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ON ABELIAN PSEUDO LATTICE ORDERED GROUPS

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Throughout this paper po-group will mean partially ordered abelian group. A subgroup H of a po-group G is an o -ideal if H is a convex, directed subgroup of G . A subgroup M of G is a value of $0 \neq g \in G$ if M is an o -ideal of G that is maximal without g . Let $\mathcal{M}(g) = \{M \subseteq G \mid M \text{ is a value of } g\}$ and $\mathcal{M}^*(g) = \bigcap \mathcal{M}(g)$. Two positive elements $a, b \in G$ are pseudo disjoint (p -disjoint) if $a \in \mathcal{M}^*(b)$ and $b \in \mathcal{M}^*(a)$, and G is a pseudo-lattice ordered group (pl -group) if each $g \in G$ can be written $g = a - b$ where a and b are p -disjoint.

The main result of §2 shows that every pl -group G is a Riesz group. That is, G is semiclosed ($ng \geq 0$ implies $g \geq 0$ for all $g \in G$ and all positive integers n), and G satisfies the Riesz interpolation property; if, whenever $x_1, \dots, x_m, y_1, \dots, y_n$ are elements of G and $x_i \leq y_j$ for $1 \leq i \leq m, 1 \leq j \leq n$, then there is an element $z \in G$ such that $x_i \leq z \leq y_j$.

In §3, we determine which Riesz groups are also pl -groups. In the final section it is shown that each pair of p -disjoint elements a, b determines an o -ideal $H(a, b)$ with the property that if $a - b = x - y$ where x and y are also p -disjoint, then $H(a, b) = H(x, y)$ and $a - x = b - y \in H(a, b)$.

The concept of a pl -group has been introduced by Conrad [1]. For each $g \in G$, $\mathcal{M}^*(g)$ exists by definition, and in particular, $\mathcal{M}^*(0) = G$. In §2 we list a number of properties of pl -groups that will be used. We adopt the notation $a \parallel b$ for $a \not\geq b$ and $b \not\geq a$. If S is a subset of a po-group G and $a \in G$, the notation $a > S$ means $a > s$ for all $s \in S$. If H is an o -ideal of a po-group G , a natural order is defined in G/H by setting $X \in G/H$ positive if X contains a positive element of G . All quotient structures will be ordered in this manner. Finally, $G^+ = \{x \in G \mid x \geq 0\}$.

2. Some properties of pl -groups. We first list a number of properties of pl -groups. The proofs of these may be found in [1]. If G is a pl -group, then

- (1) G is semiclosed.
- (2) G is directed.
- (3) The intersection of o -ideals of G is an o -ideal.

(4) If $g \in G$ and $M \in \mathcal{M}(g)$ and M' is the intersection of all o -ideals of G that properly contain M , then $g \in M'$, M'/M is o -isomorphic to a naturally ordered subgroup of the real numbers and, if $M < X \in G/M \setminus M'/M$, then $X > M'/M$.

(5) If K is an o -ideal of G , then K and G/K are pl -groups.

(6) If K is an o -ideal of G and $g \in G \setminus K$, then there is $M \in \mathcal{M}(g)$ such that $M \supseteq K$.

(7) If $g = a - b$ where a and b are p -disjoint, then $\mathcal{M}(g) = \mathcal{M}(a) \cup \mathcal{M}(b)$.

(8) A nonzero element $g \in G$ is positive if and only if $g + M > M$ for all $M \in \mathcal{M}(g)$.

(9) If a and b are p -disjoint and $g \leq a, g \leq b$, then $ng \leq a$ and $ng \leq b$ for all $n > 0$.

(10) If a and b are p -disjoint, then no value of a is comparable to a value of b .

The following set of propositions leads to the first theorem which states that every pl -group is a Riesz group.

(2.1) Let G be a po -group and $g \in G$. If $g = a - b$ where a and b are p -disjoint and $z \in G^+$ such that $z \geq g$, then each value of a is contained in a value of z , and if $a \geq z$, then z and $z - g$ are p -disjoint.

Proof. Let $M \in \mathcal{M}(a)$, then $b \in M$ and $z \geq g = a - b$ implies $z + b \geq a \geq 0$. Hence, $z \notin M$ and there is $M' \in \mathcal{M}(z)$ such that $M' \supseteq M$.

From $a \geq z \geq 0$ it follows that if $M \in \mathcal{M}(z)$, then $a \in M$. By the above, $M \in \mathcal{M}(a)$ so $b \in M$. Now $a \geq z \geq g$ implies $a - g = b \geq z - g \geq 0$ so $z - g \in M$. Similarly, if $M \in \mathcal{M}(z - g)$, then $b \in M$ so $M \in \mathcal{M}(b)$, $a \in M$ and hence, $z \in M$. Thus, z and $z - g$ are p -disjoint.

(2.2) If G is a po -group and $g = a - b = x - y$ where a and b are p -disjoint and x and y are positive, then for each

$$M \in \mathcal{M}(a)[M \in \mathcal{M}(b)]$$

there is $M' \in \mathcal{M}(x)[M' \in \mathcal{M}(y)]$ such that $M' \supseteq M$. In particular, if x and y are p -disjoint, $\mathcal{M}(a) = \mathcal{M}(x)$, $\mathcal{M}(b) = \mathcal{M}(y)$ and $a - x = b - y \in \mathcal{M}^*(g)$.

Proof. Let $g \in G$ and $g = a - b = x - y$ where a and b are p -disjoint and x and y are positive. Since $y \geq 0$, we have $x \geq g$ so for $M \in \mathcal{M}(a)$ there is, by (2.1), $M' \in \mathcal{M}(x)$ such that $M' \supseteq M$. Similarly for $M \in \mathcal{M}(b)$. If x and y are also p -disjoint then, by interchanging the roles of a and x, y and b we obtain $\mathcal{M}(a) = \mathcal{M}(x)$ and $\mathcal{M}(b) = \mathcal{M}(y)$. Thus, $b, y \in \mathcal{M}^*(a)$ and $a, x \in \mathcal{M}^*(b)$ so

$$a - x = b - y \in \mathcal{M}^*(a) \cap \mathcal{M}^*(b)$$

which is equal to $\mathcal{M}^*(g)$ by property (7).

(2.3) Suppose G is a pl -group, $g \in G$, $g = a - b$ where a and b are p -disjoint and $z \in G^+$ such that $z \geq g$. If $M \in \mathcal{M}(a - z)$, then either $M \in \mathcal{M}(z)$ and $z + M > a + M$ or M is properly contained in a value of a .

Proof. If $M \in \mathcal{M}(a - z)$, then by (4),

$$a + M > z + M \quad \text{or} \quad a + M < z + M.$$

For $M \in \mathcal{M}(z)$ and $M \in \mathcal{M}(a)$, it follows that $z + M > M$ and, from (2.1), that $a \in M$. Hence, $z + M > M = a + M$. For $M \in \mathcal{M}(z)$ and $M \in \mathcal{M}(a)$, we have $a + M = g + M \leq z + M$ so $a + M < z + M$. Now if $M \notin \mathcal{M}(z)$, then $a \notin M$ so there is $M' \in \mathcal{M}(a)$ such that $M' \supseteq M$. If $M' = M$, then M is properly contained in $M'' \in \mathcal{M}(z)$ so a and $a - z$ are in M'' and $z \in M''$, a contradiction. Thus M' properly contains M .

LEMMA 2.1. *If G is a pl -group, $g \in G$ and $z \in G^+$ such that $z \geq g$, then there is $x \in G^+$ such that $z \geq x$ and $x, x - g$ are p -disjoint. Moreover, if $g = a - b$, with a and b p -disjoint, then there exists such an x with $a \geq x$.*

Proof. Let G be a pl -group and $g \in G$. Then $g = a - b$ where a and b are p -disjoint. If $z \in G^+$ and $g \leq z$, take $x = a$ if $z \geq a$; and take $x = z$ if $z < a$. The result follows from (2.1).

If $z - a \parallel 0$, then $z - a = p - q$ where p and q are p -disjoint. We first show $\mathcal{M}(q) = \{M \in \mathcal{M}(z - a) \mid z + M < a + M\}$. Let $M \in \mathcal{M}(q)$, then $M \in \mathcal{M}(z - a)$ and $(z - a) + M = -q + M < M$ so $z + M < a + M$. Conversely, if $M \in \mathcal{M}(z - a)$ and $z + M < a + M$, then $M \in \mathcal{M}(p)$ or $M \in \mathcal{M}(q)$. If $M \in \mathcal{M}(p)$, then $q \in M$ so $(z - a) + M = p + M > M$. This implies $z + M > a + M$, a contradiction. Thus, $M \in \mathcal{M}(q)$.

Now let $x = a - q = z - p$, then $x < a$ and $x < z$. If $M \in \mathcal{M}(x)$, then $q \in M$. For if $q \notin M$, then $M \subseteq M' \in \mathcal{M}(q)$, $M' \in \mathcal{M}(z - a)$ and $z + M' < a + M'$. By (2.3), M' is properly contained in $M'' \in \mathcal{M}(a)$. Thus, $x \in M''$, $q \in M''$ so $a \in M''$ a contradiction. Therefore, $q \in M$ and hence $a \notin M$. We now have $M \neq a + M = x + q + M$ so $M < a + M = x + M$ for all $M \in \mathcal{M}(x)$. By (8), $x \geq 0$.

To complete the proof we need only show $x \geq g$, for then the result follows by (2.1). To accomplish this we show $(b - q) + M > M$ for all $M \in \mathcal{M}(b - q)$. Thus, let $M \in \mathcal{M}(b - q)$. If $M \in \mathcal{M}(q)$, then $M \in \mathcal{M}(z - a)$ and $z + M < a + M$, so $b \notin M$. By (2.3) and (10) there must exist $M' \in \mathcal{M}(b)$ such that M' properly contains M . But M' properly containing M implies $b - q, q$ and hence $b \in M'$, a contradiction. Thus, $M \notin \mathcal{M}(q)$.

Now since $b \in M$, there is $M'' \in \mathcal{M}(b)$ such that $M'' \supseteq M$. If

$M'' \neq M$, then $b - q \in M''$ so $M'' < b + M'' = q + M''$ and $q \notin M''$. By (2.3), every value of q is contained in a value of a so M'' is contained in a value of a , a contradiction. Thus $M'' = M \in \mathcal{M}(b)$, and as above, it follows that $q \in M$. Consequently, $b - q + M = b + M > M$ so by (8), $b > q$ and $x > g$. This completes the proof.

With Lemma 2.1 we are now able to prove the following.

THEOREM 2.1. *Every pl -group is a Riesz group.*

Proof. Since by (1), a pl -group is semiclosed, we need only show a pl -group G satisfies the Riesz interpolation property. Without loss of generality, we may assume, $g, u, z \in G$ and $u \geq 0, z \geq 0, u \geq g, z \geq g$. There exists, by Lemma 2.1, an element $a \in G^+$ such that $u \geq a$ with $a, a - g$ p -disjoint. Also, there is $x \in G^+$ such that $a \geq x, z \geq x$ with $x, x - g$ p -disjoint. Hence, $u \geq x \geq 0, z \geq x \geq g$ and G is a Riesz group.

We note that the above theorem and Theorem 4.8 in [1] answer affirmatively the open question posed at the end of [2].

3. Sufficient conditions for pseudo-lattice ordering. As a consequence of § 2 we have that every pl -group G is a Riesz group that satisfies

(*) for each $g \in G$, there is $a \in G^+$ such that $g \leq a$ and whenever $0 \leq x, g \leq x$ then $a \leq x + h$ for some $h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$.

To see this let $g \in G$, then g can be written $g = a - b$ where a and b are p -disjoint, so $a \in G^+$ and $g \leq a$. If $x \in G^+$ and $x \geq g$, then, since G is a Riesz group, there is $u \in G$ such that $a \geq u \geq 0$ and $x \geq u \geq g$. By (2.1), u and $u - g$ are p -disjoint and by (2.2) and (7), $a - u \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$. By setting $a - u = h$ we have $u = a - h$ so $x \geq u = a - h$ which implies $x + h \geq a$.

In this section we show that every Riesz group that satisfies (*) is a pl -group. For the remainder of this section we assume G is a Riesz group that satisfies (*).

LEMMA 3.1. *The intersection of o -ideals of G is again an o -ideal.*

Proof. Let $M_\alpha, \alpha \in J$ be o -ideals of G and $M = \bigcap_{\alpha \in J} M_\alpha$. Clearly, M is a convex subgroup of G . To show M is directed let $g \in M$. By (*) there is $a \in G$ such that $0 \leq a, g \leq a$. Now for each $\alpha \in J, M_\alpha$ is directed so M_α is a Riesz group. Thus, there are elements $y_\alpha \in M_\alpha, x_\alpha \in G$ such that $y_\alpha \geq 0, y_\alpha \geq g, a \geq x_\alpha \geq g$ and $y_\alpha \geq x_\alpha \geq 0$. Thus, $x_\alpha \in M_\alpha$ and $a \leq x_\alpha + h_\alpha$ for some $h_\alpha \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$.

Now $x_\alpha \in M_\alpha$ and $x_\alpha + h_\alpha \geq a \geq x_\alpha$ implies $a - x_\alpha \in \mathcal{M}^*(a)$. Thus, if $a \notin M_\alpha$ then there is $M' \in \mathcal{M}(a)$ such that $M' \supseteq M_\alpha$. But then $x_\alpha,$

$a - x_\alpha$ and hence $a \in M'$, a contradiction. Thus $a \in M_\alpha$ for all α , M is directed and M is an o -ideal of G .

We note that in the above we have proved that if a satisfies (*) for g and $a \geq x \geq 0, x \geq g$ then $a - x \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$.

LEMMA 3.2. *If M is an o -ideal of G , then M and G/M are Riesz groups satisfying (*).*

Proof. If M is an o -ideal of G , then M and G/M are Riesz groups by [2, p. 1393]. If $g \in M$, then let $a \in G$ such that a satisfies (*) for g . There then are elements $m \in M^+$ and $x \in G$ such that $m \geq g, a \geq x \geq g$ and $m \geq x \geq 0$, which implies $x \in M$ and $a - x \in \mathcal{M}^*(a)$. As a consequence of this latter part, $a \in M$. Now if $0 \leq y \in M$ and $g \leq y$ then there is $u \in M$ such that $y \geq u \geq 0, a \geq u \geq g$. Thus, by the remark preceding this lemma, $u = a + h$ where

$$h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$$

and hence $u - a = h \in M$. By Lemma 3.1, every o -ideal M' of M that is maximal without a [$a - g$] can be written $M' = M \cap \bar{M}$ where $\bar{M} \in \mathcal{M}(a)[\bar{M} \in \mathcal{M}(a - g)]$. Thus, it follows that h belongs to every value of a and every value of $a - g$ in M and M satisfies (*).

Now let $g + M \in G/M$, and let $a \in G$ such that a satisfies (*) for g . Then $a + M \geq M$ and $a + M \geq g + M$. If $M \leq x + M \in G/M$ and $x + M \geq g + M$, then there are elements $m_1, m_2 \in M$ such that $x + m_1 \geq 0$ and $x + m_2 \geq g$. Since M is directed, there is $m \in M$ such that $m \geq m_1, m \geq m_2$.

By (*), $a \leq (x + m) + h$ so $a + M \leq (x + M) + (h + M)$ where

$$h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g).$$

Now let X be a value of $a + M$ in G/M . Then $X = M'/M$ where M' is an o -ideal of G and $a \notin M'$. It follows that $M' \in \mathcal{M}(a)$ so $h \in M'$ and $h + M \in X$. In a similar manner, $h + M$ belongs to every value of $(a - g) + M$ in G/M . The proof is complete.

LEMMA 3.3. *Let H be the intersection of all nonzero o -ideals of G . If $x \in H^+, g \in G \setminus H$ and $g < x$, then $g < 0$.*

Proof. Suppose H is the intersection of all nonzero o -ideals of G . If $x \in H^+, g \in G \setminus H$ and $g < x$, then $a \leq x + h$ where a satisfies (*) for g and $h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$. If $a \neq 0$ and $M \in \mathcal{M}(a)$, then $M \neq 0$ so $H \subseteq M$ and $x + h \in M$. This implies $a \in M$ since $0 \leq a \leq x + h$, a contradiction. Thus, $a = 0$ and $g < 0$.

COROLLARY. *If H is the intersection of all nonzero o -ideals of G , then every positive element of $G \setminus H$ exceeds every element of H .*

Proof. Let $0 < g \in G \setminus H$ and $h \in H$. By Lemma 3.1, H is an o -ideal of G so there is $h' \in H^+$ such that $h' \geq h$. Now $h' - g \in G \setminus H$ and $h' - g < h'$ so $h' - g < 0$, $h \leq h' < g$ and the corollary follows.

As a final observation before we turn to the main proof of this section, we note that if G has no proper o -ideals then G is a subgroup of the naturally ordered real numbers. This is a special case of 4.6 in [1].

THEOREM 3.1. *A Riesz group G is a pl -group if and only if G satisfies.*

(*) *for each $g \in G$, there is $a \in G^+$ such that $g \leq a$ and whenever $0 \leq x$, $g \leq x$ then $a \leq x + h$ for some $h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$.*

Proof. Let $g \in G$ and a satisfy (*) for g . We show a and $a - g$ are p -disjoint. If $a = 0$ or $a = g$, the result easily follows so we assume $g \parallel 0$. Let $M \in \mathcal{M}(a)$ and let M' be the intersection of all o -ideals of G that properly contain M . Then M' is an o -ideal of G , $a \in M'$, M'/M is o -isomorphic to a subgroup of the naturally ordered real numbers and if $M < X \in (G/M) \setminus (M'/M)$, then $X > M'/M$.

If $(a - g) + M \geq a + M$, then there is $m \in M^+$ such that $a - g + m \geq a$, so $m \geq g$. By (*), $0 < a \leq m + h$ where $h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$.

Thus, $m + h \in M$ and $a \in M$, a contradiction. Since $(a - g) + M$ is comparable to $a + M$, we must have $(a - g) + M < a + M$, so there is $m \in M$ such that $a > (a - g) + m$. Let $m' \in M$ such that $m' < m$, $m' < 0$, then $g - m' > g$ and $g - m' > 0$. Thus, by (*), $a \leq (g - m') + h'$ where $h' \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$, and $0 < a - g \leq -m' + h' \in M$. By convexity $a - g \in M$ so $a - g \in \mathcal{M}^*(a)$.

By interchanging the roles of a and $a - g$ in the above we are led to the conclusion that $a + M < (a - g) + M$ where $M \in \mathcal{M}(a - g)$. There then is $m \in M^+$ such that $a < (a - g) + m$ so $g < m$. As always, $a \leq m + h$ with $h \in \mathcal{M}^*(a) \cap \mathcal{M}^*(a - g)$ so $a \in M$. Thus, a and $a - g$ are p -disjoint and G is a pl -group.

The necessity follows from the remarks at the beginning of this section.

4. Pseudo-disjoint elements. Throughout this section we assume G is a pl -group. We have shown if $g \in G$ and $g = a - b = x - y$ where a, b and x, y are pairs of p -disjoint elements then

$$a - x = b - y \in \mathcal{M}^*(a) \cap \mathcal{M}^*(b).$$

However, the converse of this is not true. For if $K = R_1 + R_2 + R_3$ (the cardinal sum) where each R_i is the real numbers, $i = 1, 2, 3$; then K is an l -group so, of course, a pl -group. Clearly, $(1, -1, 0) = (1, 0, 0) - (0, 1, 0)$ where $(1, 0, 0), (0, 1, 0)$ are p -disjoint. Now $(1, 0, 0)$ has exactly one value namely $M_1 = R_2 + R_3$ and $(0, 1, 0)$ has the value $M_2 = R_1 + R_3$. Thus, $R_3 = M_1 \cap M_2$ and if $0 \neq h \in R_3$ it is clear that $(1, 0, 0) + (0, 0, h) = (1, 0, h)$ and $(0, 1, 0) + (0, 0, h) = (0, 1, h)$ are not p -disjoint but $(1, -1, 0) = (1, 0, h) - (0, 1, h)$.

We now show how pairs of p -disjoint elements a, b and x, y are related, when $g = a - b = x - y$. Assume a and b are p -disjoint and let $K = \{0 \leq m \in G \mid m \leq a, m \leq b\}$. Clearly, K is convex. If $m_1, m_2 \in K$, then by the Riesz interpolation property, there is an element $m \in G$ such that $m_1 \leq m \leq a$ and $m_2 \leq m \leq b$. Moreover, $2m \geq m_1 + m_2 \geq 0$ and by (9), $2m \leq a, 2m \leq b$ since a and b are p -disjoint. Thus, $2m \in K$ so $m_1 + m_2 \in K$ and K is a convex subsemigroup of G^+ that contains 0. Let H be the o -ideal of G that is generated by K . It is well known that $H^+ = K$ and any $x \in H$ can be written $x = h_1 - h_2$ where $h_1, h_2 \in K$. Thus $H < a$ and $H < b$. We denote by $H(a, b)$ the o -ideal generated by $\{0 \leq m \in G \mid m \leq a, m \leq b\}$ for p -disjoint elements a, b .

LEMMA 4.1. *If a and b are p -disjoint and $m \in H(a, b)$, then $\mathcal{M}(a) = \mathcal{M}(a + m)$ and $\mathcal{M}(b) = \mathcal{M}(b + m)$.*

Proof. We first consider $0 \leq m \in H(a, b)$. Since $a \geq a - m \geq 0$ and $a - m \geq a - b$ (2.1) implies $a - m$ and $b - m$ are p -disjoint, so $\mathcal{M}(a) = \mathcal{M}(a - m)$, $\mathcal{M}(b) = \mathcal{M}(b - m)$ by (2.2).

If $M \in \mathcal{M}(a + m)$, then $a - m \notin M$ so there is $M' \supseteq M$ such that $M' \in \mathcal{M}(a - m) = \mathcal{M}(a)$. Since $0 \leq m \leq b \in M'$, $m \in M'$ so $M = M' \in \mathcal{M}(a)$. Conversely, if $M \in \mathcal{M}(a)$ then $0 \leq m \leq b \in M$ implies $m \in M$ so $a + m \notin M$ and $M \in \mathcal{M}(a + m)$. Hence, $\mathcal{M}(a) = \mathcal{M}(a + m)$. Similarly, $\mathcal{M}(b) = \mathcal{M}(b + m)$.

For an arbitrary element $m \in H(a, b)$ there are elements $m_1, m_2 \in H(a, b)$ such that $m_1 \leq 0$ and $m_1 \leq m, 0 \leq m_2 \leq a, m \leq m_2 \leq b$. Hence, $0 \leq a + m_1 \leq a + m$ and $0 \leq a + m \leq a + m_2$. By the above, $\mathcal{M}(a) = \mathcal{M}(a + m_1) = \mathcal{M}(a + m_2)$. If $M \in \mathcal{M}(a + m)$, then $a + m_2 \notin M$ so $M \in \mathcal{M}(a + m_2) = \mathcal{M}(a)$. Conversely, if $M \in \mathcal{M}(a)$, then $m \in M$ and $M \in \mathcal{M}(a + m_1)$ so $a + m \notin M$ and $M \in \mathcal{M}(a + m)$. Thus, for any $m \in H(a, b)$, $\mathcal{M}(a) = \mathcal{M}(a + m)$. In a similar manner $\mathcal{M}(b) = \mathcal{M}(b + m)$.

We note at this point that if $0 \leq m \in H(a, b)$, then $0 \leq m \leq a$ implies $m \in \mathcal{M}^*(b)$ and $0 \leq m \leq b$ implies $m \in \mathcal{M}^*(a)$. Consequently, $H(a, b) \subset \mathcal{M}^*(a) \cap \mathcal{M}^*(b)$.

LEMMA 4.2. *If a and b are p -disjoint in G , then $a + m$ and*

$b + m$ are p -disjoint if and only if $m \in H(a, b)$.

Proof. Let a and b be p -disjoint and $m \in H(a, b)$, since $\mathcal{M}(a) = \mathcal{M}(a + m)$, b, m and hence $b + m \in \mathcal{M}^*(a + m)$. Dually, $a + m \in \mathcal{M}^*(b + m)$ so $a + m$ and $b + m$ are p -disjoint.

Conversely, if $a + m$ and $b + m$ are p -disjoint, then $a \geq -m$, $b \geq -m$ so there is $h \in G$ such that $a \geq h \geq 0$ and $b \geq h \geq -m$. This implies $h \in H(a, b)$. Since $\mathcal{M}(a) = \mathcal{M}(a + m)$ and $\mathcal{M}(b) = \mathcal{M}(b + m)$ we have $m \in \mathcal{M}^*(a) \cap \mathcal{M}^*(b)$. Now if $M \in \mathcal{M}(a - m)$ and $a + m \in M$, then $a \notin M$, so $M \in \mathcal{M}(a) = \mathcal{M}(a + m)$ and $a + m \notin M$, a contradiction. Thus, $a + m \notin M$ so $M \in \mathcal{M}(a + m) = \mathcal{M}(a)$, $a \notin M$, $b \in M$. Therefore $M < a + M = (a - m) + M$. By (8), $a - m > 0$. A similar argument shows $b > m$. Finally, by the Riesz interpolation property, there is an element $h' \in G$ such that $a \geq h' \geq 0$ and $b \geq h' \geq m$. Thus, $h' \in H(a, b)$ and we have $h' \geq m \geq -h$ so $m \in H(a, b)$.

COROLLARY. If a and b are p -disjoint in G , then $a \wedge b = 0$ if and only if $H(a, b) = 0$.

As a consequence of Lemma 4.2 we can associate with $g = a - b$, a and b p -disjoint, the o -ideal $H(a, b)$. Moreover, $H(a, b)$ depends only on g and is independent of the representation of g as the difference of p -disjoint elements. To show this, let $g = x - y$ where x and y are also p -disjoint. Then by (2.2) $\mathcal{M}(a) = \mathcal{M}(x)$ and $\mathcal{M}(b) = \mathcal{M}(y)$. If $0 \leq k \in H(x, y)$ then $k \in \mathcal{M}^*(a) \cap \mathcal{M}^*(b)$ and $a + k, b + k$ are p -disjoint so $k \in H(a, b)$ and $H(x, y) \subseteq H(a, b)$. Dually, we can show $H(a, b) \subseteq H(x, y)$ so $H(a, b) = H(x, y)$.

Using the above we can easily show a pl -group G satisfies

(**) for each $g \in G$, there is $a \in G^+$ such that $g \leq a$ and whenever $0 \leq x$, and $g \leq x$, then $a \leq x + h$ for some $h \in H(a, a - g)$.

To see this, let $g \in G$ and a satisfy (*) for g . If $0 \leq x, g \leq x$ there is $z \in G$ such that $a \geq z \geq 0$ and $x \geq z \geq g$ since every pl -group is a Riesz group. By (2.1), z and $z - g$ are p -disjoint and since $a = z + (a - z)$ and $a - g = (z - g) + (a - z)$ we have $a - z \in H(z, z - g) = H(a, a - g)$. Therefore, $x \geq z = a - (a - z)$ so $x + (a - z) \geq a$.

We have shown, that in a pl -group G , $H(a, b)$ is the o -ideal generated by $K = \{0 \leq m \in G \mid m \leq a, m \leq b\}$ for a and b p -disjoint, and $H(a, b)^+ = K$. If we now let $H(x, y)$ be the o -ideal generated by $K = \{0 \leq m \in G \mid m \leq x, m \leq y\}$ for arbitrary positive elements x and y , it may happen that $H(x, y)^+ \neq K$ and the following example shows (**) is not sufficient for a Riesz group G to be a pl -group.

Let R be the naturally ordered real numbers and $G = R + R$. Let $(u, v) \in G$ be positive if $v > 0$ or $v = 0$ and $u = 0$. Then G is a Riesz group but G is not a pl -group. If $g = (g_1, g_2) \in G$ and $g_2 > 0$

let $a = g$; if $g_2 < 0$ let $a = 0$. In either case $H(a, a - g) = 0$ and a satisfies (**) for g . If $g_2 = 0$ and $g_1 = 0$ take $a = 0$. If $g_2 = 0$ and $g_1 \neq 0$ let $a = (a_1, a_2)$ where $a_2 > 0$. Then $a > 0$, $a > g$ and $H(a, a - g) = G$. For any $b = (b_1, b_2) \geq (0, 0)$ and $(b_1, b_2) \geq (g_1, g_2)$ we must have $b_2 > 0$. If $h = (0, a_2)$, then $(a_1, a_2) < (b_1, b_2) + (0, a_2)$ and $h \in H(a, a - g)$. Thus (**) holds.

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