Pacific Journal of Mathematics

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Vol. 27, No. 2 February 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 2, 1968

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The behavior of the analytic Feynman integral under translation has been studied in the recent work of R. H. Cameron and D. A. Storvick. The purpose of this paper is to continue the development of this transformation theory. In particular, the behavior of the analytic Feynman integral under certain linear transformations is determined and, using this linear transformation theory, a "generalized Schroedinger equation" is solved in terms of an analytic Feynman integral.

2. Preliminaries. The analytic Wiener and analytic Feynman integrals were defined in [4] and [6] as follows:

DEFINITION 2.1. Let the complex number λ_0 satisfy $Re\lambda_0 \geqq 0$ and $\lambda_0 \neq 0$, so that $\lambda_0 = |\lambda_0| \exp(i\theta)$ for some θ on the interval $[-\pi/2, \pi/2]$. Let $F(x)$ be a functional defined on $C[a, b]$ (the space of continuous functions on $[a, b]$ which vanish at a) such that the Wiener integral

$$J(\lambda) = \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x)dx$$

exists for all real λ in the interval $|\lambda_0| < \lambda < |\lambda_0| + \delta$ for some $\delta > 0$. Then if $J(\lambda)$ can be extended so that it is defined and continuous on the closed region

$$S = \{\lambda = \rho e^{i\gamma} : |\lambda_0| \leqq \rho \leqq |\lambda_0| + (1 - \gamma\theta^{-1})\delta\ , \quad \gamma \in [0, \theta] \text{ or } \gamma \in [\theta, 0]\}\ , \tag{2.1}$$

and analytic in its interior, we define

$$\int_{C[a,b]}^{anw_{\lambda_0}} F(x)dx = J(\lambda_0) \tag{2.2}$$

and we call the left member of (2.2) the analytic Wiener integral of $F(x)$ with parameter λ_0. If $\theta = 0$ we interpret S to be the interval $[\lambda_0, \lambda_0 + \delta]$, omit the analyticity requirement since the interior of S is empty, and define the analytic Wiener integral to be $J(\lambda_0^+)$. If $\lambda_0 = -i$ the integral (2.2) will be called the analytic Feynman integral and we write

$$\int_{C[a,b]}^{anf} F(x)dx = J(-i)\ .$$

The additional concept of uniform analytic Wiener (and Feynman) integrability is needed and is defined as in [6].

DEFINITION 2.2. Let λ_0 and θ be given as in Definition 2.1, let A be any nonempty set, and let $F(x \mid \alpha)$ be a functional defined on $C[a, b] \times A$. Suppose that there is a positive δ (independent of α) and a corresponding set S as given in (2.1) and a function $J(\lambda \mid \alpha)$ defined on $S \times A$ such that

(i) $J(\lambda \mid \alpha)$ is analytic in λ in the interior of S for each fixed $\alpha \in A$,

(ii) $J(\lambda \mid \alpha)$ is uniformly continuous in λ on S uniformly with respect to α in A, and

(iii) for all real $\lambda \in (|\lambda_0|, |\lambda_0| + \delta)$ and all $\alpha \in A$

$$J(\lambda \mid \alpha) = \int_{C[a,b]} F(\lambda^{-\frac{1}{2}} x \mid \alpha) dx .$$

Then we say $F(x \mid \alpha)$ is analytic Wiener integrable with parameter λ_0 uniformly with respect to α in A. If $\lambda_0 = -i$ we say $F(x \mid \alpha)$ is analytic Feynman integrable uniformly with respect to α in A.

In order to prove our results we need the following lemma regarding the analytic continuation of a function of several complex variables. This lemma is related to the results in [5] and a recent communication with R. H. Cameron informed me that he and D. A. Storvick have also obtained this result.

LEMMA 2.1. *Let Λ and M be open simply connected subsets of the λ-plane and the μ-plane, respectively, and such that the intersection of Λ and the real axis of the λ-plane is an interval I. Further, if $f(\lambda, \mu)$ is a function defined and bounded on all compact subsets of $\Lambda \times M$ such that*

(i) for every $\lambda \in I$ we have $f(\lambda, \mu)$ analytic in $\mu \in M$ and

(ii) for every $\mu \in M$ we have $f(\lambda, \mu)$ analytic in $\lambda \in \Lambda$, then $f(\lambda, \mu)$ is analytic on $\Lambda \times M$.

Proof. Let $\lambda_0 \in I$ and $\mu_0 \in M$. Let D_1 and D_2 be discs centered at λ_0 and μ_0, respectively, such that $4D_1 \subseteqq \Lambda$ and $4D_2 \subseteqq M$. Using Lemma 1 of [5, p. 7] and the note following the lemma we see that $f(\lambda, \mu)$ is analytic on $D_1 \times D_2$. Then using the generalized Hartog's lemma [2, p. 141] with $D = D_1 \times D_2$, $\tilde{D} = \Lambda \times D_2$, $\Delta(\mu) = D_1$, and $\tilde{\Delta}(\mu) = \Lambda$ we find $f(\lambda, \mu)$ analytic on $\Lambda \times D_2$. It now follows that $f(\lambda, \mu)$ is analytic on $\Lambda \times M$. That is, if $p \in \Lambda \times M$ then there exists a neighborhood of p, namely $\Lambda \times D_2$ where D_2 is a sphere about the μ coordinate of p, on which $f(\lambda, \mu)$ is analytic.

3. The analytic Feynman integral under linear transformation. The linear transformation theorem for the analytic Feynman integral is based on the following theorem of Woodward [8] which appears in this form in [7, p. 268-69]. Before stating the theorem several definitions are needed.

Let $I = [a, b]$ and by BVH denote the class of functions on $I \times I$ which are of bounded variation in the sense of Hardy-Krause. That is, $K \in BVH$ if there exists $(s_0, t_0) \in I \times I$ such that $K(s_0, t)$ and $K(s, t_0)$ are of bounded variation on I, and

$$\sup \sum_{i=1}^{m} \sum_{j=1}^{n} | K(s_i, t_j) - K(s_i, t_{j-1}) - K(s_{i-1}, t_j) + K(s_{i-1}, t_{j-1}) |$$

is finite where, of course, the supremum is taken over all grids of $I \times I$.

The Fredholm determinant for a kernel k in BVH evaluated at -1 is given by

$$D(K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_I \cdots \int_I \begin{vmatrix} K(s_1, s_1) & \cdots & K(s_1, s_n) \\ \cdots & & \cdots \\ K(s_n, s_1) & \cdots & K(s_n, s_n) \end{vmatrix} ds_1 \cdots ds_n \, .$$

THEOREM A (*Woodward*). *Let*

$$T(x)(t) = x(t) + \int_a^b \int_a^t K(u, v) du \, dx(v)$$

be a transformation defined on $C[a, b]$ and suppose that there exist functions M and J such that

$$K(s, t) = \begin{cases} M(s, t) & s < t \\ M(s, t) + J(s)/2 & s = t \\ M(s, t) + J(s) & s > t \end{cases}$$

where $M \in BVH$ on $I \times I$ and J is of bounded variation on I. Suppose further that $D(K) \neq 0$. Then the transformation T carries $C[a, b]$ onto $C[a, b]$ in a one-to-one fashion and if F is a Wiener integrable functional on $C[a, b]$ then

$$\int_{C[a,b]} F(x) dx = | D(K) | \int_{C[a,b]} F(T(x)) \exp\{-\alpha(x)/2\} dx \tag{3.1}$$

where

$$\alpha(x) = \int_I \int_I \left[K(s, t) + K(t, s) + \int_I K(u, s) K(u, t) du \right] dx(s) dx(t) \, . \tag{3.2}$$

Futhermore, if $F(T(x))$ is Wiener integrable then

$$(3.3)\qquad \int_{C[a,b]} F(T(x))dx = |D(K)|^{-1}\int_{C[a,b]} F(x)\exp\{-\beta(x)/2\}dx$$

where

$$(3.4)\qquad \beta(x) = \int_I\int_I\Big[\tilde{K}(s,t) + \tilde{K}(t,s) + \int_I \tilde{K}(u,s)\tilde{K}(u,t)du\Big]dx(s)dx(t)$$

and $\tilde{K}$ *is the Volterra reciprocal kernel of* K.

The behavior of the analytic Feynman integral under linear transformation is described in the following theorems.

THEOREM 1. *Let* Λ *and* Ω *be subsets of the* λ*-plane and* μ*-plane, respectively, such that for some positive* d *less than one*

$$\Lambda = \Big\{\lambda = \rho e^{i\theta} : |\rho - 1| < d\Big(1 - \frac{2}{\pi}|\theta|\Big),\ -\pi/2 < \theta < d\Big\}$$

and

$$\Omega = \Big\{\mu = re^{i\phi} : |r - 1| < d\Big(1 - \frac{2}{\pi}|\phi|\Big),\ -\pi/2 < \phi < d\Big\}.$$

Let

$$T(x)(t) = x(t) + \int_a^b\int_a^t K(u,v)du\,dx(v)$$

where $K(u,v)$ *satisfies the hypothesis of Theorem A and let* $\beta(x)$ *be defined as in* (3.4). *Then, if* F *is a functional on* $C[a,b]$ *satisfying the three conditions below, it follows that the integrals in* (3.5) *exist and that equality holds*:

$$(3.5)\qquad \int_{C[a,b]}^{anf} F(T(x))dx = |D(K)|^{-1}\int_{C[a,b]}^{anf} F(x)\exp\{i/2\beta(x)\}dx\,.$$

The conditions on the functional F *are*:

(1) *The integral*

$$(3.6)\qquad J(\lambda,\mu) = |D(K)|^{-1}\int_{C[a,b]}^{anw_\lambda} F(x)\exp\{-\mu/2\beta(x)\}dx$$

is bounded on all compact subsets of $\Lambda\times\Omega$ *and* $\lim_{\substack{\mu\to -i\\ \mu\in\Omega}} J(\lambda,\mu)$ *exists for all* $\lambda\in\Lambda$.

(2) *For* μ *in* Ω *and in some neighbourhood of* $\mu = -i$ *the integral*

$$|D(K)|^{-1}\int_{C[a,b]}^{anf} F(x)\exp\{-\mu/2\beta(x)\}dx$$

exists uniformly and approaches a finite as $\mu\to -i$ *inside* Ω.

(3) *$F(\lambda^{-\frac{1}{2}}x)$ is Wiener integrable for real λ in Λ.*

Proof. We will denote the subset of Ω described in the second condition on F by M. As a result of condition (2) there exists a $\delta > 0$ and a set S of the form (2.1) with $\lambda_0 = -i$ such that S is a subset of Λ and such that a function $\Phi(\lambda, \mu)$ can be defined on $S \times M$ satisfying (i) $\Phi(\lambda, \mu)$ is analytic in λ on the interior of S for every μ in M, (ii) $\Phi(\lambda, \mu)$ is uniformly continuous in $\lambda \in S$ uniformly for $\mu \in M$, (iii) for real $\lambda \in S$ and all $\mu \in M$

$$\Phi(\lambda, \mu) = |D(K)|^{-1} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\beta(\lambda^{-\frac{1}{2}}x)\}dx \,, \tag{3.7}$$

and (iv) $\lim_{\substack{\mu \to -i \\ \mu \in \Omega}} \Phi(-i, \mu)$ exists as a finite number.

From (3.6) and the monodromy theorem we have $J(\lambda, \mu)$ analytic on Λ for each $\mu \in \Omega$, and for real λ in Λ and μ in Ω we have

$$J(\lambda, \mu) = |D(K)|^{-1} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\beta(\lambda^{-\frac{1}{2}}x)\}dx \tag{3.8}$$

existing. Thus for real λ in Λ it follows from (3.8) that $J(\lambda, \mu)$ is analytic in μ on Ω. That is, let Γ be any rectifiable closed curve in Ω. Then there exists an $\varepsilon > 0$ such that $\varepsilon < d$ and $\mu \in \Gamma$ implies $0 < RE\mu < 1 + \varepsilon$. Hence (3.8) implies

$$|D(K)|^{-1} \int_{C[a,b]} |F(\lambda^{-\frac{1}{2}}x)| \exp\{-(1+\varepsilon)/2\beta(\lambda^{-\frac{1}{2}}x)\}dx \tag{3.9}$$

exists for all real λ in Λ, and since λ real implies

$$|\exp\{-\mu/2\beta(\lambda^{-\frac{1}{2}}x)\}| \leqq 1 + \exp\{-(1+\varepsilon)/2\beta(\lambda^{-\frac{1}{2}}x)\} \tag{3.10}$$

we have by applying (3.9), condition (3), and the dominated convergence theorem that $J(\lambda, \mu)$ is a continuous function of $\mu \in \Omega$ for $0 \leqq Re\mu < 1 + \varepsilon$. Moreover, for real λ in Λ

$$\begin{aligned}\int_\Gamma J(\lambda, \mu)d\mu &= \int_\Gamma |D(K)|^{-1} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\beta(\lambda^{-\frac{1}{2}}x)\}dxd\mu \\ &= |D(K)|^{-1} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \int_\Gamma \exp\{-\mu/2\beta(\lambda^{-\frac{1}{2}}x)\}d\mu dx \\ &= 0\end{aligned}$$

where the interchange of integration follows because of Fubini's theorem, (3.9), and (3.10). Since Γ was an arbitrary closed curve in Ω we have, by Morera's theorem, that $J(\lambda, \mu)$ is analytic for μ in Ω when λ is real.

Applying Lemma 2.1, since $J(\lambda, \mu)$ is bounded on compact subsets of $\Lambda \times \Omega$, we find $J(\lambda, \mu)$ analytic on $\Lambda \times \Omega$. Furthermore, by (3.7) and (3.8) we have that $J(\lambda, \mu) = \Phi(\lambda, \mu)$ for $\lambda \in \Lambda \cap S$ and $\mu \in M$. Hence $\Phi(\lambda, \mu)$ can be extended to be analytic on $D \times \Omega$ where $D = S - \{i\}$.

We now show that $\Phi(\lambda, -i) = \lim_{\substack{\mu \to -i \\ \mu \in \Omega}} \Phi(\lambda, \mu)$ is analytic in D and continuous on S. By condition (1) (since $J(\lambda, \mu) = \Phi(\lambda, \mu)$) we know $\Phi(\lambda, -i)$ exists for $\lambda \in S$, and by condition (2) we have $\Phi(\lambda, \mu)$ uniformly continuous for λ in S uniformly in $\mu \in M$. Let $\varepsilon > 0$ be given. Choose $\gamma > 0$ such that $\lambda, \tilde{\lambda} \in S$ and $|\lambda - \tilde{\lambda}| < \gamma$ imply $|\Phi(\lambda, \mu) - \Phi(\tilde{\lambda}, \mu)| < \varepsilon/3$ for all μ in M. Let $\lambda_1, \cdots, \lambda_n$ be points in S such that for every $\lambda \in S$ we have $\min\{|\lambda - \lambda_k|: k = 1, \cdots, n\} < \gamma$. Then for $\mu, \mu_0 \in M$ we have for all $\lambda \in S$

$$|\Phi(\lambda, \mu) - \Phi(\lambda, \mu_0)| \leqq |\Phi(\lambda, \mu) - \Phi(\lambda_k, \mu)| + |\Phi(\lambda_k, \mu) - \Phi(\lambda_k, \mu_0)| + |\Phi(\lambda_k, \mu_0) - \Phi(\lambda, \mu_0)|$$

where λ_k is such that $|\lambda - \lambda_k| < \gamma$. Hence for $\mu, \mu_0 \in M$

$$\sup_{\lambda \in S} |\Phi(\lambda, \mu) - \Phi(\lambda, \mu_0)| \leqq 2\varepsilon/3 + \sup_{1 \leqq k \leqq n} |\Phi(\lambda_k, \mu) - \Phi(\lambda_k, \mu_0)| .$$

However, since $\lim_{\substack{\mu \to -i \\ \mu \in S}} \Phi(\lambda, \mu)$ exists for all $\lambda \in S$ and $\varepsilon > 0$ was arbitrary we have that the convergence is uniform in λ as $\mu \to -i$ within Ω. Since each $\Phi(\lambda, \mu)$, $\mu \in M$, is analytic in D and continuous in S we have $\Phi(\lambda, -i)$ with analogous properties.

Using Theorem A, condition (1), and that $T(x)$ is linear we have for real $\lambda \in S$ that $F(T(\lambda^{-\frac{1}{2}}x)) = F(\lambda^{-\frac{1}{2}}T(x))$ is Wiener integrable and

$$(3.11) \quad \int_{C[a,b]} F(T(\lambda^{-\frac{1}{2}}x))dx = |D(K)|^{-1} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \exp\left\{-\frac{1}{2}\beta(x)\right\}dx = \Phi(\lambda, \lambda).$$

Now $\Phi(\lambda, \lambda)$ is analytic on D and since $\lim_{\substack{\lambda \to -i \\ \lambda \in M}} \Phi(\lambda, \mu)$ exists uniformly for $\mu \in M$ and $\lim_{\substack{\mu \to -i \\ \mu \in M}} \Phi(\lambda, \mu)$ exists for $\lambda \in S$ we have

$$\Phi(-i, -i) = \lim_{(\lambda, \mu) \to (-i, -i)} \Phi(\lambda, \mu)$$

existing where, of course, the limit is taken over $S \times M$. Thus $\Phi(\lambda, \lambda)$ is continuous on S and

$$(3.12) \quad \int_{C[a,b]}^{anf} F(T(x))dx = \Phi(-i, -i)$$

exists.

Now $\Phi(\lambda, -i)$ is continuous on S, analytic on D, and for real $\lambda \in S$

$$(3.13)\qquad \begin{aligned}\Phi(\lambda, -i) &= \lim_{\substack{\mu\to -i\\ \mu\in M}} \Phi(\lambda, \mu)\\ &= \lim_{\substack{\mu\to -i\\ \mu\in M}} |D(K)|^{-1}\int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x)\exp\{-\mu/2\beta(\lambda^{-\frac{1}{2}}x)\}dx\,.\end{aligned}$$

Thus since $F(\lambda^{-\frac{1}{2}}x)$ and $F(\lambda^{-\frac{1}{2}}x)\exp\{-1/2\beta(^{-\frac{1}{2}}x)\}$ are both Wiener integrable it follows that the dominated convergence theorem applies to the last limit in (3.13). Hence for real $\lambda\in S$ we have

$$\Phi(\lambda, -i) = |D(K)|^{-1}\int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x)\exp\{i/2\beta(\lambda^{-\frac{1}{2}}x)\}dx$$

and using the properties of $\Phi(\lambda, -i)$ we see

$$(3.14)\qquad |D(K)|^{-1}\int_{C[a,b]}^{anf} F(x)\exp\{i/2\beta(x)\} = \Phi(-i, -i)\,.$$

Combining (3.12) and (3.14) the theorem follows.

THEOREM 2. *Let Λ and Ω be as defined in Theorem* 1, *let*

$$T(x)(t) = x(t) + \int_a^b\int_a^t K(u, v)dudx(v)$$

where $K(u, v)$ satisfies the hypothesis of Theorem A, and let $\alpha(x)$ be defined as in (3.2). *Then, if F is a functional on $C[a, b]$ satisfying the three conditions below, it follows that the integrals in* (3.15) *exist and that equality holds*:

$$(3.15)\qquad \int_{C[a,b]}^{anf} F(x)dx = |D(K)|\int_{C[a,b]}^{anf} F(T(x))\exp\{i/2\alpha(x)\}dx\,.$$

The conditions on the functional F are:

(1) *The integral*

$$(3.16)\qquad J(\lambda, \mu) = |D(K)|\int_{C[a,b]}^{anw_\lambda} F(T(x))\exp\{-\mu/2\alpha(x)\}dx$$

is bounded on all compact subsets of $\Lambda\times\Omega$ and $\lim_{\substack{\mu\to -i\\ \mu\in\Omega}} J(\lambda, \mu)$ exists for all $\lambda\in\Lambda$.

(2) *For μ in Ω and in some neighborhood of $\mu = -i$ the integral*

$$|D(K)|\int_{C[a,b]}^{anf} F(T(x))\exp\{-\mu/2\alpha(x)\}dx$$

exists uniformly and approaches a finite limit as $\mu\to -i$ inside Ω.

(3) *$F(\lambda^{-\frac{1}{2}}T(x))$ is Wiener integrable for real λ in Λ.*

Proof. The proof follows the argument used to prove Theorem 1 very closely and will not be produced here.

Sufficient conditions on a functional $F(x)$ which assure that the three conditions of Theorem 1 are satisfied will be given in the next theorem.

THEOREM 3. *Let Λ and Ω be defined as in Theorem 1, let $D_1 = \Lambda \cup \{-i\}$, $D_2 = \Omega \cup \{-i\}$, and suppose $F(\cdot)$ is a functional defined on $C[a, b]$ such that:*

(i) *$F(\lambda^{-\frac{1}{2}}x)$ is an analytic function of λ in Λ and continuous in D_1 for each $x \in C[a, b]$.*

(ii) *$F(\lambda^{-\frac{1}{2}}x)$ is Wiener measurable for $\lambda \in D_1$ and Wiener integrable for real $\lambda \in \Lambda$.*

(iii) *the functional*

$$\sup_{(\lambda,\mu) \in D_1 \times D_2} | F(\lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\lambda\beta(x)\} | \tag{3.17}$$

is Wiener integrable where $\beta(x)$ is as given in Theorem 1. Then $F(x)$ satisfies conditions $1 - 3$ of Theorem 1 with Λ and Ω defined in terms of $d' < d$ instead of d.

Proof. Since $\exp\{-\mu/2\lambda\beta(x)\}$ is a measurable functional on $C[a, b]$ and by condition (ii) $F(\lambda^{-\frac{1}{2}}x)$ is measurable for $\lambda \in D_1$ we have by applying the dominated convergence theorem and (iii) that

$$F(\lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\lambda\beta(x)\}$$

is Wiener integrable on $(\lambda, \mu) \in D_1 \times D_2$. Further, by (iii)

$$\Phi(\lambda, \mu) = | D(K) |^{-1} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\lambda\beta(x)\} dx$$

is defined, bounded, and jointly continuous for $(\lambda, \mu) \in D_1 \times C_2$. Using Morera's theorem as in Theorem 1 we can verify that $\Phi(\lambda, \mu)$ is an analytic function of $\lambda \in \Lambda$ for each $\mu \in D_2$ Combining this with the joint continuity on $D_1 \times D_2$ and (ii) we obtain (1), (2), and (3) of Theorem 1 with Λ and Ω defined in terms of $d' < d$ instead of d.

4. In [1] a generalized analytic Feynman integral is defined for Gaussian Markov processes. Applying our linear transformation theorem we will evaluate a certain class of these integrals in terms of analytic Feynman integrals based on the Wiener integral as given in Definition 2.1. Furthermore, in Theorem 5 we are able to show that certain analytic Feynman integrals are solutions of a "generalized Schroedinger equation."

Let μ_R denote the Gaussian measure on the Borel subsets of $C[a, b]$ with mean function identically zero and covariance function $R(s, t)$ given by

$$R(s, t) = \begin{cases} u(s)v(t) & s \leqq t \\ u(t)v(s) & s \geqq t \end{cases} \tag{4.1}$$

where

$$u(a) = 0 , \tag{4.2}$$

$$v(t) > 0 \qquad \text{on } [a, b] , \tag{4.3}$$

$$u'' \text{ and } v'' \text{ exist and are continuous on } [a, b] , \tag{4.4}$$

$$v(t)u'(t) - u(t)v'(t) > 0 \qquad \text{on } [a, b] . \tag{4.5}$$

The integral of a functional $F(x)$ on $C[a, b]$ with respect to the measure μ_R will be denoted by $\int_{C[a,b]} F(x)d'_R x$ and as before, $\int_{C[a,b]} F(x)dx$ represents the Wiener integral.

Then by [7, p. 272] we know that μ_R is absolutely continuous with respect to Wiener measure on $C[a, b]$ if and only if

$$v(t)u'(t) - u(t)v'(t) = 1 \qquad \text{on } [a, b] . \tag{4.6}$$

Furthermore, the results of [7] assure us that if (4.6) holds and if $F(x)$ is a measurable functional on $C[a, b]$ such that $F(T(x))$ is Wiener integrable where

$$T(x)(\cdot) = x(\cdot) + \int_a^b \int_a^{\cdot} K(s, t)\, dsdx(t) \tag{4.7}$$

and

$$K(s, t) = \begin{cases} v'(s)/v(t) & s > t \\ v'(s)/2v(s) & s = t , \\ 0 & s < t \end{cases} \tag{4.8}$$

then

$$\begin{aligned} \int_{C[a,b]} F(x)d_R x &= \int_{C[a,b]} F(T(x))dx \\ &= \left[\frac{v(a)}{v(b)}\right]^{\frac{1}{2}} \int_{C[a,b]} F(x) \exp\left\{\frac{1}{2}\int_a^b v'(t)d[x^2(t)/v(t)]\right\} dx . \end{aligned} \tag{4.9}$$

An immediate consequence of (4.9) is that μ_R is induced by the linear transformation (4.7). That is, for all measurable subsets E of $C[a, b]$

$$\mu_R(E) = \mu_W(T(E)) \tag{4.10}$$

where μ_W is, of course, Wiener measure.

The following definition parallels Definition 2.1 and is as given in [1].

DEFINITION 4.1. Let μ_R be a Gaussian measure on $C[a, b]$ with mean function identically zero and covariance function as in (4.1) where $u(t)$ and $v(t)$ satisfy (4.2), (4.3), (4.4) and (4.5). Let the complex number λ_0 satisfy $Re\lambda_0 \geqq 0$ and $\lambda_0 \neq 0$ so that $\lambda_0 = |\lambda_0| \exp(i\theta)$ for some θ on the interval $[-\pi/2, \pi/2]$. Let $F(x)$ be a functional defined on $C[a, b]$ such that the Gaussian integral

$$J(\lambda) = \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x)d_R x$$

exists for all real λ in the interval $|\lambda_0| < |\lambda| < |\lambda_0| + \delta$, for some $\delta > 0$. Then if $J(\lambda)$ can be extended so that it is defined and continuous on the closed region S (as defined in (2.1)) and analytic in its interior, we define

$$\int_{C[a,b]}^{ang\,\lambda_0} F(x)d_R x = J(\lambda_0) \tag{4.11}$$

and we call the left member of (4.11) the analytic Gaussian integral of $F(x)$ with parameter λ_0. If $\theta = 0$ we interpret S to be the real interval $[\lambda_0, \lambda_0 + \delta]$, omit the analyticity requirement, since the interior of S is empty, and define the analytic Gaussian integral to be $J(\lambda_0^+)$. If $\lambda_0 = -i$ the integral in (4.11) will be called the generalized analytic Feynman integral and we write

$$\int_{C[a,b]}^{anf} F(x)d_R x = J(-i) .$$

The next theorem relates the analytic Feynman integral as given in Definition 2.1 and its generalization as defined above.

THEOREM 4. *Let μ_R be a Gaussian measure on $C[a, b]$ with mean function zero and covariance function as in* (4.1) *where $u(t)$ and $v(t)$ satisfy* (4.2) – (4.4) *and* (4.6). *Let $F(x)$ satisfy conditions* 1 – 3 *of Theorem* 1. *Then*

$$\int_{C[a,b]}^{anf} F(x)d_R x = \left[\frac{v(a)}{v(b)}\right]^{\frac{1}{2}} \int_{C[a,b]}^{anf} F(x) \exp\left\{-\frac{i}{2}\int_a^b v'(t)d[x^2(t)/v(t)]\right\}dx . \tag{4.12}$$

Proof. Let Λ be defined as in Theorem 1. Then by (4.9) and condition one of Theorem 1 we have for all real $\lambda \in \Lambda$ that

$$\begin{aligned} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x)d_R x &= \int_{C[a,b]} F(T(\lambda^{-\frac{1}{2}}x))dx \\ &= \left[\frac{v(a)}{v(b)}\right]^{\frac{1}{2}} \int_{C[a,b]} F(\lambda^{-\frac{1}{2}}x) \exp\left\{\frac{1}{2}\int_a^b v'(t)d[x^2(t)/v(t)]\right\}dx \qquad (4.13) \\ &= J(\lambda, \lambda) \end{aligned}$$

where $J(\lambda, \mu)$ is as defined in (3.6), $D(K) = [v(b)/v(a)]^{\frac{1}{2}}$, and $\beta(x) = -\int_a^b v'(t)d[x^2(t)/v(t)]$. Now in the proof of Theorem 1 under conditions 1-3 it is shown that $J(\lambda, \lambda)$ is an analytic function of $\lambda \in \Lambda$ which is continuous on a set S of the form (2.1) with $\lambda_0 = -i$ and $S \subseteqq \Lambda$. Hence by (4.13) we have $\int_{C[a,b]}^{anf} F(x)d_R x$ existing and

$$\int_{C[a,b]}^{anf} F(x)d_R x = \int_{C[a,b]}^{anf} F(T(x))dx \,. \tag{4.14}$$

Applying Theorem 1 we have (4.12) holding.

From (4.9) we see that if μ_R is the Gaussian measure on $C[a, b]$ as defined in Theorem 4 then the Radon-Nikodym derivative of μ_R with respect to Wiener measure, μ_w, is

$$\frac{d\mu_R}{d\mu_w} = \left[\frac{v(a)}{v(b)}\right]^{\frac{1}{2}} \exp\left\{\frac{1}{2}\int_a^b v'(t)d[x^2(t)/v(t)]\right\} . \tag{4.15}$$

Thus Theorem 4 confirms what heuristic considerations indicate. That is, we can interpret

$$\left[\frac{v(a)}{v(b)}\right]^{\frac{1}{2}} \exp\left\{-\frac{i}{2}\int_a^b v'(t)d[x^2(t)/v(t)]\right\} \tag{4.16}$$

as the derivative of the generalized analytic Feynman integral with respect to the analytic Feynman integral.

Now consider the generalized Schroedinger equation

$$\frac{i}{2}\frac{\partial^2 G}{\partial \xi^2} + \xi B(s)\frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial s} + \psi(\xi, s)G = 0 \tag{4.17}$$

for $(\xi, s) \in (-\infty, \infty) \times (a, b)$ with boundary condition

$$\lim_{s \to b} G(\xi, s) = \sigma(\xi) \tag{4.18}$$

for $-\infty < \xi < \infty$. We will show that given certain rather strong conditions on ψ, B, and σ the analytic Feynman integral of a certain functional will yield a solution of (4.17). This result differs from those in [3] or [4] in that (4.17) is a generalized Schroedinger equation and from those in [1] in that the solution is expressed in terms of an analytic Feynman integral involving the "derivative" given in (4.16) rather than a generalized analytic Feynman integral.

THEOREM 5. *Let $B(s)$ be real-valued and have continuous derivative on $[a, b]$ such that for some d, $0 < d < 1/2$, the functional*

$$\exp\left\{\left(\frac{1+d}{1-d}\right)^2\left[B(b)x^2(b) - \int_s^b x^2(t)[B^2(t) + B'(t)]dt\right]\right\} \tag{4.19}$$

is Wiener integrable on $C[s, b]$ *for* $a \leqq s \leqq b$. *Let*

$$F(\xi, s, x) = \exp\left\{\int_s^b \psi\left[x(r) + \xi\frac{v(r)}{v(s)}, r\right]dr\right\}\sigma\left[x(b) + \xi\frac{v(b)}{v(s)}\right] \tag{4.20}$$

where

(A) $$v(t) = \exp\left\{\int_a^t B(s)ds\right\}, \qquad a \leqq t \leqq b\ .$$

(B) *$\sigma(z)$ is an entire function whose first and second derivatives are integrable on $(-\infty, \infty)$ and such that $\sigma''(z)$ is of exponential type for real z.*

(C) *$\psi(z, t)$ is an entire function of z for all $t, a \leqq t \leqq b$, such that $Re(\psi(z, t))$ is bounded above uniformly in $t \in (a, t)$ for z in the infinite wedges*

$$\begin{gathered} -d \leqq \arg(z - \xi) \leqq \pi/4 \\ \pi - d \leqq \arg(z - \xi) \leqq 3\pi/4 \end{gathered} \tag{4.21}$$

for all real ξ. Further, we assume $\psi_t, \psi_z, \psi_{zz}$ are defined on $(-\infty, \infty) \times [a, b]$ with ψ_z and ψ_{zz} continuous there and such that ψ, ψ_z, ψ_{zz} are integrable on $(-\infty, \infty)$ with the integrals of their absolute values being bounded uniformly in $t, a \leqq t \leqq b$. We also assume ψ_{zz} is of exponential type in $z \in (-\infty, \infty)$ uniformly in $t \in [a, b]$. Then, if Λ is defined as in Theorem 1, $D_1 = \Lambda \cup \{-i\}$, and

$$\sup_{\lambda \in D_1}\left|\sigma\left[\lambda^{-\frac{1}{2}}x(b) + \xi\frac{v(b)}{v(a)}\right]\right|^2 \tag{4.22}$$

is Wiener integrable on $C[s, b]$, we have

$$G(\xi, s) = \left[\frac{v(s)}{v(b)}\right]^{\frac{1}{2}}\int_{C[a,b]}^{anf} F(\xi, s, x)\exp\left\{-\frac{i}{2}\int_s^b v'(t)d[x^2(t)/v(t)]\right\}dx \tag{4.23}$$

existing for $(\xi, s) \in (-\infty, \infty) \times (a, b)$. Furthermore, $G(\xi, s)$ satisfies the generalized Schroedinger equation (4.17) *with boundary condition as given in* (4.18).

Proof. Since

$$v(t) = \exp\left\{\int_a^t B(x)dx\right\}$$

we define

$$u(t) = v(t)\int_a^t [v(r)]^{-2}dr$$

for $a \leqq t \leqq b$. Then $u(t)$ and $v(t)$ satisfy (4.2), (4.3), (4.4) and (4.6). Further, if

$$U(p) = u(p) - u(s)v(p)/v(s)\,, \qquad a \leqq s \leqq p \leqq b\,, \tag{4.24}$$

and

$$R(p, g) = \begin{cases} U(p)v(g) & p \leqq g \\ U(g)v(p) & g \leqq p\,, \end{cases} \tag{4.25}$$

then $U(s) = 0$, $v(t) > 0$ on $[s, b]$, U'' and v'' exist and are continuous on $[s, b]$, and

$$v(t)U'(t) - U(t)v'(t) = 1 \qquad \text{on } [s, b]\,. \tag{4.26}$$

By μ_R we mean the Gaussian measure on $C[s, b]$ with mean function zero and covariance function as given in (4.25). Due to hypothesis (B) on $\sigma(z)$ and (C) on $\psi(z, t)$ Theorems 4.1 and 6 of [1] imply that

$$J(\xi, s) = \int_{C[a,b]}^{anf} F(\xi, s, x)d_R x \tag{4.27}$$

exists and satisfies (4.17) with the boundary condition as in (4.18). However, by (4.12) we have

$$J(\xi, s) = \left[\frac{v(s)}{v(b)}\right]^{\frac{1}{2}} \int_{C[s,b]}^{anf} F(\xi, s, x) \exp\left\{-\frac{i}{2}\int_s^b v'(t)d[x^2(t)/v(t)]\right\}dx \tag{4.28}$$

provided $F(\xi, s, x)$ satisfies condition 1–3 of Theorem 1. We use Theorem 3 for these purposes. First of all, since $\sigma(z)$ and $\psi(z, t)$ satisfy (B) and (C) it follows that conditions (i) and (ii) of Theorem 3 hold for $F(\xi, s, x)$. Now, in this case, for all $x \in C[s, b]$

$$\begin{aligned} \beta(x) &= -\int_s^b v'(t)d[x^2(t)/v(t)] \\ &= -B(b)x^2(b) + \int_s^b [B^2(t) + B'(t)]x^2(t)dt\,. \end{aligned} \tag{4.29}$$

Thus if $D_2 = \Omega \cup \{-i\}$, where Ω is defined as in Theorem 1, we have

$$\begin{aligned} &\sup_{(\lambda,\mu) \in D_1 \times D_2} |F(\xi, s, \lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\lambda\, \beta(x)\}| \\ &\leqq \sup_\lambda |F(\xi, s, \lambda^{-\frac{1}{2}}x)| \sup_{(\lambda,\mu)} |\exp\{-\mu/2\lambda\, \beta(x)\}|\,. \end{aligned} \tag{4.30}$$

However, by (4.22) and the hypothesis on ψ and σ we have

$$\sup_\lambda |F(\xi, s, \lambda^{-\frac{1}{2}}x)|^2$$

Wiener integrable. By (4.19) and (4.29) we also have

$$\sup_{(\lambda,\mu)} |\exp\{-\mu/\lambda\, \beta(x)\}|$$

Wiener integrable and hence by (4.30) we have

$$\sup_{(\lambda,\mu)\in D_1\times D_2} |F(\xi, s, \lambda^{-\frac{1}{2}}x) \exp\{-\mu/2\lambda\, \beta(x)\}|$$

Wiener integrable on $C[s, b]$ so (iii) of Theorem 3 holds and the proof is complete.

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Received November 27, 1967. Supported in part by NSF Grant GP-7181.

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Pacific Journal of Mathematics
Vol. 27, No. 2 February, 1968