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NÖRLUND SUMMABILITY OF FOURIER SERIES

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The Nörlund summability was first applied to the theory of Fourier series by E. Hille and J. D. Tamarkin. Many other mathematicians have since worked in this field. Recently T. Singh has proved a nice theorem concerning the Nörlund summability of Fourier series. In Part I, we shall give a generalization.

Absolute Nörlund summability was defined by L. McFadden and he proved a theorem concerning the absolute Nörlund summability of the Fourier series of functions of the Lipschitz class which was generalized by S. N. Lal. We shall give another generalization of McFadden's theorem in Part II.

PART I.

1. Let $\sum_{n=0}^{\infty} a_n$ be a given series and (s_n) be the sequence of its partial sums. Let (p_n) be a sequence of real numbers and $P_n = p_0 + p_1 + \cdots + p_n$. We suppose that $P_n \neq 0$ for all n. The series $\sum a_n$ is called to be summable (N, p_n) to s when $\lim_{n\to\infty} t_n$ exists and is equal to s, where

$$t_n = rac{1}{P_n} \sum_{k=0}^n p_{n-k} s_k = rac{1}{P_n} \sum_{k=0}^n p_k s_{n-k}$$
 .

 t_n is called the nth (N, p_n) mean or nth Nörlund mean.

In the special case in which $p_n = \binom{n+\alpha-1}{n} = A_n^{\alpha-1}(\alpha>0)$, the Nörlund mean reduces to the (C,α) mean. Another special case that $p_n = 1/(n+1)$, is called the Harmonic mean.

The condition for the regularity of summability (N, p_n) , is

$$p_{\scriptscriptstyle n}/P_{\scriptscriptstyle n} o 0 \quad {\rm and} \quad \sum\limits_{\scriptscriptstyle k=0}^{\scriptscriptstyle n} \mid p_{\scriptscriptstyle k} \mid = O(\mid P_{\scriptscriptstyle n} \mid) \qquad \qquad {\rm as} \;\; n o \infty \;\; .$$

If (p_n) is a positive sequence, then the second condition is satisfied. It is also easy to see that, if (p_n) is an increasing sequence, then a (C, 1) summable sequence is summable (N, p_n) .

We shall define p(t) on the interval $(0, \infty)$ such that $p(n) = p_n$ for $n = 0, 1, 2, \cdots$ and that p(t) is continuous on $(0, \infty)$ and is linear in each interval $(k, k+1)(k=0, 1, 2, \cdots)$. We put $P(t) = \int_0^t p(u)du$, then, $P(n) = (1/2)p_0 + p_1 + \cdots + p_{n-1} + (1/2)p_n \cong P_n$ as $n \to \infty$ when $P_n \to \infty$ and $p_n/P_n \to 0$.

T. Singh [4] has proved the following theorems:

Theorem S1. If (p_n) is a positive sequence such that $p_n \downarrow$ and $P_n \to \infty$ and further if

then the Fourier series of f is summable (N, p_n) to f(x) at the point x, where $\varphi_x(u) = f(x + u) + f(x - u) - 2f(x)$.

THEOREM S2. If (p_n) satisfies the conditions of Theorem S1 and

(2)
$$\Psi(t) = \int_0^t |\psi_x(u)| \, du = o(p(1/t)/P(1/t))$$
 as $t \to 0$,

then the conjugate Fourier series of f is summable (N, p_n) to

$$-\frac{1}{\pi}\int_0^\pi \frac{\psi_x(u)}{2\tan u/2}\,du$$

at the point x when the last integral exists, where

$$\psi_x(u) = f(x+u) - f(x-u).$$

We shall prove that we can replace the condition $p_n \downarrow$ by the more general one

(3)
$$\int_{-n}^{n} u \mid p'(u) \mid du = O(P_n) \quad \text{as } n \to \infty.$$

If (p_n) is monotone, then the condition (3) is equivalent to

$$np_n = O(P_n) ,$$

since

$$\int_{1}^{n} u p'(u) du = [u p(u)]_{1}^{n} - \int_{1}^{n} p(u) du = n p(n) - P_{n} + O(1).$$

If (p_n) is decreasing, then (4) is satisfied automatically. In general, condition (3) implies (4).

If the condition (4) is satisfied, then (1) implies

(5)
$$\Phi(t) = o(t) \qquad \text{as } t \to 0.$$

2. Our first theorem is as follows.

THEOREM 1. If (p_n) is a positive sequence such that $P_n \to \infty$, (3) holds, and condition (1) is satisfied, then the Fourier series of f is summable (N, p_n) to f(x) at the point x.

Proof. We write $\varphi_x(u) = \varphi(u)$ and by $t_n(x)$ we denote the *n*th Nörlund mean of the Fourier series of f at the point x, then

$$egin{align} t_n(x) - f(x) &= rac{1}{2\pi P_n} \int_0^\pi rac{arphi(t)}{\sin t/2} \Big(\sum_{k=0}^n p_{n-k} \sin{(k+1/2)t} \Big) dt \ &= rac{1}{2\pi P_n} \int_0^\pi rac{arphi(t)}{2\sin{t/2}} L_n(t) dt \ &= rac{1}{2\pi P_n} \Big(\int_0^{1/n} + \int_{1/n}^\pi \Big) = rac{1}{2\pi} (I+J) \; . \end{split}$$

By (5),

$$egin{align} |I| & \leq rac{A}{P_n} \int_0^{1/n} rac{|arphi(t)|}{t} \Big(\sum_{k=0}^n p_{n-k} \!\cdot\! kt \Big) dt \ & \leq A n \! \int_0^{1/n} |arphi(t)| \, dt = o(1) \;. \end{split}$$

We write

$$egin{align} |L_n(t)| &= \left|\sum_{k=0}^n p_k \sin{(n-k+1/2)t}
ight| \ &\leq \sum_{k=0}^{\lceil 1/t
ceil} p_k + \left|\sum_{k=\lceil 1/t
ceil}^n p_k \sin{(n-k+1/2)t}
ight| \ &= L'_n(t) + L''_n(t) \; , \end{aligned}$$

then we have, by Abel's lemma,

$$|L_n''(t)| \le A \Big\{ rac{p([1/t])}{t} + rac{p(n)}{t} + rac{1}{t} \int_{1/t}^n |p'(u)| du .$$

Therefore

$$egin{align} |J| &= \left| rac{1}{P_n} \!\! \int_{1/n}^\pi \! rac{arphi(t)}{\sin t/2} L_n(t) dt
ight| \ &\leq rac{A}{P_n} \!\! \left\{ \!\! \int_{1/n}^\pi \! rac{|arphi(t)|}{t} L_n'(t) dt + \int_{1/n}^\pi \! rac{|arphi(t)|}{t} L_n''(t) dt
ight\} \ &= rac{A}{P_n} (J_1 + J_2) \;, \end{aligned}$$

where

$$|J_{1}| \leq \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} P(1/t) dt \leq \left[\frac{\varphi(t)}{t} P(1/t) \right]_{1/n}^{\pi}$$

$$+ \int_{1/n}^{\pi} \frac{\varphi(t)}{t^{2}} P(1/t) dt + \int_{1/n}^{\pi} \frac{\varphi(t)}{t} \frac{p(1/t)}{t^{2}} dt$$

$$= o(P_{n}) + o(1) \int_{1/n}^{\pi} \frac{p(1/t)}{t^{2}} dt + o(1) \int_{1/n}^{\pi} \frac{p(1/t)}{t^{2}} dt$$

$$= o(P_{n})$$

and

$$\begin{split} J_{2} & \leq A \left\{ \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \frac{p([1/t])}{t} dt + \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \frac{p(n)}{t} dt \right. \\ & + \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^{2}} dt \int_{1/t}^{n} |p'(u)| du \right\} \\ & \leq A \left\{ \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} P(1/t) dt + p_{n} \left[\frac{\boldsymbol{\vartheta}(t)}{t^{2}} \right]_{1/n}^{\pi} \right\} \\ & + p_{n} \int_{1/n}^{\pi} \frac{\boldsymbol{\vartheta}(t)}{t^{3}} dt + \left[\frac{\boldsymbol{\vartheta}(t)}{t^{2}} \int_{1/t}^{n} |p'(u)| du \right]_{1/n}^{\pi} \\ & + \int_{1/n}^{\pi} \frac{\boldsymbol{\vartheta}(t)}{t^{3}} dt \int_{1/t}^{n} |p'(u)| du + \int_{1/n}^{\pi} \frac{\boldsymbol{\vartheta}(t)}{t^{2}} \frac{|p'(1/t)|}{t^{2}} dt \\ & \leq AJ_{1} + o(np_{n}) + o(np_{n}) + O\left(\int_{1}^{n} |p'(u)| du\right) \\ & + o\left(\int_{1}^{n} u |p'(u)| du\right) + o\left(\int_{1}^{n} u |p'(u)| du\right) \\ & = o(P_{n}) \; . \end{split}$$

Thus we get J = o(1) and then we have proved the theorem.

3. Theorem 2. If (p_n) is a positive sequence satisfying the conditions in Theorem 1 and

$$\Psi(t) = \int_0^t |\psi_x(u)| du = o(p(1/t)/P(1/t))$$
 as $t \to 0$,

then the conjugate Fourier series of f is summable (N, p_n) to

$$-\frac{1}{\pi} \lim_{n\to\infty} \int_{1/n}^{\pi} \frac{\psi_x(t)}{2 \tan t/2} dt$$

when the last limit exists.

Proof. Let $\psi_x(t) = \psi(t)$ and $\tilde{t}_n(x)$ be the *n*th (N, p_n) mean of the conjugate Fourier series of f, then

$$egin{aligned} \widetilde{t}_{n}(x) &- \left(-rac{1}{\pi} \int_{1/n}^{\pi} rac{\psi_{x}(t)}{2 an t/2} dt
ight) \ &= rac{-1}{2\pi P_{n}} \int_{0}^{1/n} rac{\psi_{x}(t)}{\sin t/2} \Big(\sum_{k=0}^{n} \Big(\cos rac{1}{2} t - \cos \Big(k + rac{1}{2} \Big) t \Big) \Big) dt \ &+ rac{1}{2\pi P_{n}} \int_{1/n}^{\pi} rac{\psi(t)}{\sin t/2} \Big(\sum_{k=0}^{n} p_{n-k} \cos \Big(k + rac{1}{2} \Big) t \Big) dt \;. \end{aligned}$$

Applying the method of proof of Theorem 1 to above integrals, we obtain the theorem.

PART II.

4. Let (t_n) be the sequence of Nörlund means of the series $\sum a_n$. If $\sum |t_n - t_{n-1}| < \infty$, then the series $\sum a_n$ is called to be summable $|N, p_n|$ or absolutely summable (N, p_n) .

L. McFadden [6] proved the following theorem.

THEOREM M. Let (p_n) be a nonnegative, decreasing and convex sequence tending to zero such that $\sum P_n^2/n^2 < \infty$. If $f \in \text{Lip } \alpha (0 < \alpha < 1)$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1/2} P_n} < \infty ,$$

then the Fourier series of f is $|N, p_n|$ summable.

This was generalized in the following form by S. L. Lal [3]:

THEOREM L. Let (p_n) be a nonnegative, decreasing and convex sequence tending to zero such that $\sum P_n^2/n^2 < \infty$. If the continuity modulus $\omega(t) = \omega(t; f)$ of f satisfies the conditions

$$\sum_{n=1}^{\infty} \frac{\omega(1/n^{\delta})}{n} < \infty \ (0 < \delta < 1)$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/2} P_n} < \infty ,$$

then the Fourier series of f is $|N, p_n|$ summable.

We shall prove the following theorem:

THEOREM 3. Let $2 \ge p > 1$, 1/p + 1/q = 1 and let (p_n) be a positive, decreasing and convex sequence tending to zero. If

$$(9) \qquad \qquad \sum_{n=1}^{\infty} p_n^p n^{p-2} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} P_n} < \infty$$

where $\omega(\delta)$ is the continuity modulus of f, and further if

$$\sum_{m=n}^{\infty} \frac{1}{m^p (\omega(1/m))^{p-1}} \leq \frac{A}{(n\omega(1/n))^{p-1}}$$

(or more specially, if $u^{-\delta}\omega(u) \downarrow$ as $u \uparrow$ for a positive $\delta < 1$), then the Fourier series of f is $|N, p_n|$ summable.

If (p_n) decreases monotonically then (9) is equivalent to

$$\hat{\xi}_1(x) = \sum_{n=1}^{\infty} p_n \cos nx$$
 and $\hat{\xi}_2(x) = \sum_{n=1}^{\infty} p_n \sin nx$

belong to L^p .

From (9) we have

$$egin{align} P_n &= \sum\limits_{k=1}^n p_k = \sum\limits_{k=1}^n p_k k^{1-2/p} \! \cdot \! k^{2/p-1} \ & \leq \left(\sum\limits_{k=1}^n p_k^p k^{p-2}
ight)^{1/p} \! \left(\sum\limits_{k=1}^n k^{(2-p)q/p}
ight)^{1/q} \leq A n^{1/p} \end{split}$$

and then

(11)
$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} \cdot n^{1/p}} \le A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} P_n} < \infty.$$

Therefore, under the condition (9), the condition (10) is stronger than (11). If p decreases from 2, then (9) becomes weaker but (10) becomes stronger.

In the case p = 1, we have the following.

THEOREM 4. Let (p_n) be a decreasing and convex sequence tending to zero, such that

and

$$\sum_{n=k}^{\infty} \frac{1}{nP_n} \leq \frac{A}{P_k}.$$

If f has the continuity modulus $\omega(\delta)$ such that

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_n} < \infty ,$$

then the Fourier series of f is summable $|N, p_n|$.

It is known ([7], Chap. V, § 1) that if (p_n) is a decreasing and convex sequence tending to zero, then $\xi_1(x)$ is integrable and that if (p_n) is decreasing and satisfies the condition (12), then $\xi_2(x)$ is integrable. We have also

$$P_n = \sum\limits_{k=1}^{n} p_k = \sum\limits_{k=1}^{n} (p_k/k) k \leqq An$$
 ,

and then we have $\sum \omega(1/n)/n < \infty$, i.e., (11) holds also in this case.

5. For the proof of Theorem 3, we use the following lemmas.

LEMMA 1. ([5]) If (p_n) is a positive and decreasing sequence, then

$$\left|\sum_{k=a}^{b} p_{k} e^{i(n-k)t}\right| \leq AP(1/t)$$

for any a and b > a and for any integer n.

LEMMA 2. ([6]) If (p_n) is a positive and decreasing sequence and $\hat{\xi}(t) = \sum_{k=0}^{\infty} p_k e^{ikt}$, then

$$|\xi(x+2t)-\xi(x)| \leq A\frac{t}{x}P\left(\frac{1}{t}\right)$$
 for all x in (t,π) .

LEMMA 3. ([2] and [1]) If $\lambda(t)$ is a positive increasing function on $(1, \infty)$ and $f \in L^p$ (1 , then

$$\begin{split} \sum_{n=2}^{\infty} \frac{1}{\lambda(2n)} \Big(\sum_{m=n}^{\infty} \rho_m^q \Big)^{1/q} \\ & \leq A \int_0^1 \frac{dt}{t^2 \lambda(1/t)} \Big(\int_0^{2\pi} |f(x+t) - f(x-t)|^p \, dx \Big)^{1/p} \end{split}$$

where $\rho_m^2 = a_m^2 + b_m^2$, a_m and b_m being the mth Fourier coefficients of f and 1/p + 1/q = 1.

6. We shall now prove Theorem 3. By the definition, we have

$$\begin{split} t_n - t_{n-1} &= \frac{1}{\pi} \int_0^{\pi} \varphi(t) \Big\{ \sum_{k=0}^n \Big(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} \Big) D_k(t) \Big\} dt \\ &= \frac{1}{\pi} \int_0^{\pi} \varphi(t) \Big\{ \sum_{k=1}^n \Big(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \Big) \cos kt \Big\} dt \\ &= \frac{1}{\pi} \int_0^{\pi} \varphi(t) \Big\{ \sum_{k=0}^{n-1} \Big(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \Big) \cos (n-k)t \Big\} dt \\ &= \frac{1}{\pi} \frac{1}{P_n P_{n-1}} \int_0^{\pi} \varphi(t) \Big\{ \sum_{k=0}^{n-1} (p_k P_n - p_n P_k) \cos (n-k)t \Big\} dt \end{split}$$

where $p_{-1} = P_{-1} = 0$, and then

$$(15) \begin{array}{c} \pi \mid t_{n} - t_{n-1} \mid \leq \frac{1}{P_{n-1}} \left| \int_{0}^{\pi} \varphi(t) \left(\sum_{k=0}^{\infty} p_{k} \cos \left(n - k \right) t \right) dt \right| \\ + \frac{1}{P_{n-1}} \left| \int_{0}^{1/n} \varphi(t) \left(\sum_{k=n}^{\infty} p_{k} \cos \left(n - k \right) t \right) dt \right| \\ + \frac{p_{n}}{P_{n} P_{n-1}} \left| \int_{0}^{1/n} \varphi(t) \left(\sum_{k=0}^{n-1} P_{k} \cos \left(n - k \right) t \right) dt \right| \\ + \frac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} \varphi(t) \left(\sum_{k=n}^{\infty} p_{k} \cos \left(n - k \right) t \right) dt \right| \\ + \frac{p_{n}}{P_{n}} \int_{1/n}^{\pi} \varphi(t) \left(\sum_{k=0}^{n-1} P_{k} \cos \left(n - k \right) t \right) dt \right| \\ = I_{n} + J_{n} + K_{n} + L_{n} , \end{array}$$

following McFadden [6]. We shall begin to estimate I_n .

$$egin{aligned} I_n & \leq rac{1}{P_{n-1}} \Big\{ \Big| \int_0^{\pi} \! arphi(t) \Big(\sum_{k=0}^{\infty} p_k \cos kt \Big) \cos nt \ dt \Big| \ & + \Big| \int_0^{\pi} \! arphi(t) \Big(\sum_{k=0}^{\infty} p_k \sin kt \Big) \sin nt \ dt \Big| \Big\} \ & = rac{1}{P_{n-1}} \Big\{ \Big| \int_0^{\pi} \! arphi(t) \hat{\xi}_1(t) \cos nt \ dt \Big| \ & + \Big| \int_0^{\pi} \! arphi(t) \hat{\xi}_2(t) \sin nt \ dt \Big| \Big\} \ & = I_{n,1} + I_{n,2} \; . \end{aligned}$$

Let

then

$$\begin{split} \mathscr{I} &= \sum_{n=2}^{\infty} |I_{n,1}| = \sum_{n=2}^{\infty} \frac{|A_n|}{P_{n-1}} = \sum_{j=1}^{\infty} \sum_{n=2j}^{2^{j+1}-1} \frac{|A_n|}{P_{n-1}} \\ & \leq \sum_{j=1}^{\infty} \left(\sum_{n=2j}^{2^{j+1}-1} |A_n|^q \right)^{1/q} \binom{2^{j+1}-1}{\sum_{n=2j}^{2^{j+1}-1}} \frac{1}{P_{n-1}^p} \right)^{1/p} \\ & \leq \sum_{j=1}^{\infty} E_{2j} \binom{2^{j+1}-1}{\sum_{n=2j}^{2^{j+1}-1}} \frac{1}{P_{n-1}^p} \right)^{1/p} \\ & \leq 4 \sum_{j=1}^{\infty} \sum_{n=2j-1}^{2^{j-1}} \frac{E_n}{n} \binom{4^{n-1}}{\sum_{m=2n}^{2^{j}}} \frac{1}{P_{m-1}^p} \right)^{1/p} \\ & = 4 \sum_{n=1}^{\infty} \frac{E_n}{n} \binom{4^{n-1}}{\sum_{m=2n}^{2^{j}}} \frac{1}{P_{m-1}^p} \right)^{1/p} \leq A \sum_{n=1}^{\infty} \frac{E_n}{n^{1/q} P_n} \;. \end{split}$$

If we put $\lambda(n)=n^{1/q}P_n$, then $\lambda(n)\uparrow \infty$ as n increases. By Lemma 3, we get

$$egin{aligned} \mathscr{I} & \leqq A \!\!\int_0^1 \!\! rac{dt}{t^2 \lambda(1/t)} \! \left(\int_{-\pi}^\pi \left| arDelta_t \!\! \left(\!\! arphi \! \cdot \! \hat{arxeta}_1
ight)
ight|^p dx
ight)^{1/p} \ & = A \!\!\int_0^1 \!\! rac{dt}{t^{1+1/p} P(1/t)} \! \left(\int_{-\pi}^\pi \left| arDelta_t \!\! \left(\!\! arphi \! \cdot \!\! \hat{arxeta}_1
ight)
ight|^p dx
ight)^{1/p} \end{aligned}$$

where

$$egin{aligned} arDelta_t(arphi\cdot \xi_1) &= arphi(x+t) \xi_1(x+t) - arphi(x-t) \xi_1(x-t) \ &= \xi_1(x+t) [arphi(x+t) - arphi(x-t)] + arphi(x-t) [\xi_1(x+t) - \xi_1(x-t)] \ &= \xi_1(x+t) arDelta_t arphi + arphi(x-t) arDelta_t \xi_1 \ , \end{aligned}$$

and then

$$egin{aligned} \mathscr{I} & \leq A \!\!\int_0^1 \!\! rac{dt}{t^{1+1/p} P(1/t)} \!\! \left(\!\! \int_{-\pi}^{\pi} |\, \xi_i(x\,+\,t) arDelta_t arphi \,|^p \, dx
ight)^{\!1/p} \ & + A \!\! \int_0^1 \!\! rac{dt}{t^{1+1/p} P(1/t)} \!\! \left(\!\! \int_{-\pi}^{\pi} |\, arphi(x\,-\,t) arDelta_t \xi_1 \,|^p \, dx
ight)^{\!1/p} \ & = \mathscr{I}' + \mathscr{I}'' \; , \end{aligned}$$

By (9) and (10), we have

$$\mathscr{I}' \leqq A \! \int_0^1 \! rac{\omega(t) dt}{t^{1+1/p} P(1/t)} \leqq A \int_1^\infty \! rac{\omega(1/u)}{u^{1/q} P(u)} du \ \leqq A \sum_{n=1}^\infty rac{\omega(1/n)}{n^{1/q} P_n} \leqq A$$

and, by Lemma 2 and (11),

$$\begin{split} \mathscr{I}'' & \leq A + \int_{0}^{1} \frac{dt}{t^{1+1/p} P(1/t)} \left(\int_{t}^{\pi} \omega(x)^{p} \left| \xi_{1}(x+2t) - \xi_{1}(x) \right|^{p} dx \right)^{1/p} \\ & \leq A + \int_{0}^{1} \frac{dt}{t^{1+1/p} P(1/t)} \left(\int_{t}^{\pi} \omega(x)^{p} \left(\frac{t}{x} P\left(\frac{1}{t} \right) \right)^{p} dx \right)^{1/p} \\ & \leq A + \int_{0}^{1} \frac{dt}{t^{1/p}} \left(\int_{t}^{\pi} \frac{\omega(x)^{p}}{x^{p}} dx \right)^{1/p} \\ & \leq A + \int_{1}^{\infty} \frac{\omega(1/u)^{1/q}}{u^{1/q}} \cdot \frac{1}{u\omega(1/u)^{1/q}} \left(\int_{1}^{u} \frac{\omega(1/v)^{p}}{v^{2-p}} dv \right)^{1/p} du \\ & \leq A + \left(\int_{1}^{\infty} \frac{\omega(1/u)}{u} du \right)^{1/q} \left(\int_{1}^{\infty} \frac{du}{u^{p}\omega(1/u)^{p/q}} \int_{1}^{u} \frac{\omega(1/v)^{p}}{v^{2-p}} dv \right)^{1/p} \\ & \leq A + A \left(\int_{1}^{\infty} \frac{\omega(1/v)}{v^{2-p}} dv \int_{v}^{\infty} \frac{du}{u^{p}\omega(1/u)^{p-1}} \right)^{1/p} \\ & \leq A + A \left(\int_{1}^{\infty} \frac{\omega(1/v)}{v^{2-p}} dv \right)^{1/p} \leq A . \end{split}$$

Thus we have proved that $\mathscr{I} = \sum_{n=2}^{\infty} |I_{n,1}| < \infty$. Similarly we can

prove that $\sum_{n=1}^{\infty} |I_{n,2}| < \infty$, and therefore $\sum_{n=1}^{\infty} I_n < \infty$.

Secondly, we shall estimate the sum $\sum_{n=1}^{\infty} J_n$. Since $\sum_{k=n}^{\infty} p_k \cos(k-n)x$ is a nonnegative function, we have

$$egin{aligned} \sum_{n=1}^\infty J_n & \leq \sum_{n=1}^\infty rac{\omega(1/n)}{P_{n-1}} \!\!\int_0^{1/n} \!\! \left(\sum_{k=n}^\infty p_k \cos{(k-n)t}
ight)\!\!dt \ & \leq \sum_{n=1}^\infty rac{\omega(1/n)}{P_{n-1}} \!\! \left(rac{p_n}{n} + \sum_{k=n+1}^\infty rac{p_n}{k-n} \sin{rac{k-n}{n}}
ight) \ & \leq A + \sum_{n=1}^\infty rac{\omega(1/n)}{P_{n-1}} p_n \leq \sum_{n=1}^\infty rac{\omega(1/n)}{n} + A \leq A \; , \end{aligned}$$

since $\sum_{k=1}^{n} (\sin kx/k)$ is uniformly bounded. Thirdly,

$$\begin{split} \sum_{n=1}^{\infty} K_n & \leq \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{\omega(1/n)}{n} \sum_{k=0}^{n-1} P_k \\ & \leq A \sum_{k=1}^{\infty} P_k \sum_{n=k+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{\omega(1/n)}{n} \\ & \leq A \sum_{k=1}^{\infty} P_k \frac{\omega(1/k)}{k} \sum_{n=k+1}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \\ & \leq A \sum_{k=1}^{\infty} \frac{\omega(1/k)}{k} \leq A \; . \end{split}$$

Finally,

$$egin{aligned} \sum_{k=n}^{\infty} \, p_k \cos{(k-n)}t &+ rac{p_n}{P_n} \sum_{k=0}^{n-1} P_k \cos{(n-k)}t \ &= \left\{ rac{p_n}{2} + \sum_{k=n}^{\infty} \left(p_k - p_{k+1}
ight) rac{\sin{(n-k+1/2)}t}{2\sin{t/2}}
ight\} \ &+ \left\{ \sum_{k=0}^{n-1} \, p_k rac{\sin{(n-k+1/2)}t}{2\sin{t/2}} - rac{1}{2} P_{n-1}
ight\} \end{aligned}$$

and then

$$egin{aligned} L_n & \leq rac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} rac{arphi(t)}{2 \sin t/2} \Big(\sum_{k=n}^{\infty} \left(p_k - p_{k+1}
ight) \sin \left(n - k + 1/2
ight) t
ight| \ & + rac{p_n}{P_n P_{n-1}} \left| \int_{1/n}^{\pi} rac{arphi(t)}{2 \sin t/2} \Big(\sum_{k=0}^{n-1} p_k \sin \left(n - k + 1/2
ight) t
ight) dt
ight| \ & + rac{1}{2} rac{p_n}{P_{n-1}} \Big(1 - rac{P_{n-1}}{P_n} \Big) \! \int_{0}^{\pi} |arphi(t)| \, dt \ & = L'_n + L'''_n + L'''_n \ , \end{aligned}$$

as in McFadden [6]. Now

$$\begin{split} \sum_{n=1}^{\infty} L'_{n} & \leq \sum_{n=1}^{\infty} \frac{p_{n} - p_{n+1}}{P_{n+1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t^{2}} dt \\ & \leq \sum_{n=1}^{\infty} \frac{p_{n} - p_{n+1}}{P_{n+1}} \sum_{k=1}^{n} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^{2}} dt + A \\ & \leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^{2}} dt \sum_{n=k}^{\infty} \frac{p_{n} - p_{n+1}}{P_{n+1}} + A \\ & \leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^{2}} \frac{p(1/t)}{P(1/t)} dt + A \leq A \sum_{k=1}^{\infty} \frac{\omega(1/n)p_{n}}{P_{n}} + A \\ & \leq A \sum_{k=1}^{\infty} \frac{\omega(1/n)}{n} + A \leq A , \\ \sum_{n=1}^{\infty} L''_{n} & \leq \sum_{n=1}^{\infty} \frac{p_{n}}{P_{n}P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t} P(\frac{1}{t}) dt \\ & = \sum_{n=1}^{\infty} \frac{p_{n}}{P_{n}P_{n-1}} \sum_{k=1}^{n} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t} P(\frac{1}{t}) dt + A \\ & \leq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t} P(\frac{1}{t}) dt \sum_{n=k}^{\infty} \frac{p_{n}}{P_{n}P_{n-1}} + A \\ & \leq \int_{0}^{1} \frac{\omega(t)}{t} dt + A \leq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} + A \leq A \end{split}$$

and

$$\sum_{n=1}^{\infty} L_n'' \le A \sum_{n=1}^{\infty} \frac{p_n^2}{P_n P_{n-1}} \le A \sum_{n=1}^{\infty} \frac{1}{n^2} \le A$$
.

Collecting above estimations, we get $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$.

7. We shall now prove Theorem 4. We start from (15). First

$$egin{aligned} I_n &= rac{1}{P_{n-1}} \left| \int_0^\pi arphi(t) \Big(\sum_{k=0}^\infty p_k \cos{(n-k)t} \Big) dt
ight| \ &\leq rac{1}{P_{n-1}} \left\{ \left| \int_0^\pi arphi(t) \hat{\xi}_1(t) \cos{nt} dt
ight| + \left| \int_0^\pi arphi(t) \hat{\xi}_2(t) \cos{nt} dt
ight|
ight\} \ &= I_{n,1} + I_{n,2} \; . \end{aligned}$$

Hence

$$\begin{split} \sum_{n=1}^{\infty} I_{n,1} & \leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} |\varphi(t+\pi/2n)\hat{\xi}_{1}(t+\pi/2n) - \varphi(t-\pi/2n)\hat{\xi}_{1}(t-\pi/2n)| dt \\ & \leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} |\hat{\xi}_{1}(t+\pi/2n)| \cdot |\varphi(t+\pi/2n) - \varphi(t-\pi/2n)| dt \\ & + \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} |\varphi(t-\pi/2n)| \cdot |\hat{\xi}_{1}(t+\pi/2n) - \hat{\xi}_{1}(t-\pi/2n)| dt \\ & = \mathcal{I}_{1} + \mathcal{I}_{2} \; , \end{split}$$

where, since ξ_1 is integrable and $| \varphi(t+\pi/2n) - \varphi(t-\pi/2n) | \leq A\omega(1/n)$,

$$\mathscr{I}_1 \leq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} < \infty$$

by the condition (14) and, since¹⁾

$$\xi_{\scriptscriptstyle 1}\!(x) = \sum\limits_{\scriptscriptstyle
u=0}^{\infty} \left(
u \, + \, 1
ight) arDelta^{\scriptscriptstyle 2} p_{\scriptscriptstyle
u} \! \cdot \! K_{\scriptscriptstyle
u}\!(x)$$
 ,

we get

$$egin{aligned} \mathscr{I}_2 & \leq \sum_{n=1}^\infty rac{1}{P_{n-1}} \int_{\pi/n}^\pi \omega(x) \mid \hat{\xi}_1(x+\pi/n) - \hat{\xi}_1(x) \mid dx + A \ & \leq \sum_{n=1}^\infty rac{1}{P_{n-1}} \sum_{
u=0}^\infty (
u+1) \varDelta^2 p_
u \int_{\pi/n}^\pi \omega(x) \mid K_
u(x+\pi/n) - K_
u(x) \mid dx + A \ & \leq \sum_{n=1}^\infty rac{1}{P_{n-1}} \Big(\sum_{
u=0}^n (
u+1) \varDelta^2 p_
u + \sum_{
u=n+1}^\infty (
u+1) \varDelta^2 p_
u \Big) \ & imes \int_{\pi/n}^\pi \omega(x) \mid K_
u(x+\pi/n) - K_
u(x) \mid dx + A \ & = \mathscr{I}_2' + \mathscr{I}_2'' + A \; . \end{aligned}$$

It is well known that

$$|K'_{\nu}(x)| \leq A\nu^2$$
 and $|K'_{\nu}(x)| \leq A/x^2$

and then

$$\int_0^{\pi/
u} | \, K_
u(x \, + \, \pi/n) \, - \, K_
u(x) \, | \, dx \leqq rac{\pi}{n} A
u^2 \cdot rac{\pi}{
u} \leqq A
u/n \; , \ | \, K_
u(x \, + \, \pi/n) \, - \, K_
u(x) \, | \leqq A/n x^2 \quad ext{in } (\pi/
u, \, \pi) \; .$$

Therefore,

$$\begin{split} \mathscr{J}_{2}' & \leq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=0}^{n} (\nu + 1) \Delta^{2} p_{\nu} \left(\int_{\pi/\nu}^{\pi/\nu} + \int_{\pi/\nu}^{\pi} \right) \omega(x) \mid K_{\nu}(x + \pi/n) - K_{\nu}(x) \mid dx \\ & \leq A \sum_{n=1}^{\infty} \frac{1}{n P_{n-1}} \left\{ \sum_{\nu=0}^{n} (\nu + 1)^{2} \Delta^{2} p_{\nu} \sum_{k=\nu}^{n} \frac{1}{k^{2}} \omega \left(\frac{1}{k} \right) + \sum_{\nu=0}^{n} (\nu + 1) \Delta^{2} p_{\nu} \sum_{k=1}^{\nu} \omega \left(\frac{1}{k} \right) \right\} \\ & \leq A \sum_{n=1}^{\infty} \frac{1}{n P_{n-1}} \left\{ \sum_{k=1}^{n} \frac{1}{k^{2}} \omega \left(\frac{1}{k} \right) \cdot k P_{k} + \sum_{k=1}^{n} \omega \left(\frac{1}{k} \right) (k(p_{k} - p_{k+1}) + p_{k+1}) \right\} \\ & \leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega \left(\frac{1}{k} \right) P_{k} \sum_{n=k}^{\infty} \frac{1}{n P_{n-1}} \\ & + A \sum_{k=1}^{\infty} \omega \left(\frac{1}{k} \right) \left\{ k(p_{k} - p_{k+1}) + p_{k+1} \right\} \sum_{n=k}^{\infty} \frac{1}{n P_{n-1}} \\ & \leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega \left(\frac{1}{k} \right) + A \sum_{k=1}^{\infty} \omega \left(\frac{1}{k} \right) \left\{ \frac{k(p_{k} - p_{k+1})}{p_{k}} + \frac{p_{k+1}}{P_{k}} \right\} \\ & \leq A \sum_{k=1}^{\infty} \frac{1}{k} \omega \left(\frac{1}{k} \right) \end{split}$$

¹⁾ $\Delta^2 p_{\nu} = p_{\nu} - 2p_{\nu+1} + p_{\nu+2}$ and K_{ν} is the ν th Fejér kernel.

by the condition (11) and the relation ([6], p. 183)

$$k(p_k - p_{k+1})/P_k \leq A/k$$
.

Further,

$$egin{aligned} \mathscr{I}_{2}'' & \leq A \sum_{n=1}^{\infty} rac{1}{n P_{n-1}} \sum_{
u = n+1}^{\infty} (
u + 1) \varDelta^{2} p_{
u} \sum_{k=1}^{n} \omega \left(rac{1}{k}
ight) \ & \leq A \sum_{k=1}^{\infty} \omega \left(rac{1}{k}
ight) \sum_{n=k}^{\infty} rac{(n+2)(p_{n+1} - p_{n+2}) + p_{n+2}}{n P_{n-1}} \ & \leq A \sum_{k=1}^{\infty} rac{1}{k} \omega \left(rac{1}{k}
ight) < \infty \;. \end{aligned}$$

Thus we have proved that $\sum I_{n,1} < \infty$. The estimation of $I_{n,2}$ is similar to that of $I_{n,1}$ and thus $\sum I_n < \infty$.

Now, $\sum_{k=n}^{\infty} p_k \cos((k-n)x)$ is a positive function and then

$$egin{aligned} \sum_{n=1}^\infty J_n & \leq \sum_{n=1}^\infty rac{\omega(1/n)}{P_{n-1}} \int_0^{1/n} \Bigl(\sum_{k=n}^\infty p_k \cos{(k-n)t}\Bigr) dt \ & \leq \sum_{n=1}^\infty rac{\omega(1/n)}{P_{n-1}} \ p_n \leq A \sum_{n=1}^\infty rac{\omega(1/n)}{n} < \infty \ . \end{aligned}$$

Convergence of $\sum_{n=1}^{\infty} K_n$ and $\sum_{n=1}^{\infty} L_n$ is proved as in the proof of Theorem 3. Thus we have established Theorem 4.

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Vol. 26, No. 2 December, 1968

Seymour Bachmuth and Horace Yomishi Mochizuki, Kosti				
Engel groups of prime power exponent		197		
Paul Richard Beesack and Krishna M. Das, Extensions of Control	Opial's			
inequality		215		
John H. E. Cohn, Some quartic Diophantine equations		233		
H. P. Dikshit, Absolute $(C, 1) \cdot (N, p_n)$ summability of a I	Fourier series and			
its conjugate series		245		
Raouf Doss, On measures with small transforms		257		
Charles L. Fefferman, L_p spaces over finitely additive med		265		
Le Baron O. Ferguson, <i>Uniform approximation by polynom</i>				
coefficients. II		273		
Takashi Ito and Thomas I. Seidman, Bounded generators of				
spaces		283		
Masako Izumi and Shin-ichi Izumi, <i>Nörlund summability o</i>				
series		289		
Donald Gordon James, On Witt's theorem for unimodular	quadratic			
forms	•	303		
J. L. Kelley and Edwin Spanier, <i>Euler characteristics</i>				
Carl W. Kohls and Lawrence James Lardy, Some ring exter		317		
representations		341		
Ray Mines, III, A family of functors defined on generalized	l primary			
groups		349		
Louise Arakelian Raphael, A characterization of integral of				
space of Borel measurable functions bounded with re				
function		361		
Charles Albert Ryavec, <i>The addition of residue classes mo</i>	dulo n	367		
H. M. (Hari Mohan) Srivastava, <i>Fractional integration and</i>				
formulae associated with the generalized Whittaker to		375		
Edgar Lee Stout, The second Cousin problem with bounde		379		
Donald Curtis Taylor, A generalized Fatou theorem for Ba				
algebras		389		
Bui An Ton, Boundary value problems for elliptic convolu	tion equations of			
Wiener-Hopf type in a bounded region		395		
Philip C. Tonne, Bounded series and Hausdorff matrices for				
convergent sequences		415		