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ON SUPPORTS OF REGULAR BOREL MEASURES

DELMA JOSEPH HEBERT, JR. AND HOWARD E. LACEY

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D. J. HEBERT AND H. ELTON LACEY

The existence of a regular Borel measure whose support is a given compact Hausdorff space X imposes definite structures on X , $C(X)$, and $C(X)^*$. In this paper a necessary and sufficient condition is given to insure that X is the support of a regular Borel measure. This involves the intersection number of a collection of open sets in X . Measures which vanish on a sigma ideal of a sigma field of subsets of X which contains a basis for the topology of X are also considered. In particular, for a certain class of compact Hausdorff spaces X , necessary and sufficient conditions are given to insure the existence of a nonatomic regular Borel measure whose support is X . The final section of the paper is devoted to a study of normal measures; i.e., measures which vanish on meager Borel sets. Normal measures on X are shown to be related to normal measures on the projective resolution of X .

NOTATION AND TERMINOLOGY. Set theoretical and topological terminology is that of [12], the terminology of linear topological spaces is that of [14], and measure theory terminology follows [11]. All spaces considered are taken to be nonempty and all measures considered are finite. If X is a compact Hausdorff space, $C(X)$ denotes the space of continuous real-valued functions on X in the supremum norm, $C(X)^*$ denotes the space of all continuous linear functionals on $C(X)$, or, equivalently, the space of all signed regular Borel measures, and $B(X)$ denotes the space of all bounded real-valued functions on X in the supremum norm.

1. Intersection numbers. The following definitions are motivated by the concept of an intersection number as given in [13]. Let X be a compact Hausdorff space and B be a Boolean algebra.

1.1. If $S = (f_1, \dots, f_n)$ is a finite sequence in $B(X)$, $i(S) = (1/n) \|\sum_{i=1}^n f_i\|$. If $A \subseteq C(X)$, then $I(A) = \inf \{i(S): S \text{ is a finite sequence in } A\}$.

1.2. If $S = (A_1, \dots, A_n)$ is a finite sequence of subsets of X , $i(S) = \max \{(k/n): \text{there is a subsequence } (A_{i_1}, \dots, A_{i_k}) \text{ of } S \text{ such that } \bigcap_{j=1}^k A_{i_j} \neq \emptyset\}$. If H is a collection of subsets of X , then $I(H) = \inf \{i(S): S \text{ is a finite sequence in } H\}$.

1.3. If $S = (E_1, \dots, E_n)$ is a finite sequence in B , then $i(S) =$

$\max \{ \{k/n\}: \text{there is a subsequence } (E_{i_1}, \dots, E_{i_n}) \text{ of } S \text{ such that } \bigwedge_{j=1}^k E_{i_j} \neq \emptyset \} \text{ and } I(H) = \inf \{i(S): S \text{ is a finite sequence in } H\}.$

The relationship between the above concepts is the following: if Y is the Stone space of B (see [10] or [18]), then for $H \subseteq B$, $I(H) = I\{h(E): E \in H\} = I\{C_{h(E)}: E \in H\}$, where h is the isomorphism of B onto the clopen (i.e., closed and open) sets of Y and $C_{h(E)}$ is the characteristic function of $h(E)$ (this notation for the characteristic function is used throughout).

The numbers $I(A)$ and $I(H)$ above are called the intersection numbers of the collections A and H respectively.

LEMMA 1.4. *Let X be a compact Hausdorff space and let F and G be nonempty subsets of the positive cone of $C(X)$. If $a > 0$ is such that, for each $f \in F$, there is a $g \in G$ with $a g \leq f$, then $a I(G) \leq I(F)$.*

Proof. Let $S = (f_1, \dots, f_n)$ be a finite sequence in F . For each $i, 1 \leq i \leq n$, there is a g_i in G such that $a g_i \leq f_i$. Thus

$$0 \leq \sum_{i=1}^n a g_i \leq \sum_{i=1}^n f_i \quad \text{and} \quad \left\| \sum_{i=1}^n a g_i \right\| \leq \left\| \sum_{i=1}^n f_i \right\|.$$

If $T = (g_1, \dots, g_n)$, then

$$a I(G) \leq a i(T) = a \left(\frac{1}{n} \right) \left\| \sum_{i=1}^n g_i \right\| = \left(\frac{1}{n} \right) \left\| \sum_{i=1}^n a g_i \right\| \leq \left(\frac{1}{n} \right) \left\| \sum_{i=1}^n f_i \right\| = i(S).$$

Hence $a I(G) \leq I(F)$.

DEFINITION 1.5. Let B be a Boolean algebra and let H be a nonempty subset of B . Then H is said to be *positive* if and only if $I(H) > 0$. Similarly, if H is a nonempty collection of nonempty subsets of a given set, then H is said to be a *positive collection* whenever $I(H) > 0$.

THEOREM 1.6. *If X is a compact Hausdorff space with topology G , then there is a regular Borel measure whose support is X if and only if $G \setminus \{\emptyset\}$ is the union of a countable family of positive collections. (See [13], Th. 4)*

Proof. Suppose $G \setminus \{\emptyset\}$ is the union of a countable family $\{G_n\}$ of positive collections. For each f in $C(X)$ and each n , let

$$A_f = \left\{ x \in X: f(x) > \frac{1}{2} \right\}$$

and $F_n = \{f \in C(X): f \geq 0, \|f\| = 1, \text{ and } A_f \text{ is in } G_n\}$, and denote the convex hull of F_n by Q_n . It is a matter of computation to show that $\|f\| \geq I(F_n)$ for each $f \in Q_n$ and that for each $f \in F_n$, $(1/2)C_{A_f} \leq f$. By Lemma 1.4, $0 < (1/2)I(G_n) \leq I(F_n)$, and by the above, $\|f\| \geq I(F_n) > 0$ for all $f \in Q_n$. There is a positive linear functional ϕ_n on $C(X)$ such that $\phi_n(f) \geq I(F_n)$ for all $f \in Q_n$. For, if $U = \{g \in C(X): \|g\| < I(F_n)\}$, then $Q_n + U$ is open and convex and $(Q_n + U) \cap -P = \emptyset$, where $P = \{f \in C(X) \mid f \geq 0\}$. By [14], p. 118, there is a continuous linear functional ϕ_n on $C(X)$ such that $\phi_n(-f) \leq \phi_n(g + h)$ for each $f \in P$, $g \in Q_n$, and $h \in U$. Since 0 is in $-P$, $0 \leq \phi_n(g + h)$ for all $g \in Q_n$, and $h \in U$. Thus ϕ_n is a positive linear functional and, without loss of generality, it is assumed that $\|\phi_n\| = 1$. Suppose $g \in Q_n$, $I(F_n) > \varepsilon > 0$, and let 1 denote the constant function 1 on X . Then $(\varepsilon - I(F_n))1$ is in U , and $0 \leq \phi_n[g + (\varepsilon - I(F_n))1] = \phi_n(g) - I(F_n)\phi_n(1) + \varepsilon\phi_n(1) = \phi_n(g) - I(F_n) + \varepsilon$ and hence, $\phi_n(g) \geq I(F_n) - \varepsilon$. Let $\phi = \sum_{n=1}^{\infty} (1/2^n)\phi_n$. Then ϕ is a positive linear functional on $C(X)$ and $\|\phi\| = 1$. Now, suppose $F \in P$ and $f \neq 0$. Then $g = f/\|f\|$ has norm 1 and thus, $g \in F_n$ for some n . Hence, $\phi(g) \geq \phi_n(g) \geq I(F_n) > 0$. It follows immediately that the regular Borel measure that corresponds to ϕ has support all of X .

Conversely, suppose X has a regular Borel measure whose support is X . Then there is a positive normalized linear functional ϕ on X such that $\phi(f) > 0$ if $f \geq 0, f \neq 0$. Let $F = \{f \in C(X) \mid f \geq 0, \|f\| = 1\}$ and $F_n = \{f \in F \mid \phi(f) > (1/n)\}$. By computation it follows that

$$\frac{1}{n} \leq \inf \{\phi(f): f \in F_n\} \leq I(F_n).$$

For each $f \in F$ and each n , let $B_f = \{x \in X: f(x) > 0\}$ and $H_n = \{B_f: f \in F_n\}$. From Urysohn's lemma it follows that for each open subset U of X there is an $f \in F$ such that $f(x) = 0$ for $x \notin U$ and $f(x) = 1$ for some $x \in U$ and thus, $f \leq C_{B_f} \leq C_U$. If $G_n = \{T \in G: U \text{ contains a member of } H_n\}$, then for each U in G_n there is an $f \in F_n$ such that $f \leq C_{B_f} \leq C_U$. By Lemma 1.4, $0 < I(F_n) \leq I(H_n) \leq I(G_n)$. Clearly $G \setminus \{\phi\} = \bigcup_{n=1}^{\infty} G_n$.

COROLLARY 1.7. *If X is a compact Hausdorff space and B is a basis for the topology G of X , then there is a regular Borel measure whose support is X if and only if $B \setminus \{\phi\}$ is the union of a countable family of positive collections.*

2. Nonatomic measures. In this section, certain conditions are shown to be sufficient for the existence of a nonatomic regular Borel measure whose support is a given compact Hausdorff space. The

study of supports of nonatomic regular Borel measures is shown to be related to the study of perfect separable compact Hausdorff spaces.

If m is a regular Borel measure on a compact Hausdorff space X , then m is called *nonatomic* if for each x in X , $m(\{x\}) = 0$.

PROPOSITION 2.1. [17] If X is a compact Hausdorff space, then there is a nonzero, nonatomic regular Borel measure on X if and only if X has a nonempty perfect subset.

A well-known topological lemma is also needed.

LEMMA 2.2. *If C is a closed subset of a topological space X , then $\text{int } C = \text{int cl int } C$.*

THEOREM 2.3. *If X is a perfect compact Hausdorff space such that there exists a regular Borel measure m on X whose support is X , then there are perfect subsets X_1 and X_2 of X such that*

- (i) *either X_1 is empty or there is a nonatomic regular Borel measure whose support is X_1 ,*
- (ii) *either X_2 is empty or X_2 is separable,*
- (iii) *$X = X_1 \cup X_2$,*
- (iv) *$\text{int } X_1$ contains $X \setminus X_2$ and $\text{int } X_2$ contains $X \setminus X_1$.*

Proof. Suppose $C = \{x_1, x_2, \dots\} = \{x \text{ in } X: m(\{x\}) > 0\}$ and $m = m_1 + m_2$ where m_1 is nonatomic and $m_2 = \sum_{n=1}^{\infty} m(\{x_n\})e_{x_n}$. Let $X_1 = \text{Supp}(m_1)$ and let $X_2 = \text{cl int Supp}(m_2)$. Since $m_1(X \setminus \text{Supp}(m_1)) = 0$ and $m_2[X \setminus \text{Supp}(m_2)] = 0$, if $A = X \setminus \text{Supp}(m_1) \cap X \setminus \text{Supp}(m_2)$, then

$$0 = m_1(A) = m_2(A) = m(A).$$

Since the complement of the support of a regular Borel measure is open and since the m -measure of a nonempty open set is positive, A is open; therefore empty. It follows that $\text{Supp}(m_2)$ contains the open set $X \setminus \text{Supp}(m_1)$: hence, $\text{int Supp}(m_2)$ contains $X \setminus X_1$, and X_2 contains $X \setminus X_1$. Therefore, $X_1 \cup X_2 = X$. Also, $\text{cl } X_2$ contains $\text{cl } X \setminus X_1$, and hence $\text{int } X_1 = X \setminus \text{cl } X \setminus X_1$ contains $X \setminus \text{cl } X_2 = X \setminus X_2$. Since $\text{int } X_2$ is $\text{int cl int Supp}(m_2)$, by 2.2 $\text{int } X_2 = \text{int Supp}(m_2)$. Hence, $\text{int } X_2$ contains $X \setminus X_1$. Suppose that x is in X_2 and U is an open set subset of X containing x . Since $X_2 = \text{cl int Supp}(m_2)$, $U \cap \text{int Supp}(m_2)$ is nonempty. It is easy to see that $\text{Supp}(m_2) = \text{cl } C$; hence, there is a member y of C such that y is also an element of the open set $U \cap \text{int Supp}(m_2)$. This implies that X_2 is separable, ($C \cap X_2$ is a countable dense subset). If x is not in $\text{int Supp}(m_2)$, then $y \neq x$. If x is in $\text{int Supp}(m_2)$, then since X is perfect, there is a member of

$U \cap \text{int Supp}(m_2)$ which is distinct from x . Hence, X_2 is perfect.

COROLLARY 2.4. *If X is a compact Hausdorff space on which there is a regular Borel measure whose support is X , and such that each countable subset of X is nowhere dense, then there is a nonatomic regular Borel measure on X whose support is X .*

Proof. First note that since each singleton is nowhere dense, X is perfect. Continuing the notation of the preceding theorem, since $\text{Supp}(m_2) = \text{cl } C$, and C is nowhere dense, $\text{int Supp}(m_2)$ is empty, and hence X_2 is empty. This implies that $X = X_1$, and the proof is finished.

According to Theorem 2.3, the problem of the existence of a nonatomic regular Borel measure whose support is a given compact Hausdorff space X , which has a measure whose support is X , may be reduced to the same problem for a separable perfect compact Hausdorff space. The theorem also makes possible the development of other sufficient conditions on such spaces via the following corollary.

COROLLARY 2.5. *If X is a perfect compact Hausdorff space with a regular Borel measure whose support is X , and such that each point of X is contained in the support of a nonatomic regular Borel measure, then there is a nonatomic regular Borel measure whose support is X .*

Proof. Suppose that X_1, X_2 , and m_1 are chosen as in the theorem. If X_2 is nonempty, let $\{x_1, x_2, \dots\}$ be a countable dense subset of X_2 . For each k , let ν_k be a nonatomic regular Borel measure whose support contains x_k . Then $\nu = \sum_{n=1}^{\infty} (1/2^k)\nu_k$ is a nonatomic regular Borel measure whose support is X_2 , and $m_1 + \nu$ is a nonatomic regular Borel measure whose support is X .

LEMMA 2.6. *If X is a perfect compact Hausdorff space, and x is an element of X such that there is a countable base of open neighborhoods of x , then there is a nonatomic regular Borel measure whose support contains x .*

Proof. Let $\{U_1, U_2, \dots\}$ be a countable base of open neighborhoods of x . For each k , U_k contains a perfect subset P_k , which by (2.1) contains the support of a nonatomic regular Borel measure ν_k . Let $\nu = \sum_{n=1}^{\infty} (1/2^k)\nu_k$. If U is an open set containing x , then U contains U_k for some k ; hence, U contains P_k , and $\nu_k(U) > 0$.

THEOREM 2.7. *If X is a perfect compact Hausdorff space which satisfies the first axiom of countability, and there is a regular Borel measure whose support is X , then there is a nonatomic regular Borel measure whose support is X .*

Proof. It follows immediately from the preceding results.

COROLLARY 2.8. *If X is a perfect compact metrizable Hausdorff space, then there is a nonatomic regular Borel measure whose support is X .*

3. J -null measures. Let B be a Boolean sigma-algebra, J be a sigma-ideal in B , and G be a subset of B . Using some of the methods in [13], conditions may be obtained for the existence of a measure on B which vanishes on J and which is bounded away from zero on G . A set G which satisfies such conditions is called *J -sigma-positive*. Let B be a sigma-field of subsets of a topological space, and let J be a sigma-ideal in B containing no open sets. If B contains a basis G of open sets, then there is a measure on B , which vanishes on J and is positive on nonempty open sets if and only if $G \setminus \{\phi\}$ is contained in a countable union of J -sigma-positive collections. This result is used along with results of § 2 to give necessary and sufficient conditions for the existence of a nonatomic regular Borel measure on a class of compact Hausdorff spaces X whose support is all of X .

Let B be a Boolean algebra and let h be the natural isomorphism of B onto the Boolean algebra of clopen subsets of the Stone space Y of B . The following lemma are basic results in [13].

LEMMA 3.1. *If G is a positive subset of B , then there is a finitely additive measure m on B such that $m(E) \geq I(G)$ for each E in G .*

LEMMA 3.2. *If m is a finitely additive measure on B and G is a subset of B , then $\inf \{m(E) : E \text{ is in } G\} \leq I(G)$.*

DEFINITION 3.3. If B is a Boolean sigma-algebra and C is a subset of B , then C is said to be a *monotone class* whenever for each monotone sequence $E_1 \leq E_2 \leq \dots$ of elements of C ; $\bigvee_{n=1}^{\infty} E_n$ is an element of C .

DEFINITION 3.4. Let B be a Boolean sigma-algebra, G be a subset of B , and $\hat{G} = \{E \text{ in } B : D \leq E \text{ for some } D \text{ in } G\}$. The collection G is said to be *sigma-positive* if and only if G is positive and $B \setminus \hat{G}$ is a monotone class.

The following construction of a measure on a sigma-algebra is based on an idea of Ryll-Nardzewski (see [13]).

LEMMA 3.5. *If B is a sigma-algebra and G is a sigma-positive subset of B , then there is a measure m on B such that $m(E) \geq I(G)$ for each E in G .*

Proof. By 3.1, there is a finitely additive measure m' on B such that $m'(E) \geq I(G)$ for each E in G . For each E in B , let $m(E) = \inf \{ \sum_{n=1}^{\infty} m'(A_n) : A_1, A_2, \dots \text{ is a pairwise disjoint sequence such that } \bigvee_{n=1}^{\infty} A_n = E \}$. Clearly, $0 \leq m(E) \leq m'(E)$ for each E in B . Let A_1, A_2, \dots be a pairwise disjoint sequence in B . If ε is a positive number, then for each n , there is a pairwise disjoint sequence C_1^n, C_2^n, \dots such that $A_n = \bigvee_{k=1}^{\infty} C_k^n$ and

$$m(A_n) \leq \sum_{k=1}^{\infty} m'(C_k^n) < m(A_n) + (\varepsilon/2^n).$$

Hence,

$$\sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(C_k^n) < \sum_{n=1}^{\infty} m(A_n) + \varepsilon.$$

Since $\bigvee_{n=1}^{\infty} A_n = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} C_k^n$, by the definition of m , $m(\bigvee_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n)$. If $m(\bigvee_{n=1}^{\infty} A_n) < \sum_{n=1}^{\infty} m(A_n)$, then there is a pairwise disjoint sequence E_1, E_2, \dots of elements of B such that $\bigvee_{n=1}^{\infty} A_n = \bigvee_{k=1}^{\infty} E_k$ and $m(\bigvee_{n=1}^{\infty} A_n) \leq \sum_{k=1}^{\infty} m'(E_k) < \sum_{n=1}^{\infty} m(A_n)$. Since for each

$$n, \bigvee_{k=1}^{\infty} E_k \wedge A_n = A_n \wedge \bigvee_{k=1}^{\infty} E_k = A_n,$$

(see [10], p. 28), $m(A_n) \leq \sum_{k=1}^{\infty} m'(E_k \wedge A_n)$. Hence,

$$\sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(E_k \wedge A_n).$$

However, since for each i and k , $\bigvee_{n=1}^i E_k \wedge A_n \leq E_k$, it follows that

$$\sum_{n=1}^i m'(E_k \wedge A_n) = m' \left(\sum_{n=1}^i [E_k \wedge A_n] \right) \leq m'(E_k).$$

Thus,

$$\sum_{n=1}^{\infty} m'(E_k \wedge A_n) \leq m'(E_k),$$

and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(E_k \wedge A_n) \leq \sum_{k=1}^{\infty} m'(E_k) < \sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m'(E_k \wedge A_n)$$

which is a contradiction. It has been shown then that m is countably additive. Suppose now that $m(E) < I(G)$ for some E in G . There is a monotone sequence $E_1 \leq E_2 \leq \dots$ such that $\bigvee_{n=1}^{\infty} E_n = E$, and $\{\sup \{m'(E_n): n \geq 0\} < I(G)$, (this is a trivial alteration of the definition of m). Thus, for each n , $m'(E_n) < I(G)$. This implies that for each n , E_n is not in the set $\hat{G} = \{E \text{ in } B: D \leq E \text{ for some } D \text{ in } G\}$. Since $B \setminus \hat{G}$ is a monotone class, E is $B \setminus \hat{G}$, which is a contradiction.

DEFINITION 3.6. Let B be a Boolean sigma-algebra, J be a sigma-ideal in B , and G be a subset of B . For each E in B , let \bar{E} denote the equivalence class modulo J to which E belongs. The collection G is said to be *J-sigma-positive* whenever $\{\bar{E}: E \text{ is in } G\}$ is a sigma-positive subset of B/J .

PROPOSITION 3.7. Let B be a sigma-algebra and let J be a sigma-ideal in B . Let \bar{E} denote the equivalence class modulo J of E for each E in B . If m is a measure on B such that $m(E) = 0$ for each E in J , and $\bar{m}(\bar{E}) = m(E)$ for each E in B , then \bar{m} is a measure on B/J . Conversely, if \bar{m} is a measure on B/J , and $m(E) = \bar{m}(\bar{E})$ for each E in B , then m is a measure on B such that $m(E) = 0$ for each E in J .

The above proposition is well known. The proof is a trivial alteration of a proof in [10], p. 65.

DEFINITION 3.8. If B is a Boolean sigma-algebra, J is a sigma-ideal in B and m is a measure on B , then m is called *J-null* whenever $m(E) = 0$ for each E in J .

COROLLARY 3-9. If B is a sigma-algebra, J is a sigma-ideal in B and G is a *J-sigma-positive* subset of B , then there is a *J-null* measure on B and a positive real number a such that $m(A) \geq a$ for each A in G .

Proof. Let $\bar{G} = \{\bar{E}: E \text{ is in } G\}$. Then \bar{G} is a sigma-positive subset of B/J . By 3.5, there is a measure \bar{m} on B/J such that $\bar{m}(\bar{E}) \geq I(\bar{G})$ for each E in G . Let $A = I(\bar{G})$ and let $m(E) = \bar{m}(\bar{E})$ for each E in B . By 3.6, \bar{m} is a measure on B which vanishes on J . If E is in G , then \bar{E} is in \bar{G} and $m(E) = \bar{m}(\bar{E}) \geq a$.

THEOREM 3.10. Let X be a topological space, B be a sigma-field of subsets of X which contains a basis G of the topology of X , and J be a sigma-ideal in B which contains no open sets. Then there is a *J-null* measure m on B such that $m(U) > 0$ for each nonempty

open subsets U of X if and only if $G \setminus \{\phi\}$ is contained in the union of a countable family of J -sigma-positive sub-collections of B .

Proof. Suppose that m is a J -null measure on B which is positive on $G \setminus \{\phi\}$. For each n , let $B_n = \{E \text{ in } B: m(E) > (1/n)\}$. For each E in B , let \bar{E} denote the equivalence class modulo J of E . Let $\bar{B}_n = \{\bar{E}: E \text{ is in } B_n\}$. By 3.7, if $\bar{m}(\bar{E}) = m(E)$ for each E in B , then \bar{m} is a measure on B/I . By 3.2, $(1/n) \leq \inf \bar{m}(\bar{E}): \bar{E} \text{ is in } \bar{B}_n \leq I(\bar{B}_n)$. Thus, \bar{B}_n is positive. Let $\bar{E}_1 \leq \bar{E}_2 \leq \dots$ be a monotone sequence of elements of B/I such that for each k , and for each \bar{E} in \bar{B}_n , it is not true that $\bar{E} \leq \bar{E}_k$. Then for each k , $\bar{m}(\bar{E}_k) \leq (1/n)$, which implies

$$\bar{m}\left(\sum_{k=1}^{\infty} \bar{E}_k\right) \leq (1/n),$$

and hence it is not true that $\bar{E} \leq \bigcup_{k=1}^{\infty} \bar{E}_k$ for some \bar{E} in \bar{B}_n . Therefore, \bar{B}_n is sigma-positive, and B_n is J -sigma-positive.

Conversely, if $\bigcup_{n=1}^{\infty} B_n$ contains $G \setminus \{\phi\}$ where for each n , B_n is J -sigma-positive, then for each n , there is a measure \bar{m}_n on B/J such that $\bar{m}_n(\bar{E}) \geq I(\bar{B}_n) > 0$ for each \bar{E} in \bar{B}_n . Let $\bar{m} = \sum_{n=1}^{\infty} (1/2^n) \bar{m}_n$. Then for each \bar{U} in $\bar{G} \setminus \{0\}$, $\bar{m}(\bar{U}) > 0$. If $m(E) = \bar{m}(\bar{E})$ for each E in B , then m vanishes on J and m is a positive on $G \setminus \{\phi\}$.

COROLLARY 3.11. *Let X be a compact Hausdorff space, B be the sigma-field of Baire subsets of X , and J be a sigma-ideal in B which contains no open sets. There is a regular Borel measure with support X such that $m(E) = 0$ for each E in J if and only if the collection G of nonempty open Baire subsets of X is contained in the union of a countable family of J -sigma-positive collections of Baire sets.*

Proof. According to 3.10, G is contained in the union of a countable family of J -sigma-positive subcollection if and only if there is a Baire measure m on X such that $m(U) > 0$ for each open Baire set U and $m(E) = 0$ for each E in J . The regular Borel extension of m is the regular Borel measure required.

The final theorem of this section combines the preceding results with those of § 2.

THEOREM 3.13. *Let X be a perfect compact Hausdorff space with topology G and with the property that each point contained in an open separable subset of X is a G_s -point. Let J be the sigma-ideal of Borel sets which contains no nonempty perfect subsets. There is a nonatomic regular Borel measure whose support is X if and only if G is contained in a countable union of J -sigma-positive collections*

of Borel sets.

Proof. Suppose that G is contained in a countable union of J -sigma-positive collections. By 3.10, there is a J -null Borel measure m' on X such that $m'(U) > 0$ for each nonempty open subset U of X . Let m'' be the restriction of m' to the Baire sets, and let m be the regular Borel extension of m'' . Then m is a regular Borel measure which vanishes on Baire sets which contain no nonempty perfect subsets. Let X_1, X_2, m_1, m_2 be as in 2.3. Suppose that X_2 is nonempty. Then X_2 is separable and since $X_2 = \text{cl int Supp}(m_2)$, $\text{int } X_2$ is nonempty. It is clear then that $\text{int } X_2$ is separable and locally compact in the relative topology. Thus, $\{x\}$ is a G_δ for each x in $\text{int } X_2$, and hence $\{x\}$ is a Baire set for each x in $\text{int } X_2$. Therefore, $m(\{x\}) = 0$ for each x in $\text{int } X_2$. If U is an open subset of $X \setminus X_1$, then since $\text{int } X_2$ contains $X \setminus X_1$, $\text{int } X_2$ contains U . Since $m(\{x\}) = 0$ for each x in U , $m_2(U) = 0$. Since $U \cap \text{Supp } m_2$ is empty, $m_1(U) = 0$. Thus, $m(U) = 0$, which contradicts the fact that m is positive on nonempty open Baire sets. Therefore, X_2 is empty, and $X_1 = \text{Supp}(m_1) = X$.

Conversely, if m is a nonatomic regular Borel measure whose support is X , then m is a J -null Borel measure which is positive on nonempty open Borel sets. By 3.10, $G \setminus \{\phi\}$ is contained in the union of a countable family of J -sigma-positive subcollections.

4. Normal measures and the projective resolution. This section develops relationships between the regular Borel measures on a compact Hausdorff space X and the regular Borel measures on the projective resolution of X .

In [8], Gleason introduces the concept of the projective resolution of a compact Hausdorff space. If P is a compact Hausdorff space, then P is *extremally disconnected* if and only if the closure of each open subset of P is open. If X and Y are compact Hausdorff spaces, then a *continuous irreducible map* of X onto Y is a continuous function f from X onto Y such that Y is not the image under f of a proper closed subset of X . Gleason's result states: If X is a compact Hausdorff space, then there is an extremally disconnected compact Hausdorff space P (called the *projective resolution* of X) and a continuous irreducible map g (called the *Gleason map*) of P onto X , and if P' is an extremally disconnected compact Hausdorff space such that X is the image of P' under a continuous irreducible map g' , then there is a homeomorphism h of P onto P' such that $g'h = g$.

If X is a topological space and U is an open subset of X , then U is called a *regular open subset* of X if and only if $U = \text{int cl } U$.

NOTATION. In this section, if Y is a compact Hausdorff space,

then $R(Y)$ denotes the collection of regular open subsets of Y , $B_0(Y)$ denotes the sigma-algebra of Borel subsets of Y , and $I(Y)$ denote the sigma-ideal of meager (i.e., contained in a countable union of nowhere dense sets) Borel subsets of Y .

If Y is a compact Hausdorff space, then the collection $R(Y)$ is a Boolean algebra with the following operations: If U and V are in $R(X)$, then $U \vee V = \text{int cl}(U \cup V)$, $U \wedge V = U \cap V$, and the complement of U is $Y \setminus \text{cl } U$; $0 = \phi$ and $1 = X$ (for a proof, see [10]). It is useful to note that each Borel subset of Y is equivalent modulo $I(Y)$ to a unique regular open set, and indeed $R(Y)$ is isomorphic to $B_0(Y)/I(Y)$, (see [10], p. 58).

In Gleason's construction, the projective resolution P of a compact Hausdorff space X turns out to be the Stone space of $R(X)$. The following sequence of lemmas throws some light upon the relationships involved.

LEMMA 4.1. *If P and X are compact Hausdorff spaces and g is a continuous irreducible map of P onto X , then for each nonempty open subset U of P , $g(U)$ has nonempty interior.*

Proof. Since U is nonempty, $P \setminus U$ is a proper closed subset of P and by the irreducibility of g , $g(P \setminus U)$ is a proper closed subset of X . If x is in $X \setminus g(P \setminus U)$, p is in P , and $g(p) = x$, then p is not in $P \setminus U$; hence p is in U and x is in $g(U)$. Therefore, $g(U)$ contains the nonempty open set $X \setminus g(P \setminus U)$.

In the remainder of this section, X is a compact Hausdorff space, P is the projective resolution of X , and g is the Gleason map of P onto X .

LEMMA 4.2. *If C is a nonempty clopen subset of P , then*

$$\text{int } g^{-1}[g(C)] = C \quad \text{and} \quad \text{cl } g^{-1}[\text{int } g(C)] = C.$$

Proof. Since P is extremally disconnected and $g(C)$ is closed, $\text{int } g^{-1}[g(C)]$ is clopen. If $D = \text{int } g^{-1}[(C)] \setminus C$ is nonempty, then $P \setminus D$ is a proper clopen subset of P containing C , and $g(P \setminus D) = X$, contradicting the irreducibility of g . Further, $g^{-1}[g(C)]$ contains the open set $g^{-1}[\text{int } g(C)]$, and hence, $C = \text{int } g^{-1}[(C)]$ contains $g^{-1}[\text{int } g(C)]$. Since C is closed, C contains the clopen set $\text{cl } g^{-1}[\text{int } g(C)]$. If $D = C \setminus \text{cl } g^{-1}[\text{int } g(C)]$ is nonempty, then it is clopen, and by (9.1), $g(D)$ nonempty interior. But since D is disjoint from $g^{-1}[\text{int } g(C)]$, $g(D)$ is contained in $g(C) \setminus \text{int } g(C)$.

LEMMA 4.3. *If U is a regular open subset of X , then $\text{cl } g^{-1}(U) = \text{int } g^{-1}(\text{cl } U)$ and $\text{cl } g^{-1}(U) \cap \text{cl } g^{-1}(X \setminus \text{cl } U)$ is empty.*

Proof. Since $g^{-1}(U)$ is a subset of $g^{-1}(\text{cl } U)$, the clopen set $\text{cl } g^{-1}(U)$ is a subset of the closed set $g^{-1}(\text{cl } U)$, hence, $\text{cl } g^{-1}(U)$ is a subset of $\text{int } g^{-1}(\text{cl } U)$. Let

$$D = \text{int } g^{-1}(\text{cl } U) \setminus \text{cl } g^{-1}(U) .$$

If D is nonempty, then $g(D)$ is contained in $\text{cl } (U) \setminus U$ which is a nowhere dense set, that $g(D)$ has nonempty interior by 4.1. Suppose

$$\text{cl } g^{-1}(U) \cap \text{cl } g^{-1}(X \setminus \text{cl } U) = \text{int } g^{-1}[\text{cl } (U)] \cap \text{int } g^{-1}[\text{cl } (X \setminus \text{cl } U)] = A$$

is nonempty. Since A is clopen, $g(A)$ has nonempty interior, and since $g[\text{int } g^{-1}(\text{cl } U)]$ is a subset of $g[g^{-1}(\text{cl } U)] = \text{cl } U$ and

$$g[\text{int } g^{-1}(\text{cl } (X \setminus \text{cl } U))]$$

is a subset of $\text{cl } (X \setminus \text{cl } U)$, $g(A)$ is a subset of $\text{cl } (U) \cap \text{cl } (X \setminus \text{cl } U)$, a nowhere dense set.

LEMMA 4.4. *If A is a subset of X , then A is nowhere dense if and only if $g^{-1}(A)$ is nowhere dense.*

Proof. If A is nowhere dense, then so is $\text{cl } A$. If $g^{-1}(\text{cl } A)$ has nonempty interior, then by (4.1), $g[g^{-1}(\text{cl } A)] = \text{cl } A$ has nonempty interior. Hence, $g^{-1}(\text{cl } A)$ is nowhere dense and contains $g^{-1}(A)$. Conversely, if D is a nowhere dense subset of P , then so is $\text{cl } D$. Suppose that $\text{cl } g(D)$ contains a nonempty open set V . Then $g^{-1}(V)$ is a nonempty open subset of $g^{-1}[\text{cl } g(D)]$ which is contained in $g^{-1}[g(\text{cl } D)]$. Since $\text{cl } D$ contains no nonempty open set, $g^{-1}(V) \setminus \text{cl } D$ contains a nonempty clopen set C . Since $g(\text{cl } D)$ contains V which contains $g(C)$, $P \setminus g(C)$ maps onto X , which is a contradiction. Hence, $\text{cl } g(D)$ is nowhere dense, and $g(D)$ is nowhere dense.

LEMMA 4.5. *If for each regular open subset U of X , $h(U) = \text{cl } g^{-1}(U)$, then h is an isomorphism of the Boolean algebra $R(X)$ onto the Boolean algebra of clopen subsets of P .*

Proof. Since P is extremally disconnected and $g^{-1}(U)$ is open, $\text{cl } g^{-1}(U)$ is clopen. If C is a nonempty clopen subset of P , then $g(C)$ is closed, and by 2.2, $\text{int } g(C)$ is a regular open set. By 4.2,

$$h[\text{int } g(C)] = \text{cl } g^{-1}[\text{int } g(C)] = C .$$

Thus, h is an onto map. If U and V are nonempty regular open subsets of X such that $h(U) = h(V)$, then suppose that $U \neq V$. Then

either $U \cap (X \setminus \text{cl } V)$ or $V \cap (X \setminus \text{cl } U)$ is nonempty (otherwise, $\text{cl } U$ contains V , $\text{cl } U$ contains $\text{cl } V$, $U = \text{int cl } U$ contains $\text{int cl } V = V$, and likewise V contains U). Suppose $W = U \cap (X \setminus \text{cl } V)$ is nonempty. Then W is a regular open set, and $h(W)$ is a nonempty subset of both $h(U)$ and $h(X \setminus \text{cl } V)$. But $h(U) = h(V)$, and by 4.3,

$$h(V) \cap h(X \setminus \text{cl } V)$$

is empty, which is contradictory. Therefore, $h(U) = h(V)$ and h is one-to-one. To show that h preserves complementation, it is necessary only to show that for each regular open set V , $h(V) \cup h(X \setminus \text{cl } V) = P$. If V is a nonempty regular open proper subset of X , then

$$\begin{aligned} \text{cl } g^{-1}(V) \cup \text{cl } g^{-1}(X \setminus \text{cl } V) &= \text{cl}[g^{-1}(V) \cup g^{-1}(X \setminus \text{cl } V)] \\ &= \text{cl}[g^{-1}(V \cup X \setminus \text{cl } V)] . \end{aligned}$$

Since $X \setminus (V \cup X \setminus \text{cl } V)$ is nowhere dense,

$$g^{-1}(X \setminus [V \cup (X \setminus \text{cl } V)]) = P \setminus g^{-1}[V \cup (X \setminus \text{cl } V)]$$

is nowhere dense, and the desired result follows. Suppose that U and V are nonempty regular open subsets of X . If

$$A = h(U \cap V) = \text{cl } g^{-1}(U \cap V) = \text{cl}[g^{-1}(U) \cap g^{-1}(V)] ,$$

then clearly both $h(U)$ and $h(V)$ contain A . Let D be a clopen subset of both $h(U)$ and $h(V)$. Then $h^{-1}(D)$ is contained in both $h^{-1}[h(U)]$ and $h^{-1}[h(V)]$. Thus $U \cap V$ contains $h^{-1}(D)$ and $A = h(U \cap V)$ contains D . Therefore, $A = h(U) \cap h(V)$. Therefore, h is one-to-one, onto, and h preserves complementation and intersection; thus, h is an isomorphism.

THEOREM 4.6. *There is a regular Borel measure whose support is X if and only if there is a regular Borel measure whose support is P .*

Proof. If μ is a regular Borel measure whose support is P , then $\nu(A) = \mu(g^{-1}(A))$ defines a regular Borel measure whose support is X , where g is the continuous irreducible map of P onto X .

Suppose ν is a regular Borel measure whose support is X . Let ϕ be the positive linear functional on $\{f \circ g : f \in C(X)\}$ defined by $\phi(f \circ g) = \nu(f)$. Then by [14], p. 20, ϕ has a positive extension ψ to all of $C(P)$. Let μ denote the regular Borel measure that corresponds to ψ . Then for each Borel set A in X , $\nu(A) = \mu(g^{-1}(A))$. Let U be a nonempty open set on P . The proof of Lemma 4.1 shows that $g^{-1}(X \setminus g(P \setminus U)) \subseteq U$ and if $U \neq P$, then $V = X \setminus g(P \setminus U)$ is nonempty.

Hence, $\mu(g^{-1}(V)) = \nu(V) > 0$. It follows that the support of μ is all of P .

REMARK. Suppose μ and ν are related by $\nu(A) = \mu(g^{-1}(A))$ for all Borel sets A in X . If ν is nonatomic and p is in P , then $\nu(g(p)) = 0$, and, thus, $0 \leq \mu(p) \leq \mu(g^{-1}(g(p))) = \nu(g(p)) = 0$ and μ is nonatomic.

DEFINITION 4.7. If Y is a compact Hausdorff space and m is a regular Borel measure on Y , then m is called a *normal* measure on Y if and only if for each meager Borel subset A of Y , $m(A) = 0$.

This concept applied to extremally disconnected spaces is equivalent to Dixmier's notation of normal measure (see [3]).

LEMMA 4.8. *If m is a normal measure on X , then m restricted to $R(X)$ is a measure on $R(X)$.*

Proof. It suffices to show that m is countably additive on $R(X)$. If U_1, U_2, \dots is a pairwise disjoint sequence of regular open sets, the least upper bound of this sequence in $R(X)$ is $\text{int cl } (\bigcup_{n=1}^{\infty} U_n)$. Since the boundary of an open set is nowhere dense,

$$m\left(\bigcup_{n=1}^{\infty} U_n\right) = m\left(\text{cl } \bigcup_{n=1}^{\infty} U_n\right),$$

and it follows from 2.2 that $m(\text{int cl } \bigcup_{n=1}^{\infty} U_n) = m(\text{cl } \bigcup_{n=1}^{\infty} U_n)$. Thus, the countable additivity on $R(X)$ follows from the countable additivity on the sigma-algebra of Borel sets.

THEOREM 4.9. *If m is a normal measure on X , then there is a unique normal measure m' on P such that for each Borel subset E of X , $m(E) = m'(g^{-1}[E])$.*

Proof. By 4.8, m defines a measure on $R(X)$. Let h be the isomorphism in 4.5 of $R(X)$ onto $R(P)$, let i be the isomorphism of $R(P)$ onto $B_0(P)/I(P)$, and let j be the natural homomorphism of $B_0(P)$ onto $B_0(P)/I(P)$. For each Borel subset F of P , let $m'(F) = m(h^{-1}i^{-1}j)(F)$. Clearly, $m'(F) = 0$ whenever F is a meager Borel subset of P . For each Borel subset E of X , let $f(E) = g^{-1}(E)$. By elementary properties of functions (see [12], p. 11), f is an isomorphism of $B_0(X)$ onto $B_0(P)$. If E is a Borel subset of X , then E is equivalent modulo $I(X)$ to a regular open subset U of X . Since the symmetric difference $U + E$ is in $I(X)$, by (4.4), $f(E + U) = f(E) + f(U)$ is in $I(P)$, and since $h(U) = \text{cl } f(U)$, $h(U)$ is equivalent to $f(E)$ modulo $I(P)$. Hence, $j[f(E)]$ is the equivalence class of $h(U)$ and

$$m'[f(E)] = m(h^{-1}i^{-1}jf)(E) = m(U) = m(E) .$$

It remains to show that m' is a regular Borel measure. Suppose e is a positive real number and F is a Borel subset of P . There is a clopen set C such that the symmetric difference $C + F$ is meager. Hence, $F \setminus C$ is meager, and there are nowhere dense Borel sets D_1, D_2, \dots such that $F \setminus C = \bigcup_{n=1}^{\infty} D_n$. Now clearly,

$$g\left(C \cup \bigcup_{n=1}^{\infty} \text{cl } D_n\right) = g(C) \cup \bigcup_{n=1}^{\infty} g(\text{cl } D_n) = E$$

is a Borel subset of X , and $m(E) = m[\text{int } g(C)] = m'(C)$. Since m is a regular Borel measure, there is an open set U containing E such that $m(U) < m(E) + e$. Thus,

$$\begin{aligned} m'(F) &= m'(C \cup F \setminus C) = m'\left(C \cup \bigcup_{n=1}^{\infty} \text{cl } D_n\right) \leq m'[g^{-1}(E)] = m(E) \\ &= m'(C) \leq m(U) = m'[g^{-1}(U)] < m(E) + e = m'(E) + e . \end{aligned}$$

Since U contains

$$g(C) \cup g(F \setminus C) = g(C \cup F \setminus C) = g(C \cup F) ,$$

$g^{-1}(U)$ contain F . The uniqueness follows from the fact that the sigma-algebra of Baire subsets of P is generated by clopen subsets (see [12], p. 999). If $m''[g^{-1}(E)] = m(E)$ for each Borel set E , then m'' agrees with m' on the clopen subsets of P , and hence on the Baire subsets of P . By [11], p. 239, the regular Borel extension of m' restricted to the Baire subsets is unique.

COROLLARY 4.10. *There is a one-to-one, order and norm preserving correspondence between the collection $N(X)$ of normal measures on X and the collection $N(P)$ of normal measures on P .*

Proof. The mapping g of P onto X defines a continuous linear norm decreasing mapping \hat{g} of $M(P)$ onto $M(X)$ such that for each Borel subset E of X , $\hat{g}(m)(E) = m[g^{-1}(E)]$ (see [15], p. 180). If \hat{g} is restricted to $N(P)$, then the correspondence of the preceding theorem shows that \hat{g} maps $N(P)$ one-to-one, onto $N(X)$. The preservation of order and norm is immediate.

COROLLARY 4.11. *There is a one-to-one, order and norm preserving correspondence between the collection of normal measures whose support is X and the collection of normal measures whose support is P .*

Proof. Let \hat{g} be the mapping defined in the previous corollary.

If m is a normal measure whose support is P , then $\hat{g}(m)$ is a normal measure. If U is a nonempty open subset of X , then $g^{-1}(U)$ is a nonempty open subset of P and $\hat{g}(m)(U) = m[g^{-1}(U)] > 0$. If m is a normal measure on X and U is a nonempty open subset of P , then U contains a nonempty clopen set C and by (4.2), $\text{cl } g^{-1}[\text{int } g(C)] = C$. Thus, $g^{-1}(m)(U) \geq g^{-1}(m)(C) \geq g^{-1}(m)[g^{-1} \text{int } \{g(C)\}] = m[\text{int } g(C)] > 0$.

Recall that each signed regular Borel measure m on a compact Hausdorff space Y may be written as $m = m^+ - m^-$. If Y is a compact Hausdorff space and m_1 and m_2 are normal measures on Y , then $m = m_1 - m_2$ is a signed regular Borel measure on Y , and since $m_1 \geq m^+ \geq 0$ and $m_2 \geq m^- \geq 0$, m^+ and m^- are also normal measures on Y . Thus, $\{m: m \text{ is a signed regular Borel measure on } Y \text{ such that } m^+ \text{ and } m^- \text{ are normal}\} = \{m: m = m_1 - m_2 \text{ where } m_1 \text{ and } m_2 \text{ are normal measures on } Y\}$. This collection is called the collection of signed normal measures on Y and is denoted by $M'(Y)$. It is easy to show that $M'(Y)$ is a closed linear subspace $M(Y)$ for each compact Hausdorff space Y . It also follows easily from 4.10 that $M'(X)$ and $M'(P)$ are linearly isometric. For a different proof of this, see [2].

The following theorem summarizes some of the results in this section in combination with some results from the theory of measure algebras.

THEOREM 4.12. *The following statements are equivalent.*

- (i) *There is a normal measure whose support is X .*
- (ii) *There is a normal measure whose support is P .*
- (iii) *There is a strictly positive (countably additive) measure of $R(X)$ (resp. $R(P)$).*
- (iv) *There is a regular Borel measure on P whose support is P and every meager set in P is nowhere dense.*
- (v) *There is a regular Borel measure on X whose support is X and every meager set in X is nowhere dense.*

Proof. Theorem 4.6 and 4.9 yield that (i) is equivalent to (ii). Since $R(P)$, $R(X)$, $B_0(X)/I(X)$, and $B_0(P)/I(P)$ are all isomorphic under an isomorphism that preserves countable operations, (i) equivalent to (iii) and (ii) is equivalent to (iii). Suppose there is a normal measure μ on P with support P . Let $B \subset P$ be a meager set. Then $\mu(B) = \mu(\text{int cl } B)$ and hence B is nowhere dense (see [3], p. 158). Thus (ii) implies (iv).

Suppose (iv) is true. Then by Theorem 4.6 there is a regular Borel measure whose support is X . If $B \subset X$ is meager, then by Lemma 4.4 $g^{-1}(B)$ is meager and thus $g^{-1}(B)$ is nowhere dense. Hence, by Lemma 4.4 again, B is nowhere dense and (iv) implies (v). A

similar argument shows that (v) implies (iv). For, by Theorem 4.6 again, if there is a regular Borel measure whose support is X , then there is one whose support is P . Suppose $B \subset P$ is closed and nowhere dense. Then $g(B)$ is closed and nowhere dense. Thus, if $B \subset P$ is meager, then $g(B)$ is meager. It follows that if each meager set in X is nowhere dense, then each meager set in P is nowhere dense. Finally, (iv) implies (iii). For, in [13] Kelley shows that if a complete Boolean algebra A has a strictly positive finitely additive measure and if each meager set in the Stone space of A is nowhere dense, then A has a strictly positive countably additive measure. Thus, if μ is a regular Borel measure whose support is P and each meager set in P is nowhere dense, then μ induces a finitely additive strictly positive measure on $R(P)$, $R(P)$ is a complete Boolean algebra, the Stone space of $R(P)$ is P , and by the above cited result, $R(P)$ has a countably additive strictly positive measure.

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