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THE ADDITION OF RESIDUE CLASSES MODULO n

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In the present paper, the following is proved:

THEOREM. Let a_1, \dots, a_m be *m* distinct, nonzero residues modulo *n*, where *n* is any natural number and where

$$m \ge 3\sqrt{6n} \exp\left\{c rac{\sqrt{\log n}}{\log \log n}
ight\},$$

where c > 0 is some large constant. Then the congruence

$$\varepsilon_1 a_1 + \cdots + \varepsilon_m a_m \equiv 0 \pmod{n}$$

is solvable with $\epsilon_i = 0$ or 1 and not all $\epsilon_i = 0$.

The method of proof is completely elementary, in that it is based upon well-known results concerning the addition of residues modulo a natural number n and upon results from elementary number theory.

In a recent paper by Erdös and Heilbronn (see [1]) the following question is investigated. Let p be a prime and a_1, \dots, a_m distinct, nonzero residue classes modulo p, and N any residue class modulo p. Let $F(N) = F(N; p; a_1, \dots, a_m)$ denote the number of solutions of the congruence

(1)
$$\varepsilon_1 a_1 + \cdots + \varepsilon_m a_m \equiv N \pmod{p}$$
,

where the ε_i are restricted to the values 0 or 1. What can be said about the function F(N)? The authors prove the following result:

THEOREM 1. F(N) > 0 if $m \ge 3\sqrt{6p}$.

They conjecture that the bound $3\sqrt{6p}$ in Theorem 1 is not best possible: $3\sqrt{6p}$ can probably be replaced by $2\sqrt{p}$. On the other hand, they show that the constant 2 cannot be replaced by any smaller constant, as shown by the example

$$a_{_1}=1\;, \qquad a_{_2}=\;-1,\;\cdots\;, \qquad a_{_m}=(-1)^{_{m-1}}\!\!\left[rac{m+1}{2}
ight].$$

Note that if $m < 2 \cdot (\sqrt{p} - 2)$, F(1/2(p - 1) = 0.

The question which now arises is what can be said about F(N) if the prime p is replaced by a composite integer n? Theorem 1 is clearly false for composite n. In fact, even the bound $m \ge -1 + n/2$ will not guarantee that F(N) > 0 for all N when n is composite. The difficulty is that all of the a_i may have a prime factor in common with n, in which case N = 1 could not be represented in the form

(1). However, this predicament does not arise when we try to represent 0 in the form (1). Therefore, it is natural to ask what condition on m will guarantee F(0) > 0 for all n. Erdös and Heilbronn conjectured that F(0) > 0 provided $m > 2\sqrt{n}$;¹ and at a conference at Ohio State University Erdös raised the question whether F(0) > 0 could be proved if one assumed the stronger hypothesis $m > K \cdot n^{(1/2)+\varepsilon}$, where ε is any positive number, and K is some absolute constant.

Since the expression $\exp \{c \cdot (\sqrt{\log n})/(\log \log n)\}$ is $O(n^{\epsilon})$ for any $\epsilon > 0$, the theorem of this paper answers Erdös' question.

2. Necessary lemmas. In order to prove the theorem a number of lemmas will be needed. They are rather straightforward modifications of those given in [1] for the case when n is a prime.

LEMMA 1. Let b_1, \dots, b_l be l distinct residues modulo n; and let B(x) denote the number of solutions of

$$x \equiv b_i - b_j \pmod{n}$$

with $1 \leq i, j \leq l$. Then $B(x + y) \geq -l + B(x) + B(y)$; i.e.,

 $l - B(x + y) \leq (l - B(x)) + (l - B(y))$.

Proof. See [1], page 150.

LEMMA 2. Let $1 \leq k \leq l \leq n/2$, $n \geq 2$, and let d_1, \dots, d_k be k distinct nonzero residues modulo n such that $(d_i, n) = 1$. Let b_1, \dots, b_l be l distinct residues modulo n. Then there is an $i, 1 \leq i \leq k$, such that

$$B(d_i) < l - k/6 ,$$

where $B(d_i)$ is the number of solutions of

$$d_i \equiv b_s - b_t \pmod{n} .$$

Proof. Let G denote the cyclic group of residues modulo n, and let $A = \{0, d_1, \dots, d_k\}$. Put $r = 1 + \lfloor (2l/k) \rfloor$. By I. Chowla's theorem on the addition of residues modulo n (see [2], Corollary 1. 2. 4 (p. 3)), one obtains

$$egin{array}{l} |2A| \geq |A| + |A| - 1 = 2k + 1 \ dots \ |rA| \geq rk + 1 \ , \end{array}$$

¹ Relative to this conjecture, we mention an unpublished result of Mann and Olson (see [3]). They have shown that if G is a group of type (p, p) and a_1, \dots, a_m are distinct elements of G, then F(g) > 0 for every $g \in G$ if $m \ge 2p = 2\sqrt{|G|}$.

provided $jA \neq G$ for $1 \leq j \leq r$. Hence, we obtain $t \geq \min(n-1, rk)$ distinct, nonzero residues c_1, \dots, c_t modulo n which can be expressed as sums of not more than r of the d_j ; and the summands need not be distinct

Since $\sum_{1 \leq s \leq t} B(c_s) \leq B(1) + \cdots + B(n-1) = l(l-1)$, there is an s such that

$$egin{aligned} B(c_s) &\leq rac{l(l-1)}{t} \ &\leq l(l-1) \max\left\{rac{1}{n-1},rac{1}{rk}
ight\} \ &\leq rac{l(l-1)}{2l-1} = rac{l}{2} rac{l-1}{l-rac{1}{r}} < rac{l}{2} \ ; \end{aligned}$$

i.e., $l - B(c_s) > l/2$.

By using induction on the conclusion of Lemma 1, we obtain

(2)
$$l - B(x_1 + \cdots + x_t) \leq \sum_{i=1}^t (l - B(x_i))$$
.

By construction, $c_s \equiv \sum_{i=1}^r \varepsilon_i d_{j_i} \pmod{n}$ is solvable with not all $\varepsilon_i = 0$. Rewrite the above expression as $c_s \equiv \sum_{i=1}^{r_1} d_{j_i} \pmod{n}$, where we have suppressed those terms in the sum for which $\varepsilon_i = 0$. Applying (2) we obtain

$$rac{l}{2} < l - B(c_s) \leqq \sum_{i=1}^r \left(l - B(d_{j_i})
ight)$$
 .

Therefore, one obtains a d_i such that

$$l-B(d_i)>rac{l}{2r_1}\geqrac{l}{2r}\geqrac{lk}{2(k+2l)}\geqrac{k}{6}$$
 ,

since $1 \leq r_1 \leq r$.

Now let $1 \leq d_1 < d_2 < \cdots < d_{\nu} \leq n-1$ be ν distinct, nonzero residues modulo n such that $(d_i, n) = 1$. For $1 \leq u \leq \nu/2$, consider all possible subsets, S_u , of u elements from the set $\{d_1, \dots, d_{2u}\}$. For each subset S_u , let $L(S_u)$ denote the number of distinct residue classes modulo nwhich can be obtained in the form $\varepsilon_1 d_1 + \cdots + \varepsilon_{2u} d_{2u}$, where not all $\varepsilon_i = 0$ and where $\varepsilon_i = 0$ or 1 and $\varepsilon_i = 0$ if d_i is not in S_u . Note that determining $L(S_u)$, we do not include the residue class 0 unless it can be expressed as the sum of $\leq u$ distinct elements of S_u .

Finally, put $L(u) = \text{Max}(L(S_u))$, where the maximum is taken over all subsets, S_u , of u elements from the set $\{d_1, \dots, d_{2u}\}$.

LEMMA 3. Let d_1, \dots, d_{ν} satisfy the properties in the above definition. If

$$\varepsilon_1 d_1 + \cdots + \varepsilon_{\nu} d_{\nu} \equiv 0 \pmod{n}$$

implies that all $\varepsilon_i = 0$, then

$$(3) L(u+1) \ge L(u) when u \ge 1$$

(4)
$$L(u) \ge u+2$$
 when $u \ge 3$, for $n \ge 4$.

Proof. (3) is obvious. In order to prove (4), it may be assumed without loss of generality that the maximum, L(u), is obtained from d_1, \dots, d_u , which are distinct modulo n by assumption. Also, $d_1 + \dots + d_u$ is distinct from them by the assumption that

$$\varepsilon_1 d_1 + \cdots + \varepsilon_{\nu} d_{\nu} \equiv 0(n)$$

is impossible unless all $\varepsilon_i = 0$. Now let $T = \{d_1 + d_i \mid 2 \leq i \leq u\}$. Each element of T is distinct from $d_1 + \cdots + d_u$, when $u \geq 3$, and from d_1 . It will be shown that at least one element of T is distinct from all of d_1, \cdots, d_u . This element, in addition to the u + 1 elements $d_1, \cdots, d_u, d_1 + \cdots + d_u$ will give u + 2 distinct residues modulo n, which proves (4), provided $u \geq 3$.

So assume that no element of T is distinct from d_1, \dots, d_u , and let $d_1 + d_i = d_j$, where j is a function of i. It is clear that

$$\{d_j \,|\, 2 \leqq j \leqq u\} = \{d_{\scriptscriptstyle 2},\, \cdots,\, d_{\scriptscriptstyle u}\}\;,$$

since no two d_j are congruent modulo n and none are congruent to d_1 . Consequently,

$$\sum\limits_{i=2}^u \left(d_1 + d_i
ight) \equiv \sum\limits_{j=2}^u d_j \pmod{n}$$
 .

Therefore, $(u-1)d_1 \equiv 0 \pmod{n}$, which is impossible since $(d_1, n) = 1$, and

$$2 \leqq u-1 <
u-1 \leqq n-2$$
 .

LEMMA 4. Let $d_1, \dots, d_u, \dots, d_\nu$ satisfy the same conditions as in Lemma 3. For $3 \leq u \leq -1 + \nu/2$, either L(u) > n/2 or

$$L(u + 1) > L(u) + \frac{u + 2}{6}$$

Proof. If L(u) > n/2 we are finished. So assume that $L(u) \le n/2$. Now let S_u be a set for which $L(u) = L(S_u)$. So we have L(u) distinct residue classes $b_1, \dots, b_{L(u)}$ modulo n which are representable as sums of distinct elements from S_u . We have $\nu - u \ge 1 + \nu/2 \ge u + 2$ other elements d_i which are not in S_u . Select u + 2 of these and, if necessary, relabel them as d_1, \dots, d_{u+2} . Since $1 \le u + 2 \le L(u) \le n/2$, we can apply Lemma 2 to the sets $\{b_1, \dots, b_{L(u)}\}$ and $\{d_1, \dots, d_{u+2}\}$, where k = u + 2, l = L(u). Hence, we obtain an $i, 1 \le i \le u + 2$ for which $B(d_i) < L(u) - (u + 2)/6$, where $B(d_i)$ is the number of representations of d_i in the form

$$d_i \equiv b_j - b_k \pmod{n} .$$

Putting $S_{u+1} = S_u \cup \{d_i\}$, we have

$$L(u+1) \ge L(S_{u+1}) = L(u) + (L(u) - B(d_i)) > L(u) + rac{u+2}{6}$$

LEMMA 5. As before, let $1 \leq d_1 < \cdots < d_{\nu} \leq n-1$ be ν distinct, nonzero residues modulo n such that $(d_i, n) = 1$. Then if $\nu \geq 3\sqrt{6n}$, the congruence

$$\varepsilon_1 d_1 + \cdots + \varepsilon_{\nu} d_{\nu} \equiv 0 \pmod{n}$$

is solvable with not all $\varepsilon_i = 0$.

Proof. Assume that $\varepsilon_1 d_1 + \cdots + \varepsilon_{\nu} d_{\nu} \equiv 0 \pmod{n}$

with $\varepsilon_i = 0$ or 1, implies

that all $\varepsilon_i = 0$. We will then obtain a contradiction. By Lemma 4, either L(u) > n/2 or

$$L(u) > \sum_{\lambda=3}^{u-1} \left(rac{\lambda+2}{6}
ight) + L(3) \geqq rac{u^2+3u+42}{12}$$
 ,

which is larger than n/2 provided $u \ge \sqrt{6n}$. Therefore, with $u \ge \sqrt{6n}$, we have L(u) > n/2 in either case. But we have $\nu \ge 3\sqrt{6n}$ distinct residues. Applying the preceding analysis to the more than $2\sqrt{6n}$ remaining residues, we obtain L(u) > n/2 for this set also.

Therefore, we have two, not necessarily disjoint, sets each with more than n/2 residues modulo n. Call these two sets A, B. By a well-known argument, either A + B = G or

$$|\,G\,| \ge |\,A\,|\,+\,|\,B\,| > n/2\,+\,n/2\,=\,n$$
 .

Therefore, A + B = G; and we conclude that 0 is representable as the sum of distinct elements from $\{d_1, \dots, d_{\nu}\}$. This contradicts our original assumption that 0 is not so represented. Therefore,

$$\varepsilon_1 d_1 + \cdots + \varepsilon_{\nu} d_{\nu} \equiv 0 \pmod{n}$$

is solvable nontrivially.

3. Proof of theorem. For each divisor d of n, let $\Phi(d) = \{a_i \mid d = (a_i, n)\}$. Put $\Phi(d) = \{c_1, \dots, c_k\}$, where h and the c_j depend on d, although this dependence is suppressed without loss of clarity.

For each $c_j \in \mathcal{Q}(d)$, we have $c_j = dc'_j$, where $(n/d, c'_j) = 1$. Furthermore, since the c_j are distinct modulo n, the c'_j are distinct modulo n/d, and they satisfy

$$1 \leq c'_1 < \cdots < c'_h \leq \left[\frac{n-1}{d}\right] = \frac{n}{d} - 1$$
.

Therefore, by Lemma 5, if $h \ge 3\sqrt{6n/d}$, the congruence

$$\varepsilon_1 c'_1 + \cdots + \varepsilon_h c'_h \equiv 0 \pmod{n/d}$$

is solvable nontrivially, in which case the congruence $\varepsilon_1 c_1 + \cdots + \varepsilon_k c_k \equiv 0 \pmod{n}$ is solvable nontrivially.

So if $m = \sum_{d/n} | \varPhi(d) | \ge \sum_{d/n} 3\sqrt{6n/d}$, then for some $d, \varPhi(d)$ will contain more than $3\sqrt{6n/d}$ distinct elements modulo n such that $\{(a_i/d), (n/d)\} = 1$. Thus, the congruence $\varepsilon_1 a_1 + \cdots + \varepsilon_m a_m \equiv 0 \pmod{n}$ will be solvable nontrivially.

We now obtain an upper bound for $\sum_{d/n} 3\sqrt{6n/d}$ in terms of *n*. Suppose $p^{e_p} || n$. Then we have

$$egin{array}{ll} \sum\limits_{d/n} 3\sqrt{6n/d} &= 3\sqrt{6n} \sum\limits_{d/n} d^{-(1/2)} \ &= 3\sqrt{6n} \prod\limits_{p/n} \left(1 + p^{-(1/2)} + \cdots + (p^{e_p})^{-(1/2)}
ight) \ &< 3\sqrt{6n} \prod\limits_{p/n} \left(1 - p^{-(1/2)}
ight)^{-1} \,. \end{array}$$

Put $f(n) = \prod_{p \neq n} (1 - p^{-(1/2)})^{-1}$ and choose the prime q = q(n) such that $\eta = \prod_{p \leq q} p \leq n < q' \prod_{p \leq q} p$, where q' is the smallest prime greater than q. Clearly $f(\eta) \geq f(n)$. Now

$$egin{aligned} \log{(f(\eta))} &= -\sum\limits_{p \leq q} \log{(1 - p^{-(1/2)})} = \sum\limits_{p \leq q}{p^{-(1/2)}} + O(1) \ &= Oigg(\sum\limits_{p \leq q}{p^{-(1/2)}}igg) = Oigg(rac{\sqrt{q}}{\log{q}}igg) \,. \end{aligned}$$

But $\log \eta = \sum_{p \leq q} \log p = \delta(q) \leq \log n$. It is well known that there exist positive constants α and β such that

$$lpha q \leq \delta(q) \leq eta q$$

for all primes q. Hence, we conclude that $\log n \ge \alpha \cdot q$. Also, $\eta' = \eta \cdot q' > n$, which implies that $\log \eta' > \log n$. But $\log \eta' = \delta(q') \le \beta q' = \beta q(q'/q) \le \gamma q$, for some constant $\gamma > 0$. Therefore, $\log q \ge \gamma_1 \cdot \log \log n$;

and so

$$f(n) \leq f(\eta) \leq \exp\left\{c rac{\sqrt{\log n}}{\log \log n}
ight\}$$
,

where c > 0 is some positive constant.

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