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**THE SECOND COUSIN PROBLEM WITH BOUNDED DATA**

EDGAR LEE STOUT

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E. L. STOUT

Given a complex manifold  $M$ , an open covering  $\mathcal{V} \equiv \{V_\alpha\}_{\alpha \in A}$ , and, for each  $\alpha \in A$  a function  $f_\alpha$  holomorphic on  $V_\alpha$  such that for all  $\alpha, \alpha' \in A$ ,  $f_\alpha f_{\alpha'}^{-1}$  is a zero-free holomorphic function on  $V_\alpha \cap V_{\alpha'}$ , the associated second Cousin problem is the problem of showing the existence of a function  $F$  holomorphic on  $M$  such that for all  $\alpha$ ,  $F f_\alpha^{-1}$  is a zero free function holomorphic on  $V_\alpha$ . In the present paper we consider an analogous problem in the case that  $M$  is the open unit polycylinder  $U^N = \{(z_1, \dots, z_N) \in \mathbb{C}^N : |z_1| < 1, \dots, |z_N| < 1\}$ , that the functions  $f_\alpha$  are required to be bounded and that the sought function  $F$  is also required to be bounded.

If  $\Omega$  is an open set in  $\mathbb{C}^N$ , we denote by  $H^\infty[\Omega]$  the algebra of functions bounded and holomorphic in  $\Omega$ . In the first section of the present paper we will establish the following result.

**THEOREM I.1.** *Let  $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$  be an open covering of  $\bar{U}^N$ , the closure of  $U^N$ , and for each  $\alpha$ , let  $f_\alpha \in H^\infty[V_\alpha \cap U^N]$ . If for all  $\alpha, \beta \in A$ ,  $f_\alpha f_\beta^{-1}$  is an invertible element of  $H^\infty[V_\alpha \cap V_\beta \cap U^N]$ , then there exists  $F \in H^\infty[U^N]$  such that for all  $\alpha$ ,  $F f_\alpha^{-1}$  is an invertible element of  $H^\infty[V_\alpha \cap U^N]$ .*

An analogous theorem is valid in the case that we consider functions with continuous boundary values; it is essentially contained in the thesis of Douady. In § II we consider this result briefly and give an application of it to an interpolation theorem.

**I. Proof of theorem I.1.** A remark concerning the statement may be in order. Although the functions  $f_\alpha$  are only defined on subsets of  $U^N$  and although we seek a function  $F$  defined only in  $U^N$ , it seems necessary to require that  $\mathcal{V}$  be an open covering of  $\bar{U}^N$  even if we require  $\mathcal{V}$  to be finite. A relevant example, with  $\mathcal{V}$  an open cover of  $U = U^1$  which consists of two sets is as follows. Let  $\lambda$  be an arc with end points 1 and  $-1$  which is contained, except for its end points, in the lower half of the open unit disc. If  $\lambda$  approaches the real axis with sufficient rapidity, the sequence  $\{1 - 1/n\}_{n=1}^\infty$  will be the zero set of a function  $f_1$  holomorphic and bounded on the set  $V_1 = \{z \in U : z \text{ lies above } \lambda\}$ . We can choose  $f_1$  to be continuous on  $V_1 \cup (\text{interior } \lambda)$  and of modulus 1 on  $\lambda$ . We take for  $V_2$  the part of  $U$  lying on and below  $\lambda$  together with the set  $\{z \in V_1 : |f_1(z)| > 1/2\}$ .

Define  $f_2$  on  $V_2$  to be identically 1. Then  $\{V_1, V_2\}$  is an open cover for  $U$ . On  $V_1 \cap V_2$ ,  $f_1 f_2^{-1}$  is bounded and bounded away from 0. But since the sequence  $\{1 - 1/n\}$  does not satisfy the Blaschke condition, it is not the zero set of any  $f \in H^\infty[U]$ .

The proof of the theorem depends on the following lemma.

**LEMMA I.2.** *Let  $f = \tilde{u} + i\tilde{v}$  be a function holomorphic in  $U^N$ . Let  $\lambda_1$  and  $\lambda_2$  be disjoint arcs in the unit circle, and define  $V_j, j = 1, 2$ , to be the union of  $U$ , the interior of  $\lambda_j$ , and the exterior, including  $\infty$ , of the unit disc. If  $\tilde{u}$  is bounded we may write, for  $z \in U^N$ ,  $f(z) = f_1(z) + f_2(z)$  where  $f_j$  is holomorphic and has bounded real part in  $V_j \times U^{N-1}$ .*

We shall defer the proof of the lemma for the moment and proceed to show how the lemma implies the theorem. It is more convenient to work in the polycube than in the polycylinder. We set

$$\mathcal{A}^+ = \{z \in \mathbb{C}^N : z_j = x_j + iy_j, -\frac{1}{2} < x_1 < 1, |x_2|, \dots, |x_N|, \\ |y_1|, \dots, |y_N| < 1\},$$

and

$$\mathcal{A}^- = \{z \in \mathbb{C}^N : -1 < x_1 < \frac{1}{2}, |x_2|, \dots, |x_N|, |y_1|, \dots, |y_N| < 1\}.$$

Assume given  $F$  an invertible element of  $H^\infty[\mathcal{A}^+ \cap \mathcal{A}^-]$ . We assert that on  $\mathcal{A}^+ \cap \mathcal{A}^-$ ,

$$(I.1.1) \quad F = F^+ F^-$$

where  $F^+$  and  $F^-$  are invertible elements of  $H^\infty[\mathcal{A}^+]$  and  $H^\infty[\mathcal{A}^-]$  respectively.

We may write  $F = e^G$  where  $G$  is holomorphic in  $\mathcal{A}^+ \cap \mathcal{A}^-$  and where  $\operatorname{Re} G$  is bounded; in general  $\operatorname{Im} G$  will not be bounded. The lemma implies that  $G = G_+ + G_-$  where  $G_+$  and  $G_-$  are holomorphic in  $\mathcal{A}^+$  and  $\mathcal{A}^-$  respectively and have bounded real parts: This can be proved in the following way.

Let  $\alpha$  be a conformal map of  $U$  onto  $\{z = x + iy : |x|, |y| < 1\}$  and  $\beta$  one from  $U$  onto  $\{z = x + iy : |x| < 1/2, |y| < 1\}$ . Then the map  $\phi : U^N \rightarrow \mathcal{A}^+ \cap \mathcal{A}^-$  given by  $\phi(z) = (\beta(z_1), \alpha(z_2), \dots, \alpha(z_N))$  is an analytic homeomorphism. Let

$$A_1 = \left\{ \frac{1}{2} + iy : |y| \leq 1 \right\}, \\ A_2 = \left\{ -\frac{1}{2} + iy : |y| \leq 1 \right\},$$

and let  $\lambda_1, \lambda_2$  be respectively the preimages of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  under  $\beta$ . Let  $V_1$  and  $V_2$  be the domains in the Riemann sphere constructed from  $\lambda_1$  and  $\lambda_2$  respectively as in the statement of the lemma. Given a point  $w, |w| > 1$ , we define  $\beta_1(w)$  to be  $1 - [\beta(1/\bar{w})]^-$ . (Here  $[\cdot]^-$  is used to indicate complex conjugation.) This definition effects a continuation of  $\beta$  to a conformal one-to-one mapping of  $V_1$  into  $\mathbb{C}$  in such a way that the range of the continued map,  $\beta_1$ , contains

$$\left\{ z = x + iy : |y| < 1, -\frac{1}{2} < x < 1 \right\}.$$

In the same way  $\beta$  continues to a conformal map  $\beta_2$  of  $V_2$  into  $\mathbb{C}$  such that  $\beta_2[V_2]$  contains

$$\left\{ z = x + iy : |y| < 1, -1 < x < \frac{1}{2} \right\}.$$

Define  $\Phi_j : V_j \times U^{N-1} \rightarrow \mathbb{C}^N$  to be the natural continuation of  $\Phi$  from  $U^N$  into the set  $V_j \times U^{N-1}$ . The range of  $\Phi_1$  contains  $\mathcal{A}^+$ , that of  $\Phi_2, \mathcal{A}^-$ .

Apply the lemma to the function  $G \circ \Phi$  to get  $G \circ \Phi = G_1 + G_2$  where  $G_j$  is holomorphic and has bounded real part in  $V_j \times U^{N-1}$ . But then  $G_j \circ \Phi_j^{-1}$  is holomorphic and has bounded real part in  $\mathcal{A}^+$  if  $j = 1$ , in  $\mathcal{A}^-$  if  $j = 2$ , and we have, since  $\Phi_1^{-1} = \Phi_2^{-1} = \Phi^{-1}$  on  $\mathcal{A}^+ \cap \mathcal{A}^-$ , that in  $\mathcal{A}^+ \cap \mathcal{A}^-$ ,  $G = G_1 \circ \Phi_1^{-1} + G_2 \circ \Phi_2^{-1}$ . This is a decomposition of  $G$  of the desired kind.

For the function  $F_+$ , we take  $\exp(G_1 \circ \Phi_1^{-1})$ , and for  $F_-$  we take  $\exp(G_2 \circ \Phi_2^{-1})$ . This choice of  $F_+$  and  $F_-$  gives the decomposition of  $F$  which we seek.

As soon as we have the decomposition (I.1.1.) for invertible elements of  $H^\infty[\mathcal{A}^+ \cap \mathcal{A}^-]$ , the theorem can be established by a patching argument familiar in this context. (See, e.g. [1].) For the rest of the argument, let us set  $\mathcal{A}_1^+ = \mathcal{A}^+, \mathcal{A}_1^- = \mathcal{A}^-$ . Suppose given  $\mathcal{V} = \{V_\alpha\}$ , an open covering of the closure of the polycube  $\mathcal{A} = \mathcal{A}_1^+ \cup \mathcal{A}_1^-$ , and for each  $\alpha, f_\alpha \in H^\infty[V_\alpha \cap \mathcal{A}]$  such that  $f_\alpha f_\beta^{-1}$  is invertible in  $H^\infty[\mathcal{A} \cap V_\alpha \cap V_\beta]$ . Suppose the theorem is false so that for no  $F \in H^\infty[\mathcal{A}]$  is it the case that for all  $\alpha, F f_\alpha^{-1}$  is invertible in  $H^\infty[V_\alpha \cap \mathcal{A}]$ . Then it cannot be that the induced problems on  $\mathcal{A}_1^+$  and  $\mathcal{A}_1^-$  are both solvable. That is to say, it cannot be the case that there exist  $F_1$  and  $F_2$  in  $H^\infty[\mathcal{A}_1^+]$  and  $H^\infty[\mathcal{A}_1^-]$  respectively such that  $F_j f_\alpha^{-1}$  is, for all  $\alpha$ , an invertible element of  $H^\infty[V_\alpha \cap \mathcal{A}_j^{\pm}]$ .

Suppose that such  $F_1$  and  $F_2$  exist. The function  $F_1 F_2^{-1}$  is then an invertible element of  $H^\infty[\mathcal{A}_1^+ \cap \mathcal{A}_1^-]$ . Since  $\mathcal{V}$  is an open cover for the closed polycube  $\bar{\mathcal{A}}$ , finitely many of the elements of  $\mathcal{V}$ , say  $V_1, \dots, V_p$  cover  $\mathcal{A}_1^+ \cap \mathcal{A}_1^-$ . Thus, there are  $\varepsilon$  and  $M > 0$  such that for

$j = 1, 2, \dots, p$ ,  $|F_1 f_j^{-1}|$  and  $|F_2 f_j^{-1}|$  are at least  $\varepsilon$  but no more than  $M$  on  $\mathcal{A}_1^1 \cap V_j$  and  $\mathcal{A}_1^2 \cap V_j$  respectively. Thus, if  $z \in \mathcal{A}_1^1 \cap \mathcal{A}_1^2$  so that  $z \in V_j$ , say, then

$$|F_1(z)F_2(z)^{-1}| = |(F_1(z)f_j(z))(F_2(z)f_j(z))^{-1}|$$

and this is no more than  $M/\varepsilon$  and no less than  $\varepsilon/M$ . Consequently, the decomposition (I.1.1.) applies to  $F_1 F_2^{-1}$  and provides  $G_1$  and  $G_2$ , invertible elements of  $H^\infty[\mathcal{A}_1^1]$  and  $H^\infty[\mathcal{A}_1^2]$  respectively, such that on  $\mathcal{A}_1^1 \cap \mathcal{A}_1^2$  we have

$$F_1 F_2^{-1} = G_1 G_2$$

whence

$$F_1 G_1^{-1} = F_2 G_2.$$

Thus, we can define  $F \in H^\infty[\mathcal{A}]$  by setting

$$F(z) = \begin{cases} F_1(z)G_1^{-1}(z) & z \in \mathcal{A}_1^1 \\ F_2(z)G_2(z) & z \in \mathcal{A}_1^2. \end{cases}$$

Consider now  $F f_\alpha^{-1}$  on  $V_\alpha$ . On  $V_\alpha \cap \mathcal{A}_1^1$ , this is  $(F F_1^{-1})(F_1 f_\alpha^{-1})$  and so is an invertible element of  $H^\infty[V_\alpha \cap \mathcal{A}_1^1]$  and on  $V_\alpha \cap \mathcal{A}_1^2$  it is  $(F F_2^{-1})(F_2 f_\alpha^{-1})$  and so is invertible in  $H^\infty[V_\alpha \cap \mathcal{A}_1^2]$ . Thus, for all  $\alpha$ ,  $F f_\alpha^{-1}$  is invertible in  $H^\infty[V_\alpha \cap \mathcal{A}]$ . By hypothesis no such  $F$  exists, so we must conclude that either  $F_1$  or  $F_2$  does not exist.

Suppose  $F_1$  does not exist. Set

$$\begin{aligned} \mathcal{A}_2^1 &= \left\{ z \in \mathcal{A}_1^1: -\frac{1}{2} < y_1 < 1 \right\} \\ \mathcal{A}_2^2 &= \left\{ z \in \mathcal{A}_1^1: -1 < y_1 < \frac{1}{2} \right\}. \end{aligned}$$

Arguing as in the last paragraph, there do not exist functions  $F_2^{(1)}$  and  $F_2^{(2)}$  in  $H^\infty[\mathcal{A}_2^1]$  and  $H^\infty[\mathcal{A}_2^2]$  respectively such that for all  $\alpha$ ,  $F_2^{(j)} f_\alpha^{-1}$  is invertible in  $H^\infty[\mathcal{A}_2^j]$ ,  $j = 1, 2$ . Thus, the induced problem is not solvable on both  $\mathcal{A}_2^1$  and  $\mathcal{A}_2^2$ .

Iterating this procedure, proceeding cyclicly through the real coordinates of  $\mathbb{C}^N$  we obtain a nested sequence  $\mathcal{A}_1^1 \supset \mathcal{A}_2^2 \supset \mathcal{A}_3^3 \supset \dots$  of cubes with diameter decreasing to zero on none of which we are able to solve the induced problem. This leads to a contradiction, though, for  $\mathcal{V}$  is an open cover for the closure of  $\mathcal{A}$  so for some  $N$  and some  $\alpha$ ,  $\mathcal{A}_n^{k_n} \subset V_\alpha$  if  $n \geq N$ , and the function  $f_\alpha$  is then a solution to the induced problem on  $\mathcal{A}_n^{k_n}$  for all  $n \geq N$ .

Thus, it remains only to prove the lemma.

*Proof of Lemma 1.2.* It is convenient to establish at the outset certain notations which will be useful throughout the proof. We will

denote by  $\mathbf{Z}^N$  the set of  $N$ -tuples of integers. Let  $E = \{n = (n_1, \dots, n_N) \in \mathbf{Z}^N: n_1 \geq 0, \dots, n_N \geq 0 \text{ or else } n_1 \leq 0, \dots, n_N \leq 0\}$ . Given  $n \in \mathbf{Z}^N$ , let  $E^+(n) = \{m \in \mathbf{Z}^N: n_1 \leq m_1, \dots, n_N \leq m_N\}$ ,  $E^-(n) = \{m \in \mathbf{Z}^N: n_1 \geq m_1, \dots, n_N \geq m_N\}$ . If  $z = (r_1 e^{i\theta_1}, \dots, r_N e^{i\theta_N})$  is in  $U^N$  and  $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathbf{R}^N$ , put

$$P(z, \varphi) = \sum_{n \in \mathbf{Z}^N} r_1^{|n_1|} \dots r_N^{|n_N|} e^{in \cdot (\theta - \varphi)}$$

and

$$K(z, \varphi) = \sum_{n \in E} r_1^{|n_1|} \dots r_N^{|n_N|} e^{in \cdot (\theta - \varphi)}.$$

Here we use  $n \cdot (\theta - \varphi)$  as abbreviation for  $n_1(\theta_1 - \varphi_1) + \dots + n_N(\theta_N - \varphi_N)$ . The kernel  $P$  is the  $N$  dimensional Poisson kernel. A short calculation shows that

$$(I.2.1) \quad K(z, \varphi) = Re \left\{ \frac{2}{(1 - z_1 e^{-i\varphi_1}) \dots (1 - z_N e^{-i\varphi_N})} \right\} - 1.$$

A preliminary reduction of the problem seems desirable. The function  $f$  of the lemma may be written as the sum  $f = f_0 + f_1$  where  $f_0(z_1, \dots, z_N) = f(0, z_2, \dots, z_N)$  and  $f_1 = f - f_0$ . The function  $f_0$  is holomorphic in the product of the Riemann sphere and  $U^{N-1}$ , and it has bounded real part. Thus, it is enough to prove the lemma with  $f$  replaced by  $f_1$ . Let  $f_1 = u + iv$ ; the function  $u$  is bounded. For  $0 \leq \rho_1, \dots, \rho_N < 1$ , we may write

$$(I.2.2) \quad u(\rho_1 e^{i\theta_1}, \dots, \rho_N e^{i\theta_N}) = \sum_{k \in E} \hat{u}(k) \rho_1^{k_1} \dots \rho_N^{k_N} e^{ik \cdot \theta}$$

where  $\hat{u}(k)$  denotes the  $k^{th}$  Fourier coefficient of the function  $u$ :

$$\hat{u}(k) = \left( \frac{1}{2\pi} \right)^N \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} u(e^{i\varphi_1}, \dots, e^{i\varphi_N}) e^{-ik \cdot \varphi} d\varphi_1 \dots d\varphi_N.$$

Since  $u = Re f_1$  and since  $f_1(0, z_2, \dots, z_N) = 0$ , it follows that  $\hat{u}(0, k_2, \dots, k_N) = 0$  for all choices of  $k_2, \dots, k_N$ . Thus the summation in (I.2.2) extends over the set  $E^+(1, 0, \dots, 0) \cup E^-(-1, 0, \dots, 0)$ .

Let  $h$  be a real function of class  $\mathcal{C}^\infty$  on the circle which vanishes identically on a neighborhood of  $\lambda_1$  and is identically one on a neighborhood of  $\lambda_2$ . Let  $h$  have Fourier expansion  $h(e^{i\varphi_1}) = \sum_{n=-\infty}^{\infty} c_n e^{in\varphi_1}$ . Note that since  $h$  is real,  $c_{-m} = \bar{c}_m$ . Define  $g_1$  and  $g_2$  by means of

$$g_1(z) = \left( \frac{1}{2\pi} \right)^N \int u(\varphi) h(e^{i\varphi_1}) \left\{ \frac{2}{(1 - z_1 e^{-i\varphi_1}) \dots (1 - z_N e^{-i\varphi_N})} - 1 \right\} d\varphi$$

and

$$g_2(z) = \left( \frac{1}{2\pi} \right)^N \int u(\varphi) \{1 - h(e^{i\varphi_1})\} \left\{ \frac{2}{(1 - z_1 e^{-i\varphi_1}) \dots (1 - z_N e^{-i\varphi_N})} - 1 \right\} d\varphi.$$

Here we have written  $u(\varphi)$  instead of  $u(e^{i\varphi_1}, \dots, e^{i\varphi_N})$  and  $d\varphi$  instead of  $d\varphi_1 \cdots d\varphi_N$ . The integrals extend over  $-\pi \leq \varphi_1, \dots, \varphi_N \leq \pi$ . Since  $h$  is identically zero on  $\lambda_1$ , it follows that the function  $g_1$  is holomorphic in  $V_1 \times U^{N-1}$ . Similarly,  $g_2$  is holomorphic in  $V_2 \times U^{N-1}$ . We shall show that  $g_1$  and  $g_2$  have bounded real parts. We have  $\operatorname{Re} g_1 + \operatorname{Re} g_2 = u$  whence for some purely imaginary constant  $\gamma$ ,  $f_1 = g_1 + (g_2 + \gamma)$  is a decomposition of  $f_1$  of the kind we seek. It is enough to establish the boundedness of  $\operatorname{Re} g_1$ .

Since  $u$  and  $h$  are real, we have

$$\begin{aligned} \operatorname{Re} g_1(z) &= \left( \frac{1}{2\pi} \right)^N \int u(\varphi) h(e^{i\varphi_1}) K(z, \varphi) d\varphi \\ (I.2.3) \quad &= c_0 \left( \frac{1}{2\pi} \right)^N \int u(\varphi) K(z, \varphi) d\varphi + \sum_{m=1}^{\infty} \left( \frac{1}{2\pi} \right)^N \int u(\varphi) [c_m e^{im\varphi_1} \\ &\quad + \bar{c}_m e^{-im\varphi_1}] K(z, \varphi) d\varphi. \end{aligned}$$

Let  $\mu_m$  be the measure on  $T^N$  whose Fourier transform  $\hat{\mu}_m$  is the characteristic function of  $\{n \in \mathbb{Z}^N: -m+1 \leq n_1 \leq -1\}$  and let  $\nu_m$  be such that  $\hat{\nu}_m$  is the characteristic function of  $\{n \in \mathbb{Z}^N: 1 \leq n_1 \leq m-1\}$ ; the measures  $\mu_m$  and  $\nu_m$  are of norm no more than  $C \log(1+m)$  for some absolute constant  $C$ . Define  $A_m$  on  $T^N$  by

$$A_m(\varphi) = c_m e^{im\varphi_1} (u - u * \mu_m)(\varphi) + \bar{c}_m e^{-im\varphi_1} (u - u * \nu_m)(\varphi)$$

where  $u * \mu$  denotes the convolution of the function  $u$  and the measure  $\mu$ . The Fourier series of the function  $u$  is

$$\sum \{\hat{u}(k) e^{ik \cdot \varphi}: k \in E^+(1, 0, \dots, 0) \cup E^-(-1, 0, \dots, 0)\}$$

so that of  $A_m$  is

$$\begin{aligned} &c_m \sum \{\hat{u}(k) e^{i(k \cdot \varphi + m\varphi_1)}: k \in E^+(1, 0, \dots, 0) \cup E^-(-m, 0, \dots, 0)\} \\ &+ \bar{c}_m \sum \{\hat{u}(k) e^{i(k \cdot \varphi - m\varphi_1)}: k \in E^+(m, 0, \dots, 0) \cup E^-(-1, 0, \dots, 0)\}. \end{aligned}$$

Since  $\hat{u} = 0$  off  $E^+(1, 0, \dots, 0) \cup E^-(-1, 0, \dots, 0)$ , it follows from the definition of  $K(z, \varphi)$  and  $P(z, \varphi)$ , that

$$\begin{aligned} &\left( \frac{1}{2\pi} \right)^N \int u(\varphi) [c_m e^{im\varphi_1} + \bar{c}_m e^{-im\varphi_1}] K(z, \varphi) d\varphi \\ &= \left( \frac{1}{2\pi} \right)^N \int A_m(\varphi) P(z, \varphi) d\varphi. \end{aligned}$$

Also, 
$$\left( \frac{1}{2\pi} \right)^N \int u(\varphi) K(z, \varphi) d\varphi = \left( \frac{1}{2\pi} \right)^N \int u(\varphi) P(z, \varphi) d\varphi.$$

Thus the series (I.2.3) is bounded term by term by the series  $2C \|u\|_{\infty} \sum_{m=0}^{\infty} |c_m| (1 + \log(1+m))$  provided  $z \in U^N$ . Since  $c_m = O(m^{-2})$ ,

it follows that  $\operatorname{Re} g_1$  is bounded in  $U^N$ .

We must show that  $\operatorname{Re} g_1(z_1, z_2, \dots, z_N)$  is bounded for  $|z_1| > 1$ ,  $|z_2|, \dots, |z_N| < 1$ .<sup>1)</sup> If  $|z_1| < 1$ , set  $z_1^* = 1/\bar{z}_1$  so  $|z_1^*| > 1$ . From the definition of  $g_1$ , we find, after a short calculation, that

$$\begin{aligned} & g_1(z_1, z_2, \dots, z_N) - g_1(z_1^*, z_2, \dots, z_N) \\ &= \left(\frac{1}{2\pi}\right)^N \int_{-\pi}^{\pi} h(\varphi_1) P(z_1, \varphi_1) d\varphi_1 \int \frac{2u(e^{i\varphi_1}, e^{i\varphi_2}, \dots, e^{i\varphi_N})}{(1 - z_2 e^{-i\varphi_2}) \dots (1 - z_N e^{-i\varphi_N})} d\varphi_2 \dots d\varphi_N. \end{aligned}$$

Here  $P(z_1, \varphi_1)$  is the one dimensional Poisson kernel. The real part of the inner integral is bounded in  $\varphi_1$ , as may be seen from the fact that if we replace in the first part of our argument the function  $h$  by the function identically 1 and then restrict the resulting function of  $z_1, \dots, z_N$  to the hyperplane  $z_1 = 0$ , our argument shows that the resulting function of  $z_2, \dots, z_N$  is uniformly bounded in  $U^{N-1}$ . Thus,

$$\operatorname{Re}\{g_1(z_1, z_2, \dots, z_N) - g_1(z_1^*, z_2, \dots, z_N)\}$$

is bounded, and the lemma is established.

**II. The case of continuous boundary values.** If  $K$  is a set in  $\mathbb{C}^N$ , let  $\mathcal{A}[K]$  denote the algebra of functions continuous on  $\bar{K}$  and holomorphic in its interior. Similarly, if  $E$  is a commutative Banach algebra with identity, let  $\mathcal{A}[K, E]$  be the algebra of all continuous  $E$ -valued functions on  $\bar{K}$  which are analytic in its interior. There are several formally different definitions of analytic  $E$ -valued functions which are in fact equivalent. For a discussion of these see [4]. For the sake of definiteness, let us say that  $F: \Omega \rightarrow E$ ,  $\Omega$  an open set in  $\mathbb{C}^N$ , is analytic if given  $z^0 = (z_1^0, \dots, z_N^0) \in \Omega$ , there is an expansion

$$F(z) = \sum e_{j_1 \dots j_N} (z_1 - z_1^0)^{j_1} \dots (z_N - z_N^0)^{j_N}$$

with coefficient  $e_{j_1 \dots j_N}$  elements of  $E$ , where the summation extends over all  $N$  tuples of nonnegative integers, and where  $\sum ||e_{j_1 \dots j_N}|| \rho^{j_1 + \dots + j_N}$  is convergent for some  $\rho > 0$ . The following theorem obtains.

**THEOREM II.1.** *Let  $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$  be an open covering of the closed unit polycylinder  $\bar{U}^N$  in  $\mathbb{C}^N$ , and for each  $\alpha$  let  $f_\alpha \in \mathcal{A}[V_\alpha, E]$ . If for all  $\alpha, \beta \in A$  there exists  $h_{\alpha\beta}$ , an invertible element of  $\mathcal{A}[V_\alpha \cap V_\beta]$ , such that on  $\bar{V}_\alpha \cap \bar{V}_\beta$   $f_\alpha = f_\beta h_{\alpha\beta}$ , then there exists  $F \in \mathcal{A}[\bar{U}^N, E]$  such that for all  $\alpha \in A$ ,  $F = f_\alpha h_\alpha$ ,  $h_\alpha$  an invertible element of  $\mathcal{A}[V_\alpha, E]$ .*

<sup>1</sup> This argument was suggested by W. Rudin as an alternative to a more complicated argument of the author.



The proof of this result depends on the fact proved by Douady<sup>2)</sup> [2, p. 48] that if  $\Delta$ ,  $\Delta^+$  and  $\Delta^-$  are as in the proof of Theorem I.1, and if  $F$  is an invertible element of  $\mathcal{A}[(\Delta^+ \cap \Delta^-), E]$  then there exist invertible  $F_+ \in \mathcal{A}[\Delta^+, E]$  and  $F_- \in \mathcal{A}[\Delta^-, E]$  such that on  $\Delta^+ \cap \Delta^-$ ,  $F = F_+ F_-$ .

Once one has this fact, it is possible to argue by contradiction and establish Theorem II.1 just as Theorem I.1 was established.

As an application of Theorem II.1, we will show that certain sets in  $T^N = \{z \in C^N: |z_1| = 1, \dots, |z_N| = 1\}$  are zero sets for elements of  $\mathcal{A}[U^N]$ .

**THEOREM II.2.** *Let  $\beta_1, \dots, \beta_N$  be strictly positive real numbers. Let  $A$  be a compact set of Lebesgue measure zero in the real line, and let  $G_A = \{t \in R^N: t \cdot \beta = t_1 \beta_1 + \dots + t_N \beta_N \in A\}$ . If  $E$  is a compact subset of  $\{(e^{it_1}, \dots, e^{it_N}): t \in G_A\}$ , then  $E$  is the zero set of an element of  $\mathcal{A}[U^N]$ .*

*Proof.* Define a map  $\gamma: C^N \rightarrow C^N$  by means of  $\gamma(z_1, \dots, z_N) = (e^{iz_1}, \dots, e^{iz_N})$ . Regarding  $R^N$  as the set of points in  $C^N$  with real coordinates, we have that  $\gamma$  carries  $R^N$  onto  $T^N$ . If  $Q_+^N$  denotes the set  $\{z \in C^N: \text{Im } z_j \geq 0 \text{ for } j = 1, \dots, N\}$ , then  $\gamma$  carries  $Q_+^N$  onto a dense subset of  $\bar{U}^N$ . The Jacobian of the map  $\gamma$  is  $e^{i(z_1 + \dots + z_N)}$  which never vanishes, so  $\gamma$  is a local homeomorphism at every point of  $C^N$ .

Assume initially that there exists a compact set  $K \subset G_A$  such that  $\gamma[K] = E$ . In general no such set  $K$  will exist, but we shall excise this difficulty later. If  $t \in R^N$ , let

$$\Delta(t, \varepsilon) = \{z \in Q_+^N: |t_1 - z_1| < \varepsilon, \dots, |t_N - z_N| < \varepsilon\};$$

this is a certain product of half discs. For each  $t^0 \in K$ , let  $\varepsilon(t^0)$  be such that  $\gamma$  carries  $\Delta(t^0, \varepsilon(t^0))^-$  homeomorphically onto a closed (relative) neighborhood  $V(t^0)$  of the point  $s^0 = \gamma(t^0)$ . There is  $\delta > 0$  such that  $\Delta'(s^0, \delta) = \{z \in \bar{U}^N: |s_1^0 - z_1| < \delta, \dots, |s_N^0 - z_N| < \delta\}$  is contained in  $V(t^0)$ .

Let  $\gamma_{t^0}^{-1}: V(t^0) \rightarrow \Delta(t^0, \varepsilon(t^0))^-$  be inverse to  $\gamma$ . By compactness, finitely many of the sets  $\gamma_{t^0}^{-1}[\Delta'(s^0, \delta/2)]$  will cover  $K$ . Let the  $\Delta'(s^0, \delta/2)$  corresponding to such a cover be  $\Delta'(s^j, \delta/2)$ ,  $j = 1, \dots, q$ , and let the corresponding  $\gamma_{t^0}^{-1}$  be  $\gamma_j^{-1}$ ,  $j = 1, \dots, q$ .

The set  $A$  is a compact Lebesgue null set, so the Rudin-Carleson theorem [6, p. 81] applied to  $Q_+$ , the upper half plane, yields a function  $F$  continuous on  $\bar{Q}_+$ , holomorphic on  $Q_+$ , which vanishes exactly on the set  $A$ . Define  $H$  on  $Q_+^N$  by  $H(z) = F(\beta_1 z_1 + \dots + \beta_N z_N)$ . The zero set of  $H$  is the set  $G_A$ .

Define  $F_j$  on  $\Delta'(s^j, \delta)$  by  $f_j = H \circ \gamma_j^{-1}$ . The function  $f_j$  is in

<sup>2</sup> I am indebted to the referee for this reference.

$\mathcal{N}[\mathcal{A}'(s^j, \delta)]$  and vanishes on  $r[K \cap \mathcal{A}(t^j, \varepsilon(t^j))^-]$ . Consequently [8], there is  $g_j \in \mathcal{N}[\mathcal{A}'(s^j, \delta)]$  which vanishes exactly on the part of  $r[K \cap \mathcal{A}(t^j, \varepsilon(t^j))^-]$  which lies in  $\mathcal{A}'(s^j, \delta/2)$ .

Let  $W_j = \{z \in \bar{U}^N: \text{for some } k = 1, \dots, N, |z_k - s_k^j| > 3\delta/4\}$ . Let  $\tilde{g}_j$  be the function identically one on  $\bar{W}_j$ . Since on  $W_j \cap \mathcal{A}'(s^j, \delta)$ ,  $g_j$  is zero free, the functions  $g_j$  and  $\tilde{g}_j$  constitute a set of Cousin II data to which Theorem II.1 can be applied. We conclude that there exists  $F_j \in \mathcal{N}[U^N]$  which vanishes exactly on the part of  $r[K \cap \mathcal{A}(t^j, \varepsilon(t^j))^-]$  which is contained in  $\mathcal{A}'(s^j, \delta/2)$ . Define  $F$  to be the product of the finitely many functions  $F_j$  so constructed. The function  $F$  has  $\gamma[K] = E$  as its zero set.

In the case that the set  $D$  does not exist, we may write  $E = \cup E_k$  where  $E_k = \gamma[K_k]$ ,  $K_k$  a compact set in  $G_A$ . The remark on page 435 of [5] or the simpler Corollary 1.2 of [8] now implies that  $E$  is the zero set of some element of  $\mathcal{N}[U^N]$ .

Let us note that in the light of Theorem 1.1 of [8], this theorem generalizes Theorem 4.6 of [7], the case that  $\mathcal{A} = \{0\}$ .

It should be noted that Theorem II.2 is contained as a special case of more general results of Forelli [3] which are obtained by studying measures orthogonal to  $\mathcal{N}[U^N]$ .

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|  |     |
|--|-----|
| Seymour Bachmuth and Horace Yomishi Mochizuki, <i>Kostrikin's theorem on Engel groups of prime power exponent</i> .....  | 197 |
| Paul Richard Beesack and Krishna M. Das, <i>Extensions of Opial's inequality</i> .....   | 215 |
| John H. E. Cohn, <i>Some quartic Diophantine equations</i> .....   | 233 |
| H. P. Dikshit, <i>Absolute <math>(C, 1) \cdot (N, p_n)</math> summability of a Fourier series and its conjugate series</i> .....                                     | 245 |
| Raouf Doss, <i>On measures with small transforms</i> .....   | 257 |
| Charles L. Fefferman, <i><math>L_p</math> spaces over finitely additive measures</i> .....   | 265 |
| Le Baron O. Ferguson, <i>Uniform approximation by polynomials with integral coefficients. II</i> .....   | 273 |
| Takashi Ito and Thomas I. Seidman, <i>Bounded generators of linear spaces</i> .....  | 283 |
| Masako Izumi and Shin-ichi Izumi, <i>Nörlund summability of Fourier series</i> .....   | 289 |
| Donald Gordon James, <i>On Witt's theorem for unimodular quadratic forms</i> .....   | 303 |
| J. L. Kelley and Edwin Spanier, <i>Euler characteristics</i> .....   | 317 |
| Carl W. Kohls and Lawrence James Lardy, <i>Some ring extensions with matrix representations</i> .....  | 341 |
| Ray Mines, III, <i>A family of functors defined on generalized primary groups</i> .....  | 349 |
| Louise Arakelian Raphael, <i>A characterization of integral operators on the space of Borel measurable functions bounded with respect to a weight function</i> ..... | 361 |
| Charles Albert Ryavec, <i>The addition of residue classes modulo <math>n</math></i> .....  | 367 |
| H. M. (Hari Mohan) Srivastava, <i>Fractional integration and inversion formulae associated with the generalized Whittaker transform</i> .....                        | 375 |
| Edgar Lee Stout, <i>The second Cousin problem with bounded data</i> .....  | 379 |
| Donald Curtis Taylor, <i>A generalized Fatou theorem for Banach algebras</i> .....   | 389 |
| Bui An Ton, <i>Boundary value problems for elliptic convolution equations of Wiener-Hopf type in a bounded region</i> .....  | 395 |
| Philip C. Tonne, <i>Bounded series and Hausdorff matrices for absolutely convergent sequences</i> .....  | 415 |