Pacific Journal of Mathematics

A CO-TOPOLOGICAL APPLICATION TO MINIMAL SPACES

Giovanni Viglino

Vol. 27, No. 1 January 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 1, 1968

A CO-TOPOLOGICAL APPLICATION TO MINIMAL SPACES

GIOVANNI VIGLINO

A space (X, τ) which satisfies a topological property P is said to be minimal-P if $T = \{\tau' \mid \tau'$ is a P-topology on X; $\tau' \lneqq \tau\} = \varnothing$. For example, a Hausdorff space (X, τ) is minimal Hausdorff if there exists no Hausdorff topology on X which is strictly weaker than τ. The purpose of this paper is to show that for certain properties one need only consider a subset of T " induced " by τ to determine if (X, τ) is minimal-P.

Notation. Let β be an open base for the space (X, τ). τ_β will denote the topology on X generated by the subbase $\{X \backslash Cl_\tau B \mid B \in \beta\}$.

REMARK. J. de Groot in his investigation for a general classification of Baire spaces considered the above topologies (cf. [1], [4]). These topologies have come to be known as co-topologies.

DEFINITIONS. A filter base is regular if it is open and equivalent to a closed filter base.

A filter base $\mathscr{U}$ is Urysohn if for each nonadherent point a, there exists a neighborhood V and $G \in \mathscr{U}$ such that $\mathrm{Cl}_\tau V \cap \mathrm{Cl}_\tau G = \varnothing$.

REMARK. In this paper, the Bourbaki convention for the topological separation properties will be observed; specifically, all spaces are assumed to be Hausdorff.

The proof of the following lemmas are left to the reader. A proof for the regular case of Lemma 1 is similar to the proof of Theorem 2 in [3].

LEMMA 1. *Let (X, τ) be a Hausdorff (Urysohn; regular) space; let $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ be a nonconvergent open (Urysohn; regular) filter base with unique adherent point x_0; let $\beta = \mathscr{N} \cup \mathscr{M}$ where*

$$\mathscr{N} = \{N \mid N \in \tau \text{ and } x_0 \in \mathrm{Cl}_\tau N\}$$

and

$$\mathscr{M} = \{M \mid M \in \tau \text{ and } M \subset X \backslash \mathrm{Cl}_\tau U_\alpha \text{ for some } \alpha \in A\} .$$

Then (i) *β is a base for τ; and*
(ii) *τ_β is a Hausdorff (Urysohn; regular) topology strictly*

weaker than τ.

LEMMA 2. *Let* (X, τ) *be a normal (completely normal) space; let* $\mathscr{U}$ *be a nonconvergent regular filter base with unique adherent point* x_0; *let* β *be defined as in Lemma* 1. *Then* τ_β *is a normal (completely normal) topology strictly weaker than* τ.

In the following theorem P denotes any of the following properties: (i) Hausdorff, (ii) Urysohn, (iii) regular, (iv) completely regular, (v) normal, (vi) completely normal, (vii) locally compact. In [2], [3], [5] it is shown that there exist minimal Hausdorff, minimal Urysohn, and minimal regular spaces which are not compact, while for properties (iv) through (vii) mininal-P is equivalent to compactness.

THEOREM. *A P-space* (X, τ) *is minimal-P if and only if* $\{\tau_\beta \mid \tau_\beta$ *is* P; $\tau_\beta \neq \tau\} = \varnothing$.[1]

Proof. Necessity, in each case, follows from the fact that $\tau_\beta \leqq \tau$ for every open base β.

Sufficiency for property (i) through (iii): Suppose (X, τ) is not minimal Hausdorff (Urysohn; regular). Then there exists an open (Urysohn; regular) filter base $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ with uniques adherent point x_0, which does not converge (see [5], [2]). By Lemma 1, there exists a base β for τ such that $\tau_\beta \lneqq \tau$ and τ_β is Hausdorff (Urysohn; regular).

Sufficiency for completely regular[2]: Suppose (X, τ) is not compact.

Let (Y, τ') denote a compact extension of (X, τ). Take and fix $p \in Y \backslash X$. Let $\mathscr{S}$ be the filter base of open neighborhoods of p, and $\mathscr{S}^*$ denote the trace of $\mathscr{S}$ in X. Considered as a filter base in (Y, τ'), $\mathscr{S}^*$ has a unique adherent point, namely p. Thus $\mathscr{S}^*$ has no adherent point in (X, τ). Fix and element x_0 in X. Let $\beta = \mathscr{N} \cup \mathscr{M}$ where $\mathscr{N} = \{N \mid N \in \tau$ and $x_0 \in Cl_\tau N\}$ and $\mathscr{M} = \{M \mid M \in \tau$ and $M \subset X \backslash Cl_\tau S^*$ for some $S^* \in \mathscr{S}^*\}$. One can show β is an open base for τ. Similarly one can show that $\mathscr{K} = \{X/Cl_\tau H \mid H \in \mathscr{N} \cup \mathscr{M}\}$ is a base for τ_β.

We will now show that $\tau_\beta \neq \tau$ and (X, τ_β) is completely regular. Let us first note that since (X, τ) is regular and since $\mathscr{N} \subset \beta$, then $G \in \tau_\beta$ whenever $G \in \tau$ and $x_0 \notin G$. Hence if f is continuous on (X, τ) then f is continuous everywhere on (X, τ_β) except possibly at x_0. Now there exists $S^* \in \mathscr{S}^*$ such that $x_0 \notin Cl_\tau S^*$. Since τ is regular, then there exists $U \in \tau$ such that $x_0 \in U$ and $Cl_\tau U \cap Cl_\tau S^* = \varnothing$. Since any element of τ_β which contains x_0 must meet S^*, then $U \notin \tau_\beta$. Thus

[1] The result for $p =$ Hausdorff was independently obtained by G. Strecker.

[2] The technique used by Berri in [2] to show that a space is compact if it is minimal completely regular is extensively used in this proof.

$\tau_\beta \neq \tau$.

We complete the proof by showing τ_β is completely regular. Take $b \in X$ and $X \backslash Cl_\tau H \in \mathscr{K}$ where $b \notin X \backslash Cl_\tau H$ and $H \in \mathscr{N} \cup \mathscr{M}$. We wish to show there exists a continuous, real-valued function f on (X, τ_β), such that $f(b) = 1$ and $f(x) = 0$ for all $x \in Cl_\tau H$. Suppose $H \in \mathscr{N}$. Then $x_0 \in Cl_\tau H$. Let $S^* \in \mathscr{S}^*$ be such that $b \notin Cl_\tau S^*$; Since (X, τ) is regular, then there exists $V \in \tau$ such that $b \in V$ and

$$Cl_\tau V \cap Cl(H \cup S^*) = \varnothing .$$

Since (X, τ) is completely regular, then there exists a continuous, real-valued function f such that $f(b) = 1$ and $f(x) = 0$ for all $x \in X \backslash V$. By a previous remark, f is continuous at every point of (X, τ_β) except possibly at $x = x_0$. We will now show f is continuous at $x = x_0$. Now for all $x \in X \backslash V, f(x) = 0$. Since $Cl_\tau V \cap Cl_\tau(H \cup S^*) = \varnothing$, then $f(x) = 0$ for all $x \in X \backslash Cl_\tau V$. Thus f is continuous at all $x \in X \backslash Cl_\tau V$, and hence at all $x \in Cl_\tau(H \cup S^*)$. Therefore f is continuous at x_0.

Similarly one can show that if $H \in \mathscr{M}$, then there exists a real-valued continuous function f on (X, τ_β) such that $f(b) = 1$ and $f(x) = 0$ for each $x \in Cl_\tau H$.

Sufficiency for properties (v) and (vi): Suppose the normal (completely normal) space (X, τ) is not compact. Then X is not minimal regular since a minimal regular normal (completely normal) space is minimal completely regular. Hence there exists a nonconvergent regular filter base $\mathscr{U}$ with a unique adherent point x_0. By Lemma 2, there exists a base β for τ such that $\tau_\beta \lneqq \tau$ and τ_β is normal (completely normal).

Sufficiency for locally compact: Suppose (X, τ) is not minimal locally compact (i.e., not compact). Let (Y, τ') denote the Alexandroff compactification of X with $Y = X \cup \{p\}$ where $p \notin X$. Fix an element x_0 in X and construct $\beta = \mathscr{N} \cup \mathscr{M}$ as in the proof of sufficiency for completely regular spaces. One can show $\tau_\beta \lneqq \tau$ and τ_β is locally compact, and in fact, compact.

REFERENCES

1. J. Aarts and J. de Groot, et al., *Colloquium co-topologie*, Mathematisch Centrum (Amsterdam), syllabus zwA, 1964.
2. M. P. Berri, *Minimal topological spaces*, Trans. Amer. Math. Soc. **108** (1963), 97-105.
3. M. P. Berri, and R. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc. **14** (1963), 454-458.
4. J. de Groot, *Subcompactness and the Baire category theorem*, Nederl. Akad. Wetensch. Indag. Math. **25** (1963), 761-767.
5. H. Herrlich, *T_ν-Abgeschlossenheit und T_ν-Minimalität*, Math. Zeitschr. 88 (1965), 285-294.

Received August 23, 1967, and in revised form October 20, 1967. Some of the results in this paper appear in the author's doctoral dissertation written under the supervision of Professor Robert McDowell at Washington University.

WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI
WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 27, No. 1 January, 1968