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## **ON WITT'S THEOREM FOR UNIMODULAR QUADRATIC FORMS**

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# ON WITT'S THEOREM FOR UNIMODULAR QUADRATIC FORMS

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**In this paper we give an integral generalization of Witt's theorem for quadratic forms. If  $J$  and  $K$  are sublattices of a unimodular lattice  $L$ , we investigate conditions under which an isometry from  $J$  to  $K$  will extend to an isometry of  $L$ .**

Let  $L$  be a free  $\mathbf{Z}$ -module (that is a lattice) of finite rank and  $\Phi: L \times L \rightarrow \mathbf{Z}$  a unimodular symmetric bilinear form on  $L$ . We denote  $\Phi(\alpha, \beta)$  by  $\alpha \cdot \beta$ , so that  $\alpha \cdot \beta = \beta \cdot \alpha$ . A bijective linear mapping  $\varphi: J \rightarrow K$ , where  $J$  and  $K$  are sublattices of  $L$ , is called an *isometry* if  $\varphi(\alpha) \cdot \varphi(\beta) = \alpha \cdot \beta$  for  $\alpha, \beta \in J$ . Witt's theorem concerns the extension of such an isometry to an isometry of  $L$  (onto  $L$ ). The set of isometries of  $L$  form the *orthogonal group*  $O(L, \mathbf{Z})$  of  $L$ .

Vectors  $\alpha$  and  $\beta$  in  $L$  are called *orthogonal* if  $\alpha \cdot \beta = 0$ ;  $\alpha^2$  denotes  $\alpha \cdot \alpha$ , the *norm* of  $\alpha$ . Any nonzero vector  $\alpha \in L$  may be written as  $\alpha = d\beta$  with  $\beta \in L, d \in \mathbf{Z}$  maximal. If  $d = 1$ ,  $\alpha$  is called *primitive*;  $d$  is the *divisor* of  $\alpha$ . It is clear that an isometry  $\varphi$  of  $L$  must leave invariant the divisors of all vectors; that is,  $\alpha$  and  $\varphi(\alpha)$  have the same divisor.

A sublattice  $U$  of  $L$  is called *primitive* if all the vectors of  $U$  which are "primitive in  $U$ " are also "primitive in  $L$ ". In particular the basis vectors of  $U$  must be primitive (in  $L$ ). In considering the extension of an isometry  $\varphi: J \rightarrow K$  to an isometry of  $L$ , it clearly suffices to consider the case where  $J$  and  $K$  are primitive sublattices.

A primitive vector  $\alpha \in L$  is called *characteristic* if  $\alpha \cdot \beta \equiv \beta^2 \pmod{2}$  for all  $\beta \in L$ . Again it is clear that an isometry must map a characteristic vector into a characteristic vector.

Let  $r(L)$  and  $s(L)$  denote the rank and signature of  $L$ . Then we shall prove the following.

**THEOREM.** *Let  $\varphi: J \rightarrow K$  be an isometry between the primitive sublattices  $J$  and  $K$  of  $L$ , where*

$$(1) \quad r(L) - |s(L)| \geq 2(r(J) + 1).$$

*Then  $\varphi$  extends to an isometry of  $L$  if and only if:*

*$\alpha$  a characteristic vector  $\Leftrightarrow \varphi(\alpha)$  a characteristic vector (for each  $\alpha$  in  $J$ ).*

This result is a generalization of Wall [1]; in fact we shall use

similar arguments and many of the results contained in Wall's paper.

1. Let  $\langle \alpha_1, \alpha_2, \dots, \alpha_m \rangle$  denote the lattice spanned by the vectors  $\alpha_1, \alpha_2, \dots, \alpha_m$ . If  $L$  is the orthogonal direct sum of the sublattices  $U$  and  $V$  we write  $L = U \oplus V$ . In this case we say  $U$  (or  $V$ ) *splits*  $L$ .  $U^\perp$  will denote the orthogonal complement of  $U$ .

We show first how to reduce the proof to the case where  $s(L) = 0$ . Let  $s(L) = s$ . We consider the case  $s > 0$  ( $s < 0$  is similar). Enlarge the lattice  $L$  to

$$L' = L \oplus \langle \zeta_1 \rangle \oplus \dots \oplus \langle \zeta_s \rangle$$

where  $\zeta_i^2 = -1$ ,  $1 \leq i \leq s$ , so that  $s(L') = 0$ . Let

$$J' = J \oplus \langle \zeta_1 \rangle \oplus \dots \oplus \langle \zeta_s \rangle$$

and

$$K' = K \oplus \langle \zeta_1 \rangle \oplus \dots \oplus \langle \zeta_s \rangle.$$

$J'$  and  $K'$  are primitive sublattices of  $L'$ . Furthermore if  $L$  satisfies (1)

$$r(L') - s(L') = r(L) + s \geq 2(r(J') + 1).$$

Also, extending  $\varphi$  to  $J'$  by  $\varphi(\zeta_i) = \zeta_i$ , we see immediately that  $\alpha \in J'$  is characteristic if and only if  $\varphi(\alpha) \in K'$  is characteristic. (Notice that if  $\alpha \in L'$  is characteristic, all the coefficients of the  $\zeta_i$  in  $\alpha$  must be odd.) If, therefore, we establish the theorem when the signature is zero, we know  $\varphi$  extends to an isometry of  $L'$ . Restricting back to  $L$  will establish the general result.

From now on we assume  $s(L) = 0$ . Let  $H$  denote a *hyperbolic plane* of the form  $\langle \lambda, \mu \rangle$  where  $\lambda^2 = \mu^2 = 0$  and  $\lambda \cdot \mu = 1$ ; and let  $I$  denote a sublattice of the form  $\langle \xi, \rho \rangle = \langle \xi \rangle \oplus \langle \xi - \rho \rangle$  where  $\xi^2 = \xi \cdot \rho = 1$  and  $\rho^2 = 0$ . Then it is well known that any unimodular lattice of zero signature is either an orthogonal direct sum of  $H$ 's (if *improper*) or an orthogonal direct sum of  $I$ 's (if *proper*); see Wall [1, Th. 5]. We might also mention that if  $L$  is improper there are no primitive characteristic vectors.

Before proving the theorem we give an example to show the necessity of the restriction (1) we have placed on the ranks of  $L$  and  $J$ .

EXAMPLE. Let

$$L = H_1 \oplus H_2 \oplus \dots \oplus H_n$$

where  $H_i = \langle \lambda_i, \mu_i \rangle$ ,  $1 \leq i \leq n$ . Take

$$J = \langle \lambda_1, \dots, \lambda_{n-1}, \lambda_n + uv\mu_n \rangle$$

and

$$K = \langle \lambda_1, \dots, \lambda_{n-1}, u\lambda_n + v\mu_n \rangle$$

where  $u$  and  $v$  are integers ( $\neq \pm 1$ ) such that  $(u, v) = 1$ . We shall show that the isometry  $\varphi: J \rightarrow K$  defined by

$$(2) \quad \begin{aligned} \varphi(\lambda_i) &= \lambda_i, & 1 \leq i \leq n-1, \\ \varphi(\lambda_n + uv\mu_n) &= u\lambda_n + v\mu_n, \end{aligned}$$

does not extend to an isometry of  $L$ . For if it did, (2) and the conditions  $\lambda_i \cdot \varphi(\mu_n) = \varphi(\lambda_i) \cdot \varphi(\mu_n) = \lambda_i \cdot \mu_n = 0$ ,  $1 \leq i \leq n-1$ , would force

$$\varphi(\mu_n) = x_1\lambda_1 + x_2\lambda_2 + \dots + x_{n-1}\lambda_{n-1} + x\lambda_n + y\mu_n$$

and

$$\begin{aligned} \varphi(\lambda_n) &= -uvx_1\lambda_1 - uvx_2\lambda_2 - \dots - uvx_{n-1}\lambda_{n-1} \\ &\quad + u(1 - vx)\lambda_n + v(1 - uy)\mu_n \end{aligned}$$

for some integers  $x_1, \dots, x_{n-1}, x, y$  as yet undetermined. But  $\varphi(\mu_n)^2 = \mu_n^2 = 0$  implies that  $xy = 0$ ; while  $\varphi(\lambda_n) \cdot \varphi(\mu_n) = 1$  implies  $xv + yu = 1$ . These two conditions are incompatible with our choice  $u, v \neq \pm 1$ . Thus we need, at least,  $r(L) > 2r(J)$ .

We shall now proceed with the proof of the theorem. There will be three stages in the proof.

(i) First we establish the result when  $L$  is improper. In this case there are no characteristic vectors to consider.

(ii) Secondly, we consider  $L$  proper, but with  $J$  and  $K$  containing no characteristic vectors.

(iii) Finally, we treat the general proper case.

NOTATION. The following notation will be used for an isometry. Let

$$L = \langle \alpha_1, \alpha_2, \dots, \alpha_m \rangle \oplus U = \langle \beta_1, \beta_2, \dots, \beta_m \rangle \oplus U$$

where  $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j$ ,  $1 \leq i, j \leq m$ . Then

$$\theta: \langle \alpha_1, \alpha_2, \dots, \alpha_m \rangle \rightarrow \langle \beta_1, \beta_2, \dots, \beta_m \rangle$$

is the isometry of  $L$  defined by  $\theta(\alpha_i) = \beta_i$ ,  $1 \leq i \leq m$ , with  $\theta$  restricted to  $U$  being the identity map.

Many of the isometries will be used repeatedly. We will label them  $\theta_1, \theta_2, \dots$  as they are defined so that we may refer back to them.



Repeating this process, this time in  $\langle \lambda_1, \mu_1 \rangle^\perp$ , we may obtain a new  $\beta$  this time with  $(b_s, \dots, b_h, a, b) = 1$ . Ultimately, we obtain a  $\beta$  with  $(a, b) = 1$ , so that we may finish by using lemma 1 as before.

It now suffices to prove the theorem with  $J = \langle \alpha_1, \dots, \alpha_m \rangle$ . We shall prove the theorem by induction on  $r(J)$ . When  $r(J) = 1$ , the result follows from Wall (our Lemma 1). For the general case we may assume  $K$  has the form  $\langle \alpha_1, \dots, \alpha_{m-1}, \alpha \rangle$ , with  $\varphi: J \rightarrow K$  being the mapping defined by  $\varphi(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-1$ , and

$$(5) \quad \varphi(\alpha_m) = \alpha = \sum_{i=1}^{m-1} (x_i \lambda_i + y_i \mu_i) + u \lambda_m + v \mu_m.$$

(It suffices to consider  $u\lambda_m + v\mu_m$  by Lemma 1). It remains to find an isometry  $\psi \in o(L, \mathbf{Z})$  such that  $\psi(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-1$ , and  $\psi(\alpha_m) = \alpha$ .

We show first that we may take  $u = 1$ . Using Lemma 1, we may assume  $u$  divides  $v$ . Now  $\alpha - \sum_{i=1}^{m-1} x_i \alpha_i$  is primitive (since  $K$  is a primitive lattice), so that

$$(6) \quad (u, z_{m-1}, \dots, z_2, z_1) = 1$$

where

[illegible]

We apply the isometry  $\theta_i$  again, but with  $h$  replaced by  $m - 1$  and  $k(=h + 2)$  by  $m + 1$ . As before  $\theta_i(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m - 1$ , but now

$$\theta_1(\alpha) = \alpha + z_1 \mu_{m+1}.$$

Using Lemma 1 on  $u\lambda_m + v\ell_m + z_1\ell_{m+1}$  in  $\langle \lambda_m, \ell_m \rangle \oplus \langle \lambda_{m+1}, \ell_{m+1} \rangle$ , we may replace  $\alpha$  by a new  $\alpha$  in which  $u$  divides  $z_1$ . By repeating this argument, now in  $\langle \lambda_1, \ell_1 \rangle^\perp$ , we can get a new  $u$  again, this time also dividing  $z_2$ . Eventually, from (6), we may reduce  $u$  to 1.

Finally, we reduce the  $x_1, \dots, x_{m-1}$  in (5), in turn to zero. Apply the isometry

$$\begin{aligned} \theta_2: & \langle \lambda_1, \ell_{l_1} \rangle \oplus \langle \lambda_2, \ell_{l_2} \rangle \oplus \cdots \oplus \langle \lambda_{m-1}, \ell_{l_{m-1}} \rangle \oplus \langle \lambda_m, \ell_{l_m} \rangle \rightarrow \\ & \langle \lambda_1 - x_1 c_1 \ell_{l_m}, \ell_{l_1} + x_1 \ell_{l_m} \rangle \oplus \langle \lambda_2 - x_1 a_{12} \ell_{l_m}, \ell_{l_2} \rangle \\ & \oplus \cdots \oplus \langle \lambda_{m-1} - x_1 a_{1m-1} \ell_{l_m}, \ell_{l_{m-1}} \rangle \\ & \oplus \langle \lambda_m - x_1 \lambda_1 + x_1 c_1 \ell_{l_1} + x_1 a_{12} \ell_{l_2} + \cdots + x_1 a_{1m-1} \ell_{l_{m-1}} + x_1^2 c_1 \ell_{l_m}, \ell_{l_m} \rangle. \end{aligned}$$

Then we have  $\theta_i(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-1$ , and

$$\begin{aligned}\theta_2(\alpha) &= (y_1 + x_1c_1)\mu_1 + x_2\lambda_2 + \cdots + x_{m-1}\lambda_{m-1} \\ &\quad + (y_{m-1} + x_1a_{m-1})\mu_{m-1} + \lambda_m + w\mu_m,\end{aligned}$$

so that the coefficient of  $\lambda_1$  is now zero. By repeating this process all the coefficients of  $\lambda_1, \dots, \lambda_{m-1}$  may be reduced to zero. But then, using the conditions  $\alpha_i \cdot \alpha = \alpha_i \cdot \alpha_m$  for  $1 \leq i \leq m-1$ , and  $\alpha^2 = \alpha_m^2$ , we find that we have succeeded in mapping  $\alpha$  into  $\alpha_m$ , while leaving  $\alpha_i$ ,  $1 \leq i \leq m-1$ , invariant. This completes the proof of the theorem when  $L$  is improper.

3. For the rest of this paper  $L$  will be considered to be a proper lattice with zero signature. Thus we have

$$L = \langle \xi_1, \rho_1 \rangle \oplus \cdots \oplus \langle \xi_n, \rho_n \rangle$$

where  $\xi_i^2 = \xi_i \cdot \rho_i = 1$  and  $\rho_i^2 = 0$  for  $1 \leq i \leq n$ . By (1) we must have  $n \geq 2$ . A primitive vector  $\alpha = \sum_{i=1}^n (a_i \xi_i + b_i \rho_i)$  is characteristic if and only if  $a_i \equiv 0 \pmod{2}$  and  $b_i \equiv 1 \pmod{2}$  for each  $i$ . (We see this by applying the condition  $\alpha \cdot \beta \equiv \beta^2 \pmod{2}$  with  $\beta$  ranging through the basis vectors  $\xi_i, \rho_i$ ).

**LEMMA 2.** *A primitive vector  $\alpha \in L$  may be embedded in a binary sublattice  $B$  which splits  $L$ . If  $\alpha$  is characteristic then  $B$  is proper and  $B^\perp$  is improper. If  $\alpha$  is not characteristic, then  $B$  is proper if  $\alpha^2$  is odd, and  $B$  is improper if  $\alpha^2$  is even.*

*Proof.* From Wall [1, p. 333], if  $\alpha^2 = 2a + 1$  (and hence  $\alpha$  is not characteristic), we can map  $\alpha$  into  $\xi_1 + a\rho_1$ . Thus an isometric image of  $\alpha$  is contained in  $\langle \xi_1, \rho_1 \rangle$ . Apply the inverse isometry to  $L$ . This will embed  $\alpha$  in the inverse image of  $\langle \xi_1, \rho_1 \rangle$ . If  $\alpha$  is not characteristic and  $\alpha^2 = 2a$ , then we may map  $\alpha$  into

$$\beta = (a-1)\rho_1 + \xi_1 + \xi_2.$$

Then  $\beta \cdot \rho_2 = 1$ . Put  $\zeta = \beta - a\rho_2$ , so that  $\zeta^2 = 0$  and  $\zeta \cdot \rho_2 = 1$ . Then  $\beta \in H = \langle \zeta, \rho_2 \rangle$ , a binary sublattice splitting  $L$ . Thus  $\alpha$  may similarly be embedded in an improper binary sublattice which splits  $L$ .

Finally, we consider the case where  $\alpha$  is characteristic with norm  $8b$ . Take a splitting of  $L$  of the form

$$L = \langle \xi \rangle \oplus \langle \eta \rangle \oplus H_2 \oplus \cdots \oplus H_n$$

where  $\xi^2 = -\eta^2 = 1$ . The vector  $\beta = (2b+1)\xi + (2b-1)\eta$  is characteristic with norm  $8b$ . Therefore  $\alpha$  may be mapped by an isometry into  $\beta \in \langle \xi \rangle \oplus \langle \eta \rangle$ , and the result follows as before. This completes the proof of the lemma.

We will now consider the case where  $J$  and  $K$  do not contain characteristic vectors. We obtain an embedding of an isometric image of  $J$  as close as possible to that obtained in § 2. Suppose we have already obtained  $\psi(J) = \langle \alpha_1, \dots, \alpha_h, \beta_1, \dots, \beta_k \rangle$  where  $\alpha_1, \dots, \alpha_h$  are of the form given in (3) and thus embedded in a sublattice

$$L_h = \langle \lambda_1, \mu_1 \rangle \oplus \dots \oplus \langle \lambda_h, \mu_h \rangle$$

which splits  $L$ . Assuming that  $k \geq 3$ , we now show how to obtain  $\alpha_{h+1}$  (and as a special case  $\alpha_1$ , to start the construction).

At least one of the three vectors  $\beta_1, \beta_3, \beta_1 + \beta_3$  must have even norm. We may therefore assume, changing the basis of  $\psi(J)$  if necessary, that  $\beta_1^2$  and  $\beta_2^2$  are even. Write

$$\beta_i = \sigma_i + d_i \tau_i, \quad 1 \leq i \leq k,$$

where  $\tau_i \in L_h^\perp$  is primitive and  $\sigma_i \in L_h$ . It is possible that the  $\tau_i$ , while not characteristic vectors in  $L$ , may be characteristic vectors in  $L_h^\perp$ . However, replacing  $\beta_1$  by a linear combination of  $\beta_1$  and  $\beta_2$  if necessary, we may assume  $\tau_1$  at least is not characteristic in  $L_h^\perp$ . (We may achieve this by eliminating a suitable basis vector  $\rho$  between  $\tau_1$  and  $\tau_2$ ). There are two cases to consider.

*Case 1.*  $\tau_1^2$  even. Then by Lemma 2,  $\tau_1$  may be embedded in an improper binary sublattice  $H_1$  of  $L_h^\perp$ . Since  $k \geq 2$ , we have from (1) that the rank of  $(L_h \oplus H_1)^\perp$  is at least 4. Therefore, there exists another hyperbolic plane  $H_2$  such that

$$L = L_h \oplus H_1 \oplus H_2 \oplus U.$$

But now  $\langle \alpha_1, \dots, \alpha_h, \beta_1 \rangle \subseteq L_h \oplus H_1 \oplus H_2$  and we may transform  $\beta_1$  into the form  $\alpha_{h+1}$  using the results already established for improper lattices in § 2.

*Case 2.*  $\tau_1^2 = 2a + 1$  odd. Then since  $\beta_1^2$  is even,  $d_1^2 \tau_1^2$  is also even. As in the proof of Lemma 2,  $\tau_1$  may be embedded in a sublattice  $I = \langle \xi, \rho \rangle$  with  $\tau_1 = \xi + a\rho$ . Again, from (1), we know the rank of  $(L_h \oplus I)^\perp$  is at least 4, so that we may write  $L$  in the form

$$L = L_h \oplus I \oplus H \oplus U$$

where  $H = \langle \lambda, \mu \rangle$  is a hyperbolic plane. Adding a linear combination of  $\alpha_1, \dots, \alpha_h$  to  $\beta_1$ , we may assume  $\beta_1$  has the form

$$\beta_1 = \sum_{i=1}^h b_i \mu_i + d_1 (\xi + a\rho)$$

where  $(b_1, \dots, b_h, d_1) = 1$ . The next step is to apply isometries to



$L_k \oplus I \oplus H$  that leave  $\alpha_1, \dots, \alpha_k$  invariant, but change  $\beta_1$  into a form as above with  $d_1 = 1$ . As in § 2, we may use  $\theta_1$  on  $L_k \oplus H$  and Lemma 2 to achieve this. Applying  $\theta_1$  on  $L_k \oplus H$ , we transform  $\beta_1$  into  $\beta_1 + b_1\mu$ , so that  $d_1\tau_1$  becomes  $d_1\tau_1 + b_1\mu = d'\tau'$  (say), where  $d' = (d_1, b_1)$ . If now  $\tau'^2$  is even we use case 1. Otherwise, as in Lemma 2, we transform  $\tau'$  into  $\xi + a'\rho$ , and repeat the argument, this time introducing  $b_2\mu$  by working in  $\langle \lambda_1, \mu_1 \rangle^\perp$ . Ultimately, since we may reduce  $d_1$  to 1, we must get a form with  $\tau_1^2$  even, so that we can use Case 1.

In this manner we may apply a succession of isometries to  $J$  until we obtain  $\psi(J) = \langle \alpha_1, \dots, \alpha_{m-2}, \beta, \gamma \rangle$  where  $\alpha_1, \dots, \alpha_{m-2}$  are embedded in an improper sublattice  $L_{m-2}$  of  $L$ . Furthermore, we may assume  $\beta^2$  is even. Write  $\beta = \sigma + d\tau$  where  $\tau \in L_{m-2}^\perp$  is primitive, and  $\sigma \in L_{m-2}$ . By adding a linear combination of  $\alpha_1, \dots, \alpha_{m-2}$  to  $\beta$ , we may assume

$$(8) \quad \beta = \sum_{i=1}^{m-2} b_i \mu_i + d\tau$$

and since  $J$  is primitive, we have  $(b_1, \dots, b_{m-2}, d) = 1$ .  $\tau$  may or may not be a characteristic vector in  $L_{m-2}^\perp$ . We show first how to reduce  $d$  to unity. By Lemma 2  $\tau$  may be embedded in a binary lattice  $B$ . Again by (1), the rank of  $(L_{m-2} \oplus B)^\perp$  is at least 4, so that we may write

$$L = L_{m-2} \oplus B \oplus H \oplus U$$

where  $H = \langle \lambda, \mu \rangle$  is a hyperbolic plane. Using  $\theta_1$  on  $L_{m-2} \oplus H$  and Lemma 2, we reduce  $d$  to 1 as before. Then  $\tau^2$  is even.

If  $\tau$  is not characteristic in  $L_{m-2}^\perp$  we may use the argument of case 1 above to transform  $\beta$  into  $\alpha_{m-1}$ . Suppose therefore  $\tau$  is characteristic in  $L_{m-2}^\perp$ . But we know  $\beta$  is not characteristic in  $L$ . In (8), with  $d = 1$ , it therefore follows that at least one of the coefficients  $b_i$  must be odd. For if they were all even,  $\beta$  would be characteristic in  $L$ . Say  $b_s$  is odd. We apply an isometry of type  $\theta_1$  to

$$\langle \lambda_s, \mu_s \rangle \oplus \langle \lambda_{s+1}, \mu_{s+1} \rangle \oplus \dots \oplus \langle \lambda_{m-2}, \mu_{m-2} \rangle \oplus H.$$

Then  $\theta_1(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-2$ , and  $\theta_1(\beta) = \beta + b_s\mu$ . Then  $\tau$  becomes  $\tau + b_s\lambda$  which is no longer characteristic in  $L_{m-2}^\perp$ . Therefore  $\beta$  may always be transformed into the form  $\alpha_{m-1}$  as before.

It therefore suffices to consider the case  $J = \langle \alpha_1, \dots, \alpha_{m-1}, \gamma \rangle$ . We treat  $K = \varphi(J)$  in a similar manner. Since the norms of the vectors  $\varphi(\alpha_1), \dots, \varphi(\alpha_{m-1})$  are even, and they are not characteristic vectors, they may be embedded in an improper sublattice  $L'_{m-1}$  which splits  $L$ . Adding hyperbolic planes to  $L_{m-1}$  and  $L'_{m-1}$  (they exist

since the rank of  $L_{m-1}^\perp$  is at least 4) and applying our theorem, already established for the improper case, we may assume  $\varphi(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-1$ . Thus it suffices to consider  $K$  of the form  $\langle \alpha_1, \dots, \alpha_{m-1}, \delta \rangle$ . There are now two cases depending on whether  $\gamma^2 = \delta^2$  is odd or even.

*Case 1.*  $\gamma^2 = \delta^2$  odd. Using Lemma 2 and  $\alpha_1, \dots, \alpha_{m-1}$  to eliminate the coefficients of  $\lambda_1, \dots, \lambda_{m-1}$ ,  $\gamma$  may be written as

$$(9) \quad \gamma = \sum_{i=1}^{m-1} u_i \mu_i + d(\xi' + a\rho')$$

where  $(u_1, \dots, u_{m-1}, d) = 1$ .  $L$  may be split thus

$$L = L_{m-1} \oplus \langle \xi', \rho' \rangle \oplus \langle \xi, \rho \rangle \oplus U.$$

We show first how to reduce  $d$  to unity. Apply the isometry

$$\begin{aligned} \theta_3: \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \dots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \oplus \langle \xi, \rho \rangle \rightarrow \\ \langle \lambda_1 - c_1 \rho, \mu_1 + \rho \rangle \oplus \langle \lambda_2 - a_{12} \rho, \mu_2 \rangle \oplus \dots \oplus \langle \lambda_{m-1} - a_{1m-1} \rho, \mu_{m-1} \rangle \\ \oplus \langle \xi - \lambda_1 + c_1 \mu_1 + a_{12} \mu_2 + \dots + a_{1m-1} \mu_{m-1} + c_1 \rho, \rho \rangle. \end{aligned}$$

We may easily check that  $\theta_3(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-1$ . Furthermore  $\theta_3(\gamma) = \gamma + u_1 \rho$ . Mapping  $d(\xi' + a\rho') + u_1 \rho$  back into  $\langle \xi', \rho' \rangle$  we may restore  $\gamma$  to the form (9), but now with  $d$  dividing  $u_1$ . Now repeating this process in  $\langle \lambda_1, \mu_1 \rangle^\perp$ , we may obtain a new  $\gamma$  with  $d$  also dividing  $u_2$ . Since  $(u_1, \dots, u_{m-1}, d) = 1$  we ultimately reach a form with  $d = 1$ .

Using again Lemma 2, we may arrange for  $\delta$  to have the form

$$(10) \quad \delta = \sum_{i=1}^{m-1} (x_i \lambda_i + y_i \mu_i) + f(\xi' + e\rho').$$

We may assume  $\delta - \sum_{i=1}^{m-1} x_i \alpha_i$  is primitive (since  $K$  is primitive) and therefore, using the notation of (7)

$$(f, z_1, z_2, \dots, z_{m-1}) = 1.$$

Applying  $\theta_3$ , we find  $\theta_3(\delta) = \delta + z_1 \rho$ . By the usual chain of arguments we may assume  $f = 1$  in (10).

Finally we apply isometries that reduce  $x_1, \dots, x_{m-1}$  in turn to zero. Define

$$\begin{aligned} \theta_4: \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \dots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \\ \oplus \langle \xi', \rho' \rangle \rightarrow \langle \lambda_1 - c_1 x_1 \rho', \mu_1 + x_1 \rho' \rangle \\ \oplus \langle \lambda_2 - x_1 a_{12} \rho', \mu_2 \rangle \oplus \dots \oplus \langle \lambda_{m-1} - x_1 a_{1m-1} \rho', \mu_{m-1} \rangle \\ \oplus \langle \xi' - x_1 \lambda_1 + x_1 c_1 \mu_1 + x_1 a_{12} \mu_2 + \dots + x_1 a_{1m-1} \mu_{m-1} + x_1^2 c_1 \rho', \rho' \rangle. \end{aligned}$$

Then  $\theta_4(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-1$ , and

$$\begin{aligned}\theta_4(\delta) &= (y_1 + x_1 c_1)\mu_1 + x_2 \lambda_2 + \cdots + x_{m-1} \lambda_{m-1} \\ &\quad + (y_{m-1} + x_1 a_{1m-1})\mu_{m-1} + \xi' + e'\rho' .\end{aligned}$$

We have thus reduced the coefficient of  $\lambda_1$  to zero. Proceeding in this manner we may reduce all the coefficients of  $\lambda_1, \dots, \lambda_{m-1}$  to zero. Using the conditions  $\alpha_i \cdot \gamma = \alpha_i \cdot \delta$  and  $\gamma^2 = \delta^2$ , we find we have mapped  $\delta$  into  $\gamma$ , and hence  $K$  into  $J$ , by an isometry of  $L$ . This completes the proof in this case.

*Case 2.*  $\gamma^2 = \delta^2$  even. Write  $\gamma = \sigma + d\tau$  where  $\tau \in L_{m-1}^\perp$  is primitive and  $\sigma \in L_{m-1}$ . We first show that we may take  $d = 1$ . We use a combination of the previous methods. We may assume  $\gamma$  has the form (compare (8))

$$\gamma = \sum_{i=1}^{m-1} u_i \mu_i + d\tau$$

where  $(u_1, \dots, u_{m-1}, d) = 1$ . If  $\tau$  is characteristic in  $L_{m-1}^\perp$ , we may embed  $\tau$  in a proper binary lattice  $B$  such that

$$L_{m-1}^\perp = B \oplus H_1 \oplus \cdots \oplus H_t .$$

Applying the isometry  $\theta_1$  on  $L_{m-1} \oplus H_1$ , as before, we may assume  $d$  divides  $u_1$ . If  $\tau$  is not characteristic in  $L_{m-1}^\perp$ , we embed  $\tau$  in a binary lattice  $B$  so that  $L$  splits thus

$$L = L_{m-1} \oplus B \oplus \langle \xi, \rho \rangle \oplus U .$$

Applying  $\theta_3$  on  $L_{m-1} \oplus \langle \xi, \rho \rangle$ , as before, we may assume  $d$  divides  $u_1$ . Proceeding in this manner we reduce  $d$  to unity. Then  $\tau^2$  is even and may be embedded in a hyperbolic plane  $H$  (after another isometry if  $\tau$  is characteristic in  $L_{m-1}^\perp$ ), so that, in fact,  $\gamma$  takes the form  $\alpha_m$  given in (3).

By similar reasoning  $\delta$  may be written

$$\delta = \sum_{i=1}^{m-1} (x_i \lambda_i + y_i \mu_i) + d\tau ,$$

$d$  reduced to unity, and  $\tau$  embedded in  $H$ . Finally we reduce the coefficients  $x_1, \dots, x_{m-1}$  to zero by applying  $\theta_2$ , exactly as at the end of § 2.

This completes the proof of the theorem when  $J$  and  $K$  contain no characteristic vectors.

4. It remains for us to consider the case where  $J$  and  $K$  contain characteristic vectors. As in § 3,  $L$  has the form

$$L = \langle \xi_1, \rho_1 \rangle \oplus \cdots \oplus \langle \xi_n, \rho_n \rangle$$

where  $\xi_i^2 = \xi_i \cdot \rho_i = 1$  and  $\rho_i^2 = 0$ ,  $1 \leq i \leq n$ .

We may choose a basis for  $J$  that contains only one characteristic vector; for example, eliminate the coefficients of  $\rho_1$  in all but one of the basis vectors. Applying the results of the previous section, it therefore suffices to consider the special case where

$$J = \langle \alpha_1, \dots, \alpha_{m-2}, \beta, \gamma \rangle \quad \text{and} \quad K = \langle \alpha_1, \dots, \alpha_{m-2}, \beta, \delta \rangle$$

with the  $\alpha_i$  as in (3),  $\delta = \varphi(\gamma)$  is characteristic, and with  $\beta$  either  $\alpha_{m-1}$  (if  $\beta^2$  is even) or of the form given in (9) with  $d = 1$ . There are therefore two cases to consider depending on whether the norm of  $\beta$  is even or odd.

*Case 1.*  $\beta^2$  even; so that  $\beta = \alpha_{m-1}$  and  $J = \langle \alpha_1, \dots, \alpha_{m-1}, \gamma \rangle$ .  $\gamma$  may be assumed to have the form

$$\gamma = \sum_{i=1}^{m-1} u_i \mu_i + d(2\xi + (2e - 1)\rho)$$

(after using the  $\alpha_i$  to eliminate the coefficients of the  $\lambda_i$ , and Lemma 2 to simplify the component of  $\gamma$  in  $L_{m-1}^\perp$ ).  $L$  may now be written

$$L = L_{m-1} \oplus \langle \xi, \rho \rangle \oplus U$$

where  $U$  is an orthogonal sum of hyperbolic planes. By the usual argument we may reduce  $d$  to unity. Similarly, we can transform  $\delta$  into

$$\delta = \sum_{i=1}^{m-1} (x_i \lambda_i + y_i \mu_i) + 2\xi + (2f - 1)\rho.$$

It therefore remains to transform  $\delta$  into a form where the coefficients of  $\lambda_i$  are zero. Since  $\delta$  is characteristic

$$x_i = \delta \cdot \mu_i \equiv \mu_i^2 \equiv 0 \pmod{2}, \quad 1 \leq i \leq m - 1.$$

Now apply the isometry

$$\begin{aligned} \theta_5: \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \dots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \\ \oplus \langle \xi, \rho \rangle \rightarrow \left\langle \lambda_1 - \frac{1}{2} x_1 c_1 \rho, \mu_1 + \frac{1}{2} x_1 \rho \right\rangle \\ \oplus \left\langle \lambda_2 - \frac{1}{2} x_1 a_{12} \rho, \mu_2 \right\rangle \oplus \dots \oplus \left\langle \lambda_{m-1} - \frac{1}{2} x_1 a_{1m-1} \rho, \mu_{m-1} \right\rangle \\ \oplus \left\langle \xi - \frac{1}{2} x_1 \lambda_1 + \frac{1}{2} x_1 c_1 \mu_1 + \frac{1}{2} x_1 a_{12} \mu_2 + \dots \right. \\ \left. + \frac{1}{2} x_1 a_{1m-1} \mu_{m-1} + \frac{1}{4} x_1^2 c_1 \rho, \rho \right\rangle. \end{aligned}$$

Then  $\theta_5(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m - 1$ , and

$$\begin{aligned}\theta_5(\delta) &= (y_1 + x_1 c_1)\mu_1 + x_2 \lambda_2 + \cdots + x_{m-1} \lambda_{m-1} \\ &\quad + (y_{m-1} + x_1 a_{1m-1})\mu_{m-1} + 2\xi + (2f' - 1)\rho.\end{aligned}$$

We have thus reduced the coefficient of  $\lambda_1$  to zero. Proceeding in this manner, we may reduce all the coefficients of the  $\lambda_i$  in turn to zero. Finally, since  $\alpha_i \cdot \gamma = \alpha_i \cdot \delta$  and  $\gamma^2 = \delta^2$ , the coefficients of  $\delta$  now match those in  $\gamma$ , so that we have mapped  $\delta$  into  $\gamma$ , and so  $K$  into  $J$ . This completes the proof in this case.

*Case 2.*  $\beta^2$  odd. Then  $\beta$  may be chosen as

$$\beta = \sum_{i=1}^{m-2} b_i \mu_i + \xi + b\rho.$$

Using  $\alpha_1, \dots, \alpha_{m-2}$  and  $\beta$  to eliminate the coefficients of  $\lambda_1, \dots, \lambda_{m-2}$  and  $\xi$ , we may write  $\gamma$  as

$$\gamma = \sum_{i=1}^{m-2} u_i \mu_i + u\rho + d(2\xi' + (2e - 1)\rho').$$

$L$  is now split into the form

$$L = L_{m-2} \oplus \langle \xi, \rho \rangle \oplus \langle \xi', \rho' \rangle \oplus H \oplus U$$

where  $H = \langle \lambda, \mu \rangle$  and  $U$  is an improper lattice (see Lemma 2). We now reduce the coefficient  $d$  to unity. Isometries on  $L_{m-2} \oplus \langle \xi, \rho \rangle \oplus H$  of the type

$$\begin{aligned}\theta_6: & \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-2}, \mu_{m-2} \rangle \oplus \langle \xi, \rho \rangle \\ & \oplus \langle \lambda, \mu \rangle \rightarrow \langle \lambda_1 - c_1 \mu, \mu_1 + \mu \rangle \oplus \langle \lambda_2 - a_{12} \mu, \mu_2 \rangle \oplus \cdots \\ & \oplus \langle \lambda_{m-2} - a_{1m-2} \mu, \mu_{m-2} \rangle \oplus \langle \xi - b^1 \mu, \rho \rangle \\ & \oplus \langle \lambda - \lambda_1 + c_1 \mu_1 + a_{12} \mu_2 + \cdots + a_{1m-2} \mu_{m-2} + b_1 \rho + c_1 \mu, \mu \rangle\end{aligned}$$

leave  $\alpha_1, \dots, \alpha_{m-2}$  and  $\beta$  invariant.  $\gamma$  is transformed into  $\gamma + u_1 \mu$ , so that with the usual argument we may assume  $d$  divides  $u_1$ . We may transform  $\gamma$  in this manner into a form where  $(u, d) = 1$ .

Since  $\gamma$  is characteristic we know  $\gamma \cdot \xi' \equiv 1 \pmod{2}$ , and hence that  $d$  is odd. Now apply the isometry

$$\begin{aligned}\theta_7: & \langle \xi, \rho \rangle \oplus \langle \lambda, \mu \rangle \rightarrow \langle \xi - 2b\mu, \rho + 2\mu \rangle \\ & \oplus \langle \lambda - 2\xi + 2(1 + b)\rho + 2(2b + 1)\mu, \mu \rangle.\end{aligned}$$

This leaves  $\alpha_1, \dots, \alpha_{m-2}$  and  $\beta$  invariant and transforms  $\gamma$  into  $\gamma + 2u\mu$ . Since  $(2u, d) = 1$ , we may reduce  $d$  to 1 in  $\gamma$ .

As above we may also put  $\delta = \varphi(\gamma)$  into the form

$$\delta = \sum_{i=1}^{m-2} (x_i \lambda_i + y_i \mu_i) + v\xi + w\rho + 2\xi' + (2f - 1)\rho'.$$

Since  $\delta$  is characteristic, we have  $x_i \equiv y_i \equiv 0 \pmod{2}$  for each  $i$ ,  $v \equiv 0 \pmod{2}$  and  $w \equiv 1 \pmod{2}$ .

It now remains to reduce the coefficients  $x_1, \dots, x_{m-2}, v$  to zero. First apply the isometry

$$\begin{aligned} \theta_8: \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \dots \oplus \langle \lambda_{m-2}, \mu_{m-2} \rangle \oplus \langle \xi, \rho \rangle \oplus \langle \xi', \rho' \rangle \rightarrow \\ \langle \lambda_1 - \frac{1}{2}x_1c_1\rho', \mu_1 + \frac{1}{2}x_1\rho' \rangle \oplus \langle \lambda_2 - \frac{1}{2}x_1a_{12}\rho', \mu_2 \rangle \oplus \dots \\ \oplus \langle \lambda_{m-2} - \frac{1}{2}x_1a_{1m-2}\rho', \mu_{m-2} \rangle \oplus \langle \xi - \frac{1}{2}x_1b_1\rho', \rho \rangle \\ \oplus \langle \xi' - \frac{1}{2}x_1\lambda_1 + \frac{1}{2}x_1c_1\mu_1 + \frac{1}{2}x_1a_{12}\mu_2 + \dots \\ + \frac{1}{2}x_1a_{1m-2}\mu_{m-2} + \frac{1}{2}x_1b_1\rho + \frac{1}{4}x_1^2c_1\rho', \rho' \rangle. \end{aligned}$$

Then  $\theta_8(\alpha_i) = \alpha_i$  for  $1 \leq i \leq m-2$ ,  $\theta_8(\beta) = \beta$ , and in  $\theta_8(\delta)$  the coefficient of  $\lambda_1$  is zero. Working now in  $\langle \lambda_1, \mu_1 \rangle^\perp$  we reduce the coefficient of  $\lambda_2$  to zero. We may therefore assume

$$x_1 = x_2 = \dots = x_{m-2} = 0.$$

The final step, the reduction of  $v$  to zero appears to be more difficult. If  $v \equiv 0 \pmod{4}$  we may apply the isometry

$$\begin{aligned} \theta_9: \langle \xi, \rho \rangle \oplus \langle \xi', \rho' \rangle \rightarrow \langle \xi - \frac{1}{2}vb\rho', \rho + \frac{1}{2}v\rho' \rangle \\ \oplus \langle \xi' - \frac{1}{2}v\xi + \frac{1}{2}v(1+b)\rho + t\rho', \rho' \rangle \end{aligned}$$

where  $2t = (1/4)v^2(1+2b)$ . (If  $v \equiv 2 \pmod{4}$  then  $t \notin \mathbb{Z}$ ). Then  $\theta_9$  leaves  $\alpha_1, \dots, \alpha_{m-2}$  and  $\beta$  invariant, while the coefficient of  $\xi$  in  $\theta_9(\delta)$  is reduced to zero. From the various products  $\delta \cdot \alpha_i = \gamma \cdot \alpha_i$ ,  $1 \leq i \leq m-2$ ,  $\delta \cdot \beta = \gamma \cdot \beta$  and  $\delta^2 = \gamma^2$  we see that all the coefficients of  $\delta$  (actually an isometric image of our original  $\delta$ ) now match those of  $\gamma$ . Thus we have mapped  $\delta$  into  $\gamma$  and so  $K$  into  $J$ .

If, however,  $v \equiv 2 \pmod{4}$  we must modify the above argument. We first change the basis of  $L$  so that  $G = \langle \xi', \rho' \rangle \oplus \langle \lambda, \mu \rangle$  becomes  $G = \langle \xi_1, \rho_1 \rangle \oplus \langle \xi_2, \rho_2 \rangle$  where  $\xi_i^2 = \xi_i \cdot \rho_i = 1$  and  $\rho_i^2 = 0$  for  $i = 1, 2$ . Since the characteristic vector  $2\xi' + (2f-1)\rho'$  in  $G$  can be mapped into any other characteristic vector of  $G$  by an isometry, we may assume  $\delta$  has the form

$$\delta = \sum_{i=1}^{m-2} y_i \mu_i + v\xi + w\rho + 2\xi_1 + (2e_1 - 1)\rho_1 + 2\xi_2 + (2e_2 - 1)\rho_2$$

where  $e_1$  is chosen such that

$$2e_1 + w - 1 \equiv 0 \pmod{4},$$

(recall that  $w \equiv 1 \pmod{2}$  since  $\delta$  is characteristic).

We now apply the isometry

$$\begin{aligned} \theta_{10}: \langle \xi, \rho \rangle \oplus \langle \xi_1, \rho_1 \rangle \rightarrow \\ \langle (1-b)\xi + b(1+b)\rho + b\xi_1 + b(b-1)\rho_1, \\ \xi - b\rho - \xi_1 + (1-b)\rho_1 \rangle \\ \oplus \langle -b\xi + b(1+b)\rho + (1+b)\xi_1 + b(b-1)\rho_1, \\ -\xi + (1+b)\rho + \xi_1 + b\rho_1 \rangle. \end{aligned}$$

Again  $\alpha_1, \dots, \alpha_{m-2}$  and  $\beta$  are left invariant by  $\theta_{10}$ . But the coefficient of  $\xi$  is changed from  $v$  to  $v' = v - vb + w - 2b - (2e_1 - 1)$ . But now

$$\begin{aligned} v' &\equiv 2 - 2b + w - 2b - 2e_1 + 1 \\ &\equiv 2e_1 - 1 + w \equiv 0 \pmod{4}. \end{aligned}$$

After restoring  $G$  to the form  $\langle \xi', \rho' \rangle \oplus \langle \lambda, \mu \rangle$  we are in a position to finish the proof by means of the isometry  $\theta_9$  as above.

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