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Let E be a closed subset of the unit circle $C = \{z: |z| = 1\}$ and denote by B_E the algebra of all functions which are bounded and continuous on the set $X = \{z: |z| \leqq 1, z \notin E\}$ and analytic in the open disc $D = \{z: |z| < 1\}$. An interpolation set for B_E is a relatively closed subset S of X with the property that if α is a bounded and continuous function on S (all functions are complex-valued), there is a function f in B_E such that $f(z) = \alpha(z)$ for every $z \in S$. The main result of the paper characterizes the interpolation sets for B_E as those sets S for which $S \cap D$ is an interpolation set for H^∞ and $S \cap (C - E)$ has Lebesgue measure 0. If, in addition, $S \cap D = \phi$ then S is a peak interpolation set for B_E. Also, through a construction process inspired by recent work of J. P. Kahane, it is shown that the existence of peak points for a sup norm algebra of continuous functions on a compact, connected space implies the existence of infinite interpolation sets relative to the algebra and certain of its weak extensions.

The solution of the interpolation problem in the space $H^\infty = B_C$ of bounded analytic functions on D is due to Lennart Carleson [5], and due to A. Beurling and Walter Rudin in the disc algebra $A = B_\phi$ [10]. Concerning the latter case see also the notes of Lennart Carleson [5] and the last problem in Hoffman's book [8]. Their results are given by the following two theorems.

THEOREM C. *A sequence $\{z_k\}$ of distinct points in D is an interpolation set for H^∞ if and only if it is uniformly separated*[1], *that is, if and only if there exists a positive number δ such that*

$$\prod_{j=1;\, j\neq k}^{\infty} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| \geqq \delta \qquad (k = 1, 2, \cdots) . \tag{1}$$

Whenever this condition holds, a constant $m(\delta)$ exists with the property that for any bounded sequence $\{w_k\}$ there is an f in H^∞ such that $f(z_k) = w_k$ $(k = 1, 2, \cdots)$ and $\|f\| \leqq m(\delta) \sup_k |w_k|$.

THEOREM B-R. *A closed subset S of $\bar{D}$ is an interpolation set for A if and only if*

(i) *$S \cap D$ is uniformly separated,*

[1] This terminology is due to Professor Peter Duren.

and

(ii) $S \cap C$ *has Lebesgue measure* 0.

In the terminology introduced above our characterization of the interpolation sets for the algebra B_E takes the following form.

THEOREM 1. *The relatively closed subset S of X is an interpolation set for B_E if and only if*

(i) $S \cap D$ *is uniformly separated,*

and

(ii) $S \cap (C - E)$ *has Lebesgue measure* 0.

For example, suppose $E = \{1\}$ and S is the union of the sequences $a_k = 1 - 2^{-k}$ $(k = 1, 2, \cdots)$ with any sequence $\{b_k\}$ of distinct points on C converging to 1 $(b_k \neq 1)$. For a proof that $\{a_k\}$ is uniformly separated, see [8, p. 204]. Our result then applies and asserts that for any pair of bounded sequences $\{\alpha_k\}$ and $\{\beta_k\}$ there exists a function f in H^∞, continuous on $\bar{D} - \{1\}$, such that $f(a_k) = \alpha_k$ and $f(b_k) = \beta_k$ $(k = 1, 2, \cdots)$. For $S = \{b_k\}$ alone this is a result of E. L. Stout [11, Lemma 4.1].

Our proof of Theorem 1, presented in § 2, depends on Theorem C and the generalized Rudin-Carleson theorem [4]. We also show in § 2 that the interpolation sets for B_E which are subsets of $C - E$ have the property that every bounded continuous function α on S has an extension f in B_E with $||f|| = ||\alpha||$ (all norms are supremum norms on the relevant domains). In § 3 we present an argument which, in particular, shows that the existence of peak sets for the disc algebra A implies the existence of infinite interpolation sets for H^∞.

2. Interpolation in B_E.

First we shall deal with those interpolation sets for B_E which are contained in D. Naturally, such sets are countable.

LEMMA 1. *A sequence $\{z_k\}$ of distinct points in D is an interpolation set for B_E if and only if it is uniformly separated and all of its limit points belong to E. If this condition is satisfied then there is a constant $m(\delta/2)$ such that if $\{w_k\}$ is a bounded sequence there exists an f in B_E such that*

(i) $f(z_k) = w_k$ $(k = 1, 2, \cdots)$,

(ii) $||f|| \leqq m(\delta/2) \sup_k |w_k|$.

Proof. If $\{z_k\}$ is an interpolation set for B_E it is certainly one also for H^∞ and is therefore uniformly separated by Theorem C. And if $e^{i\theta}$ is a limit point of $\{z_k\}$ there is a function in B_E which is discontinuous there; hence E contains all the limit points of $\{z_k\}$.

Now suppose that the sequence $\{z_k\}$ of distinct points in D is uniformly separated with relevant constant δ and has all its limit points in E. It is no restriction to suppose in addition that no z_k is zero. (To avoid the situation covered by Theorem C we assume that E is a proper subset of the unit circle.) The Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{z}_k}{|z_k|} \frac{z_k - z}{1 - \bar{z}_k z} \tag{2}$$

and each of its subproducts represent functions analytic in the complement of the compact set K consisting of E together with the points $1/\bar{z}_k$ [8]. Thus B is analytic and of unit modulus at each point which belongs to one of the arcs $\beta_1, \beta_2, \cdots$ in C complementary to E. This means that each point of β_n is the center of a small disc contained in the complement of K and on which B is analytic and $|B(z)| \leqq 2$. Cover β_n by a countable and locally finite (relative to β_n) collection of such discs and let β'_n be that part of the boundary of the union of these discs which lies outside D. The set β'_n is a Jordan arc having the same endpoints as β_n, and, except for these endpoints it is contained in the complement of $\bar{D}$. Now let D^* be the simply connected domain containing D whose boundary is E together with the nonintersecting arcs β'_n $(n = 1, 2, \cdots)$. Clearly $|B(z)| \leqq 2$ for $z \in D^*$.

Let $B(D^*)$ denote the space of functions bounded and analytic on D^*. If we can show that $\{z_k\}$ is an interpolation set for $B(D^*)$ the proof will be complete since $B(D^*) \subset B_E$. (In this connection compare Stout's general characterization of interpolation sets [11, Th. 5.9].) To this end choose a conformal map ϕ from D^* onto D and set

$$\phi(z_k) = y_k,\ f_k = B_k \circ \phi^{-1}\ (k = 1, 2, \cdots)$$

where B_k is the Blaschke product B with the k^{th} factor removed. For each $k, f_k \in H^\infty$, $\|f_k\| \leqq 2$, $|f_k(y_k)| \geqq \delta$ (see (1)) and

$$|f_k(y_j)| = 0\ (j \neq k)\ .$$

If C_{ks} is the finite Blaschke product (see (2)) associated with the points $y_1, y_2, \cdots, y_{k-1}, y_{k+1}, \cdots, y_s$ $(1 \leqq k \leqq s,\ s = 2, 3, \cdots)$, we have

$$\frac{\delta}{|C_{ks}(y_k)|} \leqq \frac{|f_k(y_k)|}{|C_{ks}(y_k)|} \leqq \left\| \frac{f_k}{C_{ks}} \right\| \leqq 2\ ,$$

that is,

$$\prod_{j=1;\, j\neq k}^{s} \left| \frac{y_j - y_k}{1 - y_j y_k} \right| \geqq \delta/2\ .$$

This proves that $\{y_k\}$ is uniformly separated in D. Hence if $\{w_k\}$ is a bounded sequence, there exists an f in H^∞ such that $f(y_k) = w_k$ $(k = 1, 2, \cdots)$ and $\|f\| \leqq m(\delta/2) \sup |w_k|$. The function $f \circ \phi$ is bounded and analytic on D^*, $f \circ \phi(z_k) = w_k$ and $\|f \circ \phi\| \leqq m(\varepsilon/2) \sup |w_k|$. This completes the proof.

A remark is in order concerning Lemma 1. In [1] Akutowicz and Carleson considered the general question of analytic continuation of interpolating functions. In the course of their work it was shown that if $\{z_k\}$ is an interpolation set for H^∞ which clusters on the closed set E, then there exists a solution to the interpolation problem which has an analytic continuation to a larger domain obtained by pushing out through proper subarcs of finitely many of the complementary arcs $\beta_1, \beta_2, \cdots$ [1, Th. 4]. Note that the interpolation function $f \circ \phi$ of the preceding argument is analytic in a domain which contains all the complementary arcs β_n.

For a proof of the following lemma see [2, Th. 1.2].

LEMMA 2. *Let $T: X \rightarrow Y$ be a linear and continuous map from the Banach space X into the normed linear space Y. Suppose there exist constants $\delta < 1$ and M such that for each $y \in Y$ with $\|y\| \leqq 1$, there exists an $x \in X$ such that*

$$\|Tx - y\| \leqq \delta, \|x\| \leqq M .$$

Then $TX = Y$. If $\|y\| \leqq 1$, there exists an x such that $Tx = y$ and $\|x\| \leqq M(1 - \delta)^{-1}$.

LEMMA 3. *The relatively closed subset K of $C - E$ is an interpolation set for B_E if, and only if, K has measure 0.*

Proof. Clearly every such interpolation set for B_E must be of measure 0.

For the converse we need to know that any relatively closed subset K of $C - E$ of measure 0 can be written as the disjoint union of compact sets

$$K = \bigcup_{n=1}^{\infty} K_n$$

in such a way that there exist disjoint open sets $O_n \subset C - E$ which satisfy the inclusions

$$K_n \subset O_n \, (n = 1, 2, \cdots) .$$

Because K is nowhere dense in $C - E$ it is possible to replace any finite disjoint collection of open arcs $J_1, J_2, \cdots, J_s$ which cover E by another collection of open arcs $I_p \subset J_p \, (p = 1, 2, \cdots, s)$ which cover

E and have all their endpoints in $C - E \cup K$. Hence there exists a sequence $G_1 \supset G_2 \supset G_3 \supset \cdots$ such that $E = \bigcap_{n=1}^\infty G_n$ and each G_n is a finite disjoint collection of open arcs, all of whose endpoints lie in $C - E \cup K$. Define

$$K_1 = K \cap (C - G_1), O_1 = C - \bar{G}_1$$

and, for $n > 1$,

$$K_n = K \cap (\bar{G}_{n-1} - G_n), O_n = G_{n-1} - \bar{G}_n .$$

Now let α be a bounded, complex-valued continuous function on K with $\|\alpha\| = 1$. Denote the restriction of α to K_n by α_n and fix $\delta, 0 < \delta < 1$. According to the general Rudin-Carleson interpolation theorem [4] we may choose positive continuous functions Δ_n $(n = 1, 2, \cdots)$ on C such that

(a) $\Delta_n = |\alpha_n| + \delta/2^n$ on K_n,

(b) $\Delta_n = \delta/2^n$ on $C - O_n$,

(c) $0 < \Delta_n \leqq \|\alpha\| + \delta/2^n$ everywhere;

then select functions $f_n \in A$ (the disc algebra) having the following properties:

(d) $f_n = \alpha_n$ on K_n,

(e) $|f_n| \leqq \Delta_n$ on C.

For the function f defined by

$$f(z) = \sum_{n=1}^{\infty} f_n(z) \qquad (z \in X) , \tag{3}$$

we make the following claims:

(i) $f \in B_E$,

(ii) $\|f\| \leqq 1 + \delta$,

(iii) $\sup_{z \in K} |f(z) - \alpha(z)| \leqq \delta$.

It follows from (b), (c) and (e) that the series (3) converges for every $z \in C$ and that its partial sums are bounded by δ for

$$z \in C - \bigcup_{n=1}^{\infty} O_n$$

and by $1 + \delta$ if $z \in O_n$ for some positive integer n. Therefore the series converges pointwise on X to an H^∞ function with norm satisfying (ii). Further, (b) and (e) show that convergence in (3) is uniform on any compact subset of $C - E$ because such a set misses all but a finite number of the sets O_n.

Thus f is continuous on $C - E$, hence continuous on X and (i) holds. In order to establish (iii), suppose $z \in K$; then $z \in K_p$ for some positive integer p so, by (d), $f(z) - \alpha(z) = \sum_{n \neq p} f_n(z)$ and, by (b) and (e), $|f(z) - \alpha(z)| \leqq \sum_{n \neq p} \delta 2^{-n} < \delta$ as required.

Let $C(K)$ be the Banach space of bounded continuous functions on K, and let T be the restriction mapping from B_E into $C(K)$. Conditions (i), (ii) and (iii) above show that Lemma 2 applies. Hence if $\delta < 1$ and $\alpha \in C(K)$, there exists an f in B_E such that $f = \alpha$ on the set K and $\|f\| \leqq (1+\delta)(1-\delta)^{-1}\|\alpha\|$. This is the desired conclusion.

LEMMA 4. *Let K be a relatively closed subset of $C - E$ of measure* 0. *Then the ideal*

$$J(K) = \{f \in B_E: f(K) = 0\}$$

has an approximate unit.

Proof. The implication is that there exists a net $\{e_\gamma\}$ in $J(K)$ such that $\|e_\gamma\| \leqq 1$ and $e_\gamma \to 1$ uniformly on closed subsets of X disjoint from the set K.

We assume the notation and decomposition of Lemma 3 except that each of the sets O_n is replaced by

$$V_n = \{re^{i\theta}: e^{i\theta} \in O_n, 1 - \frac{1}{n} < r \leqq 1\}.$$

The sets V_n are pairwise disjoint, open in X, and $K_n \subset V_n$. Choose numbers c_n, $0 < c_n < 1$, so that $\sum_{n=1}^\infty c_n < \infty$ and functions $g_n \in A$ such that $\|g_n\| \leqq 1$, g_n vanishes exactly on K_n and $|1 - g_n| < c_n$ on $X - V_n$ (the existence of such functions follows immediately from the construction given in [8, Chapter 6, p. 80]). Define g by

$$g(z) = \prod_{n=1}^\infty g_n(z) \qquad (z \in X).$$

The inequality

$$\left|1 - \prod_{n=N}^{N+p} g_n(z)\right| \leqq \prod_{n=N}^{N+p}(1 + |1 - g_n(z)|) - 1$$
$$\leqq \exp \sum_{n=N}^{N+p} c_n - 1,$$

valid for $z \notin \bigcup_{n=N}^\infty V_n$, shows that the product defining g converges uniformly on compact subsets of X. Thus $g \in B_E$ and

$$|1 - g(z)| \leqq \exp \sum_{n=1}^\infty c_n - 1$$

for $z \in X - \bigcup_{n=1}^\infty V_n$. This argument shows that if $\varepsilon > 0$ and S is a closed subset of X disjoint from K, then proper choices for V_n and c_n yield a g in $J(K)$ such that $\|g\| \leqq 1$, g vanishes precisely on K in

X and $|1 - g(z)| < \varepsilon \ (z \in S)$.

Proof of Theorem 1. If the relatively closed subset S of X is an interpolation set for B_E, then clearly $S \cap D$ is countable and $S \cap C$ has measure 0. The proof that $S \cap D$ is uniformly separated is identical with the corresponding proof for H^∞ [8, p. 196].

For the converse let the relatively closed subset S of X be the union of a uniformly separated sequence $\{z_k\}$ in D and a subset K of $C - E$ of measure 0. Let α be a bounded, continuous function on S with $\|\alpha\| \leqq 1$. By Lemma 3 there exists an f_1 in B_E such that $f_1 = \alpha$ on K and $\|f_1\| \leqq 3/2$; hence $\alpha_1 = \alpha - f_1$ vanishes on K and $\|\alpha_1\| \leqq 5/2$. Lemma 1 guarantees the existence of a constant c, depending only on the points z_k, and a function h in $B_{E \cup K}$ such that

$$h(z_k) = \alpha_1(z_k) \ (k = 1, 2, \cdots), \ \|h\| \leqq c \cdot 5/2 .$$

Since α_1 is continuous on S and vanishes on K, we may choose open sets U_n in X such that $K_n \subset U_n \subset V_n$ $(n = 1, 2, \cdots)$ (we assume the decomposition and notation of Lemma 4) and $|\alpha_1| < 1/4$ on the set $\bigcup_{n=1}^\infty U_n$. Now choose, by Lemma 4, a function g in $J(K)$ such that $\|g\| \leqq 1$ and $|1 - g(z)| < 1/4$ for $z \notin \bigcup_{n=1}^\infty U_n$; hence the product gh belongs to B_E and for all k

$$|g(z_k)h(z_k) - \alpha_1(z_k)| = |(g(z_k) - 1)\alpha_1(z_k)| < 3/4 ,$$

since $|g(z_k) - 1| < 1/4, |\alpha_1(z_k)| \leqq 5/2$ for $z_k \notin \bigcup U_n$ and

$$|1 - g(z_k)| \leqq 2, |\alpha_1(z_k)| < 1/4$$

otherwise. This proves that the function $f = f_1 + gh$ has the following properties:

(i) $f \in B_E$

(ii) $\|f\| \leqq 3/2 + 5/2 \cdot c = M$

(iii) $\sup_{z \in S} |f(z) - \alpha(z)| \leqq 3/4$.

In view of Lemma 2 the proof is complete.

THEOREM 2. *Let K be the closed subset of $C - E$ of measure* 0. *Then any bounded continuous function α on K has an extension g in B_E with precisely the same norm as α; in fact g can be chosen in B_E so that $g = \alpha$ on K and*

$$|g(z)| < \|\alpha\|, z \in X - K .$$

Proof. We have already established the following weaker result (see Lemma 3): if $\varepsilon > 0$, α has an extension g_ε in B_E such that $\|g_\varepsilon\| \leqq (1 + \varepsilon)\|\alpha\|$. In addition, Lemma 4 asserts that K is a strong hull in the Banach function algebra B_E [7], that is, for each closed subset S of X disjoint from K and each $\varepsilon > 0$ there exists a function

f in β_E such that $f(K) = 0$, $||f|| \leqq 1$, and $|1 - f(S)| < \varepsilon$. The conditions guarantee the existence of the required function g [see 7, Th. 4.6].

3. Peak points and interpolating sequences.

Previous to Carleson's paper [5], Gleason and Newman had constructed examples (unpublished) proving the existence of infinite interpolation sets for H^∞. In this section we present a process, depending only on the existence of peak points in the underlying algebra, which constructs infinite interpolation sets in some rather general H^∞ spaces. According to Bishop's minimal boundary theorem [3] peak points always exist for a sup norm algebra defined on a compact metric space.

Let A be a sup norm algebra on the compact Hausdorff space X and suppose that the function $F \in A$ peaks at x, that is,

$$F(x) = 1 \text{ and } |F(y)| < 1 \qquad (y \in X,\ y \neq x)\ .$$

Let S_x be the set of all bounded and continuous functions f on $X - \{x\}$ for which there exists a constant m and a sequence $\{f_n\}$ in A with $||f_n|| \leqq m$ and such that $f_n \to f$ uniformly on compact subsets of $X - \{x\}$. B_x is the uniform closure of S_x.

THEOREM 3. *Suppose P is a connected subset of X and $x \in \bar{P} - P$. Then there is an infinite sequence $\{z_k\}$ of distinct points in P which interpolates for B_x, that is, the map $T: B_x \to l^\infty$ defined by $Tf = \{f(z_k)\}$ is an onto map.*

Proof. Choose δ, $0 < \delta < 1/4$, so that the closed set

$$U_1 = \{y: |F(y) - 1| \geqq \delta\}$$

intersects P. Set $F_1 = F$ and $n(1) = 1$. We wish to construct an increasing sequence $n(1) < n(2) < \cdots$ of integers which obey

$$n(k+1) > kn(k) \qquad (k = 1, 2, \cdots) \tag{4}$$

and for which the sets

$$\begin{aligned} U_k &= \{y: |F_k(y) - 1| \geqq \delta/2^{k-1}\}\ , \\ V_k &= \{y: |F_k(y)| < \delta/2^{k-1}\} \end{aligned} \tag{5}$$

associated with the functions

$$F_k = F^{n(k)} \tag{6}$$

satisfy

$$U_1 \subset V_2 \subset U_2 \subset V_3 \cdots \tag{7}$$

and

$$(8) \qquad X - \{x\} = \bigcup_{i=1}^{\infty} U_i \, .$$

In order to construct F_2 let $n(2) > 1$ be an integer so large that

$$|F^{n(2)}| < \delta/2$$

on U_1 and define $F_2 = F^{n(2)}$. Notice that $U_1 \subset V_2 \subset U_2$.

Suppose $n(1), n(2), \cdots, n(k)$ have been chosen. Choose

$$n(k+1) > kn(k)$$

so large that $|F^{n(k+1)}| < \delta/2^k$ on the closed set U_k and define $F_{k+1} = F^{n(k+1)}$. Clearly $U_k \subset V_{k+1} \subset U_{k+1}$. The existence of the required sequence $\{n(k)\}$ follows by induction. If a point y belongs to none of the sets V_k then, by (4) and (6), $|F(y)| > \delta^{1/n(k+1)} 2^{-k/n(k+1)} \longrightarrow 1$ as $k \longrightarrow \infty$ showing that $y = x$. Hence (8) holds also.

For each integer k choose a point z_k from the set $P \cap (V_{k+1} - U_k)$, this being possible because U_k and $X - V_{k+1}$ are disjoint closed sets both of which intersect the connected set P. Fix a bounded sequence $\{w_k\}$, $\|w\| \leqq 1$, and define g by

$$(9) \qquad g = \sum_{p=1}^{\infty} w_p(F_p - F_{p+1}) \, .$$

The series converges uniformly on compact subsets of $X - \{x\}$ since any such set is eventually captured by a V_k. In order to establish bounds on the partial sums for the series (9) notice that

$$X - \{x\} = U_1 \cup (U_2 - U_1) \cup (U_3 - U_2) \cup \cdots ,$$

(the sets in the union being pairwise disjoint) and for any point y,

$$g(y) = \sum_{p=1}^{k-1} w_p(F_p(y) - F_{p+1}(y)) + w_k(F_k(y) - F_{k+1}(y))$$
$$+ w_{k+1}(F_{k+1}(y) - F_{k+2}(y)) + \sum_{p=k+2}^{\infty} w_p(F_p(y) - F_{p+1}(y)) \, .$$

If $y \in U_{k+1} - U_k$, we have the inequalities

$$\text{(A)} \qquad \left| \sum_{p=1}^{k-1} w_p(F_p(y) - F_{p+1}(y)) \right| \leqq \sum_{p=1}^{k-1} (|F_p(y) - 1| + |F_{p+1}(y) - 1|)$$
$$< \sum_{p=1}^{k-1} (\delta/2^{p-1} + \delta/2^p) \, ;$$

$$\text{(B)} \qquad |w_p(F_p(y) - F_{p+1}(y))| \leqq 2 \qquad (p = k, k+1) \, ;$$

$$\text{(C)} \qquad \left| \sum_{p=k+2}^{\infty} w_p(F_p(y) - F_{p+1}(y)) \right| < \sum_{p=k+2}^{\infty} (\delta/2^{p-1} + \delta/2^p) \, .$$

Hence $4 + 2\delta \sum 2^{-p} = 4 + 4\delta$ is a bound for the partial sums. Inequalities (B) and (C) give the same bounds when $y \in U_1$. Therefore $g \in S_x$.

In order to estimate $g(z_k) - w_k$ subtract w_k from both members of (9) and replace y by z_k in (A) and (C). In place of (B) we have the inequalities

$$\text{(B}'\text{)}\qquad \begin{aligned} &|\, w_k(F_k(z_k) - 1 - F_{k+1}(z_k))\,| \leqq \delta/2^{k-1} + \delta/2^k\ , \\ &|\, w_{k+1}(F_{k+1}(z_k) - F_{k+2}(z_k))\,| \leqq \delta/2^k + \delta/2^{k+1} \end{aligned}$$

because $z_k \in V_{k+1} - U_k$ implies $|\, F_{k+1}(z_k)\,| < \delta/2^k$, $|\, F_{k+2}(z_k)\,| < \delta/2^{k+1}$ and $|\, F_k(z_k) - 1\,| < \delta/2^{k-1}$. Addition of (A), (B′) and (C) gives

$$|\, g(z_k) - w_k\,| < 4\delta \qquad (k = 1, 2, \cdots)\ .$$

In summary, we have shown that for any $w = \{w_n\} \in l^\infty$ with $||\, w\,|| \leqq 1$ there exists a function g in S_x with $||\, g\,|| \leqq 4 + 4\delta$ and

$$\sup_k |\, g(z_k) - w_k\,| \leqq 4\delta\ .$$

Since $\delta < 1/4$ Lemma 2 applies; hence there exists a function f in B_x such that

$$f(z_k) = w_k \qquad (k = 1, 2, \cdots)\ .$$

This completes the proof.

The preceding argument shows that the series (9) converges absolutely and has uniformly bounded partial sums for every bounded sequence $\{w_k\}$ and will therefore converge uniformly on X provided $\lim w_k = 0$. This means that we may again apply Lemma 2, this time with T identified as the map $f \to \{f(z_k)\}$ from A into the space c of convergent sequences.

COROLLARY. *Let A be a sup norm function algebra on the compact Hausdorff space, let P be a connected subset of X and let $x \in \bar{P} - P$ be a peak point for A. Then there exists an infinite sequence of distinct points in P which converges to x and has the property that for every convergent sequence $\{c_k\}$ there exists an f in A such that $f(z_k) = c_k$ $(k = 1, 2, \cdots)$.*

Let m be a positive Baire measure on X which is multiplicative on the sup norm algebra A and not equal to point evaluation at x. Clearly, the functions in S_x are elements of $H^2(dm)$, the closure of A in $L^2(dm)$, and therefore

$$S_x \subset H^\infty(dm) = L^\infty(dm) \cap H^2(dm)\ .$$

Since the norm in $H^\infty(dm)$ is the essential supremum norm relative to m, it follows that $B_x \subset H^\infty(dm)$. Thus, under the assumptions of Theorem 3, we can make the following rather weak statement: there exist infinite interpolation sets for $H^\infty(dm)$ whenever point evaluations on the set P extend to homomorphisms of $H^\infty(dm)$.

Finally, we remark that in so far as we know the Carleson corona theorem, the question of whether D is dense in the maximal ideal space of B_E, is open in case E is a proper nonempty subset of C.

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