \theta_1 + \theta_2 < \pi/2 - \delta_2$  or  $-(\pi/2) < \theta_1, \theta_2 < 0, -(\pi/2) + \delta_2 < \theta_1 + \theta_2 < \delta_1 < 0$ .

The inequalities used in Example 6.3 for the validity of  $\{a_k\} \in A(2)$  can also be used here. It should be noted that the class of transforms defined by Dauns and Widder [1] is the case  $\theta_1 = \theta_2$  here.

EXAMPLE 6.5. Let  $a_{2n-1} = n^\gamma(1+i)$ ,  $a_{2n} = n^\gamma(1-i)$ ,  $\gamma > 1/2$ . In this case  $\{a_k\} \notin A(2)$  (since (1.6) is not satisfied) but clearly  $\{a_k\} \in C(2)$ . In this case  $\beta$  of Theorem 5.4 is easily computed as  $S_{2m} = (1 + o(1))4\gamma m^{-2\gamma+1}$ ,  $Q_m = (1 + o(1))4\gamma m^{-4\gamma-1}$   $m \rightarrow \infty$  and therefore

$$\left(-\gamma + \frac{1}{2}\right)\beta \leq -\gamma + \frac{1}{4}$$

that is  $\beta \geq 1 + 1/2(2\gamma - 1)$ . From this one can see easily that: (a) When  $\gamma = 1$  it is enough to have at  $t = x$   $\varphi(t) - \varphi(x) = o(|t - x|^{1/2})$  for Theorem 5.4.

(b) When  $\gamma > 3/4$  it is enough to have  $\varphi(t) - \varphi(x) = o(t - x)$   $t \rightarrow x$  or it is enough for  $\varphi(t)$  to have a left and right derivative at  $t = x$ .

EXAMPLE 6.6.  $a_{2n-1} = n^\gamma(1+i)$ ,  $a_{2n} = -n^\gamma(1+i)$ . For  $\gamma > 3/4$   $\{a_k\} \in B(2, 1/3)$ . The following remarks will constitute generalizations of the Theorems of § 5 in various directions.

REMARK 6.1. In Theorem 5.1  $|\varphi(t)| \leq Ke^{M|t|}$  can be replaced by  $\left|\int_0^t \varphi(t)dt\right| \leq Ke^{M|t|}$  if for every  $\delta > 0$  if

$$(6.1) \quad (S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2} \exp(-\delta S_m^{-1/2}) = o(1) \quad m \rightarrow \infty.$$

This result can be achieved by a proper change of Theorem 4.2 that will yield now



$$(6.2) \quad |G'_{2m}(t)| \leq M(S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2} \exp(-AS_{2m}^{-1/2} |t|) .$$

REMARK 6.2. In Theorems 5.3 and 5.4 condition (3) can be replaced by  $\left| \int_0^t \varphi(t) dt \right| \leq Ke^{M|t|}$  if either  $Q_m^{(1)} \geq LQ_m$  or if for all  $\eta > 0$

$$(Q_m^{(1)}Q_m)^{-1/4} \exp\left(-\eta \sum_{m+1}^{\infty} |a_k|^{-1-\delta}\right)^{-1/(1+\delta)} = o(1) \quad m \rightarrow \infty$$

for Theorem 5.3 and  $(Q_m^{(1)}Q_m)^{-1/4} \exp(-\eta S_{2m}^{-1/2}) = o(1)$   $m \rightarrow \infty$  for Theorem 5.4. For the above generalization slight improvements of Theorems 4.6 and 4.7 are needed in case  $Q_m^{(1)} \geq LQ_m$  is not satisfied.

REMARK 6.3. If  $S_{2m}^{-1/2} \leq KQ_m^{1/4}$ , then in Theorem 5.4  $\varphi(t) - \varphi(x) = o(1)$   $t \rightarrow x$  can be replaced by

$$\int_x^{x+h} [\varphi(t) - \varphi(x)] dt = o(h) \quad h \rightarrow 0 .$$

REMARK 6.4. If in Theorem 5.3 (5) is replaced by

$$\varphi(t) - \varphi(x+) = o(|t - x|^{\beta-1}) \quad t \rightarrow x+$$

and

$$\varphi(t) - \varphi(x-) = o(|t - x|^{\beta-1}) \quad t \rightarrow x- ,$$

then

$$\lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+) + \varphi(x-)] .$$

REMARK 6.5. If in Theorem 5.3 we have

$$\left( \sum_{m+1}^{\infty} |a_k|^{-1-\delta} \right)^{1/(1+\delta)} \leq K_1 Q_m ,$$

then  $\varphi(t) - \varphi(x) = o(1)$   $t \rightarrow x$  can be replaced by

$$\int_x^{x+h} [\varphi(x \pm t) - \varphi(x \pm 0)] dt = o(h) \quad h \rightarrow 0+$$

and then

$$\lim_{m \rightarrow \infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+0) + \varphi(x-0)] .$$

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