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AN EXTENDED FORM OF THE MEAN-ERGODIC THEOREM

LYNN CLIFFORD KURTZ AND DON HARRELL TUCKER

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Suppose X is a reflexive Banach space and V is a continuous linear operator in X such that $\|V^n\| \leq M$ for some $M(n=0, 1, 2, \dots)$. If N is the null space of $I - V$ and R is the closure of the range of $I - V$, then the mean-ergodic theorem states that

$$\lim_{n \rightarrow \infty} \frac{(I + V + \dots + V^{n-1})x}{n} = Px,$$

where P is the projection associated with N and R ; the convergence is in the norm of X . This is pointwise C_1 -summability of the sequence $\{V^k\}_{k=0}^\infty$ to P , and it suggests a similar theorem for more general Hausdorff summability methods. The purpose of this note is to demonstrate a wide class of operator-valued Hausdorff summability methods which contain the sequence $\{V^k\}_{k=0}^\infty$ in their wirkfelder and sum it to certain transforms of the projection operator P . This result shows much more clearly the sense in which convergence actually has meaning for such a sequence $\{V^k\}_{k=0}^\infty$.

Denote by $C(X)$ the space of X -valued continuous functions on $[0, 1]$ and by T_1 the bounded linear transformation from $C(X)$ into X given by $T_1(f) = \int_0^1 f(t)dt$. The mean-ergodic theorem states that

$$T_1\left(\sum_{k=0}^n \binom{n}{k} (t^k(1-t)^{n-k} V^k \cdot x)\right) \xrightarrow{n \rightarrow \infty} T_1(P \cdot x).$$

In this setting, the main theorem of this paper states a much stronger type of convergence; namely, that for any bounded linear operator T from $C(X)$ into a Banach space Y such that the generating function for T is continuous at 0 and 1, it is true that

$$T\left(\sum_{k=0}^n \binom{n}{k} t^k(1-t)^{n-k} V^k \cdot x\right) \xrightarrow{n \rightarrow \infty} T(P \cdot x).$$

In general one cannot expect much in the way of further relaxations on the operators T , i.e., on the functions which generate such operators. For example if the condition of continuity at 1 is removed, then this allows a generating function $K(t) = 0$ for $t < 1$, $K(1) = 1$ and this generates the Hausdorff method corresponding to ordinary convergence. In general the sequence $\{V^k \cdot x\}$ does not converge.

A nice presentation of the mean ergodic theorem as stated above is to be found in Lorch [2, pp. 54-56]. Suppose Y is a Banach space

and $\mu = \{\mu_k\}_{k=0}^\infty$ is a sequence of elements of $B[X, Y]$ such that the Hausdorff method $H = \rho\mu\rho$ generated by μ is regular relative to some $L \in B[X, Y]$. (See [1] for notation and terminology. Reference 8 in [1] is reference [3] of this paper.) It follows from [1] that there exists a function K on $[0, 1]$ with values in $B^+[X, Y]$ such that K satisfies the Gowurin ω -property,

$$K(0) = 0, K(1) = L \quad \text{and} \quad \mu_n = \int_0^1 dK(t) \cdot t^n \quad \text{for } n = 0, 1, 2, \dots$$

THEOREM. *If K is continuous at $t = 0$ and $t = 1$, then $\{V^k\}_{k=0}^\infty$ is pointwise H -summable to LP , i.e., $H_n\{V^k\} \cdot x$ converges in the norm of Y to LPx for each $x \in X$.*

The essential ingredient of the proof of the theorem is the following lemma.

LEMMA. *If $\{s_k\}_{k=0}^\infty$ is a bounded sequence of elements of a linear normed space S and $0 < a \leq t \leq b < 1$, then*

$$\left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (s_k - s_{k+1}) \right\|_S$$

converges uniformly to zero for $t \in [a, b]$.

*Proof of the lemma.*¹ Suppose $\|s_k\| \leq N'$ for $k = 0, 1, 2, \dots$, then set

$$\begin{aligned} A_n(t) &= \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (s_k - s_{k+1}) \\ &= \sum_{k=1}^n \left[\binom{n}{k} t^k (1-t)^{n-k} - \binom{n}{k-1} t^{k-1} (1-t)^{n-k+1} \right] s_k \\ &\quad + (1-t)^n s_0 - t^n s_{n+1} \\ &= \sum_{k=1}^n \binom{n}{k} t^k (1-t)^{n-k} \left[1 - \frac{k}{n-k+1} \cdot \frac{1-t}{t} \right] s_k \\ &\quad + (1-t)^n s_0 - t^n s_{n+1} \\ &= \frac{1}{t} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left[\frac{t - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right] s_k - t^n s_{n+1} . \\ \|A_n(t)\|_S &\leq \frac{1}{t} \left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left[\frac{t - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right] s_k \right\|_S + t^n \|s_{n+1}\|_S \end{aligned}$$

¹ The proof presented here is incorrect. See part 2 for a corrected proof.

where $0 < a \leq t \leq b < 1$, and hence

$$\|A_n(t)\|_S \leq \frac{N'}{t} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n} \cdot \frac{n}{n+1} \right|}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} + t^n \cdot N'.$$

Let $f_n(x, t)$ be given by

$$f_n(x, t) = \left| x - t \cdot \frac{n}{n+1} \right| / \left(1 - t \cdot \frac{n}{n+1} \right)$$

and $C_n(t)$ by

$$C_n(t) = \frac{1}{t} B_n[f_n(x, t)]|_{x=t}$$

where B_n denotes the n -th Bernstein polynomial. The above inequality may now be written

$$\|A_n(t)\|_S \leq N' |C_n(t)| + t^n \cdot N'$$

and the second term converges uniformly to zero for $t \in [a, b]$.

The first term is treated as follows. By a direct calculation it can be shown that for each $x \in [0, b]$, the collection $\{f_n(x, t)\}$ is equi-uniformly continuous in t for $t \in [0, b]$, that is to say, if $\varepsilon > 0$, then there exists $\delta > 0$ such that $|f_n(x, s) - f_n(x, t)| < \varepsilon/2$ for all $s, t \in [0, b]$ such that $|s - t| < \delta$ and for all n .

Consider a fixed $t \in [0, b]$ and set $A = \{k: |k/n - t| < \delta\}$ and $B = \{0, 1, \dots, n\} - A$. Then

$$\begin{aligned} & |B_n[f_n(x, t)] - f_n(x, t)| \\ & \leq \left(\sum_A + \sum_B \right) \left| \binom{n}{k} t^k (1-t)^{n-k} \left\{ \frac{x - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right\} - f_n(x, t) \right| \\ & \sum_A \leq \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

Set $Q = \max_{0 \leq t, x \leq b} f_n(x, t)$ for $n = 0, 1, 2, \dots$ and the second term can be treated as follows:

$$\sum_B \leq 2Q \sum_B \binom{n}{k} t^k (1-t)^{n-k} \frac{(k - nt)^2}{n^2 \delta^2} \leq \frac{2Q}{n^2 \delta^2} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (k - nt)^2$$

which, as is well known, converges uniformly to zero for $t \in [0, 1]$. Hence, there exists an integer N_0 such that $\sum_B < \varepsilon/2$ for $n > N_0$ and further such that $|B_n[f_n(x, t)] - f_n(x, t)| < \varepsilon$ for $n > N_0$, both

inequalities holding uniformly for $0 \leq t \leq b$. Collecting all these items together yields

$$\lim_{n \rightarrow \infty} C_n(t) = \frac{1}{t} \frac{|t - t|}{1 - t} = 0$$

uniformly for $t \in [a, b]$, and hence $\|A_n(t)\|_s \rightarrow 0$ uniformly on $[a, b]$.

Proof of the theorem. Let

$$T_n = H_n\{V^k\}_{k=0}^\infty = \sum_{k=0}^n \binom{n}{k} \Delta^{n-k} \mu_k V^k = \int_0^1 dK(t) \cdot \left[\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k \right].$$

Since N, R is a complementary pair in X , it is sufficient to investigate the behavior of T_n on each of these sets.

Suppose $f \in N$, i.e., $Vf = f$, then

$$\begin{aligned} T_n f &= \int_0^1 dK(t) \cdot \left[\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k f \right] = \int_0^1 dK(t) \cdot \left[\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} f \right] \\ &= \int_0^1 dK(t) \cdot f = [K(1) - K(0)]f = Lf = LPf. \end{aligned}$$

Now suppose $f \in R$ and $\varepsilon > 0$, then there exists g and h such that $f = g - Vg + h$ where $\|h\| < \varepsilon/4[1 + W_0^1 K]M$. For this f ,

$$\begin{aligned} T_n f &= \int_0^1 dK(t) \cdot \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \\ &\quad + \int_0^1 dK(t) \cdot \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k h = I + II. \\ \|II\|_Y &\leq W_0^1 K \cdot \max_{0 \leq t \leq 1} \left| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \right| \max_{0 \leq k \leq n} \|V^k h\|_X \\ &\leq W_0^1 K \cdot M \cdot \varepsilon/4[1 + W_0^1 K]M < \frac{\varepsilon}{4} \quad \text{for all } n. \end{aligned}$$

$$\begin{aligned} \|I\|_Y &= \|I\|_{Y^{**}} = \left\| \int_0^a + \int_a^b + \int_b^1 \right\|_{Y^{**}} \\ &\leq \left\| \int_0^a \right\|_{Y^{**}} + \left\| \int_a^b \right\|_{Y^{**}} + \left\| \int_b^1 \right\|_{Y^{**}}. \end{aligned}$$

It is necessary to regard the norms on the right as Y^{**} norms because these integrals may exist only as elements in Y^{**} and not as elements in Y (see the remarks following Theorem 1 [3, p. 950].)

$$\left\| \int_0^a dK(t) \cdot \left[\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right] \right\|_{Y^{**}} \leq W_0^a K \cdot 2M \cdot \|g\|_X$$

and

$$\left\| \int_b^1 dK(t) \cdot \left[\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right] \right\|_{Y^{**}} \leq W_b^1 K \cdot 2M \cdot \|g\|_X.$$

Since K is assumed continuous at $t = 0$ and $t = 1$, there are values for a and b sufficiently near, but distinct from 0 and 1 respectively, such that each of $W_a^a K$ and $W_b^b K$ less than $\varepsilon/8M[1 + \|g\|]$. With these values of a and b , there is n sufficiently large, by the above lemma, that

$$\max_{a \leq t \leq b} \left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right\|_X \leq \varepsilon/2[1 + W_a^b K].$$

Collecting all this together yields

$$\|T_n f\|_Y \leq \varepsilon$$

for all n sufficiently large. Thus

$$\lim_{n \rightarrow \infty} \|T_n f\|_Y = \lim_{n \rightarrow \infty} \|T_n f - LPf\| = 0$$

since

$$LPf = \theta_Y$$

and this completes the proof.

In case that $Y \equiv X$ and H is regular relative to I , then H sums $\{V^k\}_{k=0}^\infty$ to P . In particular, any regular scalar-valued Hausdorff method whose generating function K is continuous at $t = 0$ and $t = 1$ will sum $\{V^k\}_{k=0}^\infty$ to P . The case treated in [2], corresponds to the case here where $K(t) = tI$, i.e., the C_1 method. The following example illustrates the theorem for a nonscalar-valued Hausdorff method.

Suppose $X = Y = H$, a Hilbert space. Suppose also that K is a bounded resolution of the identity such that $K(0) = 0$, $K(1) = I$, K is continuous at 0 and 1 in the operator norm, and K satisfies the Gowurin ω -property. The approximating sums for integrals of the form $\int_0^1 t^n dK(t)$ converge to the integral in the operator norm [2], hence they converge in the sense given by Tucker [3]. Consider the moment sequence $\{\mu_n\}_{n=0}^\infty$ given by $\mu_n = \int_0^1 t^n dK(t)$. As shown in [2], μ_1 is a self-adjoint operator in H , and if we denote it by A , it follows that $\mu_n = A^n$ ($n = 0, 1, 2, \dots$) where $\mu_0 = K(1) = A^0 = I$. If $\{V^n\}_{n=0}^\infty$ is a sequence of operators as given in the theorem, and $H(\mu)$ is the Hausdorff summability method generated by $\{\mu_n\} = \{A^n\}$, then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k} (\Delta^{n-k} A^k) V^k x = Px$$

for all $x \in H$, the limit being taken in the norm of H .

PART 2

It has been pointed out that the proof of the lemma given above is incorrect. It can be corrected in the following manner. As given,

$$\|A_n(t)\|_s \leq \frac{N'}{t} \sum_{k=0}^n \left(\binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right) + t^n N'.$$

Proceed as follows. For $0 < a \leq t \leq b < 1$

$$\|A_n(t)\|_s \leq \frac{N'}{a} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} + b^n N'.$$

Suppose $\varepsilon > 0$ and pick δ such that $0 < \delta < \{(1-b)\varepsilon/2/(1+\varepsilon/2)\}$. For $t \in [a, b]$, set

$$A_t = \left\{ k : \left| t - \frac{k}{n+1} \right| < \delta \right\} \quad \text{and} \quad B_t = \left\{ k : \left| t - \frac{k}{n+1} \right| \geq \delta \right\}.$$

Then

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \\ &= \left(\sum_{A_t} + \sum_{B_t} \right) \left(\binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right). \end{aligned}$$

Consider the sums separately.

$$\begin{aligned} \sum_{A_t} &\leq \sum_{A_t} \binom{n}{k} t^k (1-t)^{n-k} \frac{\delta}{1-b-\delta} \leq \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \\ \sum_{B_t} &= \frac{1}{1-t} \sum_{B_t} \binom{n}{k} \frac{n+1}{n+1-k} t^k (1-t)^{n-k+1} \left| t - \frac{k}{n+1} \right|. \end{aligned}$$

For $k \in B_t$,

$$\left| \frac{k}{n+1} - t \right| \leq 1 \leq \frac{((n+1)t - k)^2}{\delta^2(n+1)^2},$$

so

$$\begin{aligned}
\sum_{B_t} &\leq \frac{1}{(1-t)\delta^2(n+1)^2} \sum_{k=0}^n \binom{n}{k} \frac{n+1}{n+1-k} t^k (1-t)^{n-k+1} [(n+1)t-k]^2 \\
&= \frac{1}{(1-t)\delta^2(n+1)^2} \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n-k+1} [(n+1)t-k]^2 \\
&= \frac{1}{(1-t)\delta^2} \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n-k+1} \left(t - \frac{k}{n+1}\right)^2 \\
&= \frac{1}{(1-t)\delta^2} \cdot \frac{t(1-t)}{n+1} \leq \frac{b}{(n+1)\delta^2}.
\end{aligned}$$

Collecting this together gives

$$\|A_n(t)\|_s \leq \frac{N'}{a} \left(\frac{\varepsilon}{2} + \frac{b}{(n+1)\delta^2} + b^n N' \right), \quad \text{for } 0 < a \leq t \leq b < 1,$$

which proves the lemma.

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