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A MODULAR TOPOLOGICAL LATTICE

DON E. EDMONDSON

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The purpose of this paper is to present a construction of a compact connected topological lattice which is modular and not distributive. As a special case there will result the example which is a two dimensional subset of R^3 , not embeddable in R^2 .

The existence of such an example is related to structure questions in topological lattices considered by Dyer and Shields [3], Anderson [1], and others.

The first step is to present a general method for constructing a class of modular lattices. Let D denote a distributive lattice which is a chain, S a nonempty set, and L the S-fold product lattice of D. That is $L = \{f \mid f: S \rightarrow D\}$ and $f \leq g$ if and only if $f(s) \leq g(s)$ for every $s \in S$. It is known that (L, \leq) is a distributive lattice with its operations \vee and \wedge characterized by $[f \vee g](s) = f(s) \vee g(s)$ and

$$[f \land g](s) = f(s) \land g(s)$$

for every $s \in S$. Define

 $M = \{f \in L \mid \text{there exists } r \in S \text{ such that } s, t \in S - \{r\} \text{ implies}$

$$f(s) \leq f(r)$$
 and $f(s) = f(t)$.

For intuition about M and the arguments that follow, note that M simply consists of all of the constant functions of L and the functions of L which are essentially constant in the sense that they assume but two values — the larger value at exactly one point.

If the order of $L, \leq ,$ is restricted to M, it will be established through a sequence of lemmas that (M, \leq) is a modular lattice. Recall a lattice (M, \lor, \land) is modular if and only if for every $a, b, c \in M$, $b \leq a$ implies that $a \land (b \lor c) = b \lor (a \land c)$.

LEMMA 1. If $f \in M$ and f is not constant, there exists a unique $r \in S$ such that $s, t \in S - \{r\}$ implies f(s) < f(r) and f(s) = f(t).

The proof of the lemma is immediate from the definition of M, and consequently for $f \in M$ and not constant, define *index* f to be the unique element described in Lemma 1.

LEMMA 2. (M, \leq) is a sub \wedge -semilattice of (L, \leq) .

It suffices to show that if $f, g \in M$, then $f \wedge g \in M$. If f and g are

constant then $f \wedge g$ is constant and therefore in *M*. If *f* is constant and *g* is not, let b = index g. Then $s, t \in S - \{b\}$ implies

$$[f \land g](s) = f(s) \land g(s) \leq f(s) \land g(b) = [f \land g](b)$$

and likewise $[f \land g](s) = [f \land g](t)$ and thus $f \land g \in M$. If f and gg are both not constant, let a = index f and b = index g. If a = b, then $s, t \in S - \{a\}$ implies $[f \land g](s) \leq [f \land g](a)$ and $[f \land g](s) = [f \land g](t)$. If $a \neq b$, $[f \land g](a) = f(a) \land g(a)$, $[f \land g](b) = f(b) \land g(b)$, and for

$$x \in S - \{a, b\} [f \land g](x) = f(b) \land g(a) = [f \land g](a) \land [f \land g](b)$$
 .

Then since D is a chain, $[f \land g](x) = [f \land g](a)$ or $[f \land g](x) = [f \land g](b)$ depending upon which is minimal and therefore $f \land g \in M$.

LEMMA 3. If a, b, c are distinct elements of S and $f \in M$, then $f(a) \wedge f(b) = f(b) \wedge f(c) = f(c) \wedge f(a)$.

The facts of the lemma are an immediate consequence of the definition and is stated as a lemma for convenient reference.

DEFINITION. For $f, g \in M$ define $f \nabla g: S \to D$ by the following (i) if f is constant or g is constant, or if f and g are both not constant and index f = index g, then $f \nabla g = f \vee g$,

(ii) if f and g are both not constant and index $f \neq index g$, let a = index f and b = index g, then

$$[f \bigtriangledown g](x) = f(x) \lor g(x) \quad ext{for} \quad x \in \{a, b\}$$

 $[f \bigtriangledown g](x) = [f(a) \lor g(a)] \land [f(b) \lor g(b)] \quad ext{for} \quad x \in S - \{a, b\}.$

LEMMA 4. If f, $g \in M$, then (1) $f \nabla g \in M$ and $f \vee g \leq f \nabla g$, and (2) $h \in M$, $f \leq h$, $g \leq h$ implies $f \nabla g \leq h$.

In case (i) of the definition of $f \nabla g$, easily $f \nabla g \in M$ and the other results are immediate from $f \nabla g = f \vee g$. In case (ii) let a = index f and b = index g, then since D is a chain $[f \nabla g](x) = [f \nabla g](a)$ or $[f \nabla g](x) = [f \nabla g](b)$ for $x \in S - \{a, b\}$. So in this case also $f \nabla g \in M$ and $f \vee g \nabla g$. Also relative to this case, if $h \in M$, $f \leq h$, and $g \leq h$, then $f(x) \vee g(x) = [f \nabla g](x)$ for x = a or x = b. But from Lemma 3, $x \in S - \{a, b\}$ implies $h(a) \wedge h(b) \leq h(x)$ and thus for

$$x \in S - \{a, b\} [f \bigtriangledown g](x) = [f \bigtriangledown g](a) \land [f \bigtriangledown](b) \leq h(a) \land h(b) \leq h(x)$$
 .

Therefore $f \nabla g \leq h$.

LEMMA 5. If $f, g, h \in M, a, b \in S, a \neq b, [f \lor g](x) = h(x)$ for $x \in \{a, b\}$

and $[f \lor g](x) \leq h(a) \land h(b) = h(x)$ for $x \in S - \{a, b\}$, then $h = f \bigtriangledown g$.

From the hypothesis $f \vee g \leq h$ and therefore from Lemma 4 $f \nabla g \leq h$. But $h(a) = [f \vee g](a) \leq [f \nabla g](a)$ and $h(b) \leq [f \nabla g](b)$. Then from Lemma 3, for $x \neq a$ and $x \neq b$

$$h(x) = h(a) \land h(b) \leq [f \bigtriangledown g](a) \land [f \lor g](b) \leq [f \lor g](x)$$

and $h \leq f \nabla g$.

THEOREM 1. (M, \leq) is a modular lattice with operations \forall and \land .

Lemmas 2 and 4 establish that (M, \leq) is a lattice with operations $\overline{\bigtriangledown}$ and \wedge , it remains to establish that it is modular. Let $f, g, h \in M$ and $f \leq g$. It suffices to establish $g \wedge (f \overline{\bigtriangledown} h) \leq f \overline{\bigtriangledown} (g \wedge h)$ since in any lattice $f \overline{\bigtriangledown} (g \wedge h) \leq g \wedge (f \overline{\bigtriangledown})$. The argument will be a case argument.

If $f \bigtriangledown h = f \lor h$, then

$$g \land (f \ \overline{\bigtriangledown} \ h) = g \land (f \lor h) = f \lor (g \land h) \leq f \ \overline{\lor} \ (g \land h)$$

since L is itself modular and $g \wedge h \in M$ allows Lemma 4 to apply.

If $h \leq g$, then $f \bigtriangledown h \leq g$ and $g \land (f \bigtriangledown h) = f \bigtriangledown h = f \bigtriangledown (g \land h)$. If $f \leq h$, then $f \leq g \land h$ and $g \land (f \bigtriangledown h) = g \land h = f \bigtriangledown (g \land h)$.

If f and h are not constant, a = index f, b = index g, $a \neq b$, $h \neq g$, and $f \leq h$. Then f(b) < f(a) and h(a) < h(b). Further, $f(a) \leq h(a)$ implies $f \leq h$ and therefore h(a) < f(a). Also $h(b) \leq g(b)$ and $h(a) < f(a) \leq$ g(a) implies $h \leq g$ and therefore $h \leq g$ implies g(b) < h(b). Therefore in this case $h(a) < f(a) \leq g(a)$ and $f(b) \leq g(b) < h(b)$. Hence

$$egin{aligned} [g \wedge (f ar ee h)](a) &= g(a) \wedge [f(a) \lor h(a)] = f(a) \ &= f(a) \lor [g(a) \wedge h(a)] = [f \lor (g \wedge h)](a) \ . \end{aligned}$$

Likewise

$$[g \land (f \bigtriangledown h)](b) = g(b) = [f \lor (g \land h)](b)$$
.

If $x \in S - \{a, b\}$, then

$$egin{aligned} & [g \wedge (f ar ar h)](x) = g(x) \wedge [f(a) ee h(a)] \wedge [f(b) ee h(b)] \ & = g(x) \wedge f(a) \wedge h(b) = g(x) \wedge g(a) \wedge f(a) \wedge h(b) \ & = g(b) \wedge g(a) \wedge f(a) \wedge h(b) = f(a) \wedge g(b) \ & = [g \wedge (f ar ar h)](a) \wedge [g \wedge (f ar ar h)](b) \;. \end{aligned}$$

But $[f \lor (g \land h)](x) \leq [g \land (f \bigtriangledown h)](x)$ and $g \land (f \bigtriangledown h) \in M$, therefore by Lemma 5 $g \land (f \bigtriangledown h) = f \bigtriangledown (g \land h)$. COROLLARY. If card S < 3, M is a distributive lattice. If $3 \leq \text{card}$ S and $2 \leq \text{card } D$, then M is a modular nondistributive lattice.

If card S < 3, then M = L and M is distributive. If $3 \leq \text{card } S$ and $2 \leq \text{card } D$, let s_1, s_2, s_3 be three distinct elements of S and c < dbe two elements of D. Define f_1, f_2, f_3 by $f_i(x) = d$ if $x = s_i$ and f(x) = c for $x \in S - \{s_i\}$. Also define g and k by g(s) = d for every $s \in S$ and k(s) = c for every $s \in S$. Then $f_1 \wedge f_2 = f_2 \wedge f_3 \wedge f_1 = k$ and $f_1 \bigtriangledown f_2 = f_2 \bigtriangledown f_3 = f_3 \bigtriangledown f_1 = g$ and $\{f_1, f_2, f_3, g, k\}$ is a modular five sublattice of M. Therefore M is not distributive [2].

At this stage the algebraic nature of M has been established, in the section that follows the topological nature of M will be studied. It will be assumed in the following that D is topological chain, that is D is a Hausdorff topological space with the operations \lor and \land continuous [3]. If L is considered with the product topology, it is as usual a topological lattice and M may be considered as a topological space in the relative topology that it inherits from L. In this context, the following theorem results.

```
THEOREM 2. If D is a topological chain, then
(1) M is a closed subset of L,
(2) M is compact if D is compact, and
(3) M is connected if D is connected.
```

Since with card $S \leq 2$, M = L, it suffices to consider $3 \leq \text{card } S$ and to establish (1) and (3).

(1) L-M is open for if $f \notin M$, then f is not constant and there exist distinct $a, b, c \in S$ such that f(b) < f(a) and f(b) < f(c). Then since D is a chain $f(b) < f(a) \land f(c)$. If there exists $z \in D$ such that $f(b) < z < f(a) \land f(c)$, define $W = \{g \in L \mid z < g(a), z < g(c), \text{ and } g(b) < z\}$ and define $W = \{g \in L \mid f(b) < g(a), f(b) < g(c), \text{ and } g(b) < f(a) \land f(c)\}$ if no such z exists. In either case, $f \in W$, W is open, and $W \cap M = \emptyset$.

(3) If D is connected, consider the map $T: D \to M$ where for each $d \in D$ $T(d) = k_d$ and k_d is the constant function generated by d. Clearly T is continuous and K the set of all constant functions is a connected subset of M. If $f \in M - K$, let a = index f, $m = \max f$, and $r = \min f$ define the map H from $[r, m] = \{x \in D \mid r \leq x \leq m\}$ into M by $H(x) = f_x$ where $f_x(a) = x$ and $f_x(s) = r$ for $s \in S - \{a\}$. Again H is continuous and since [r, m] is connected then the range of H is a connected subset of M containing f and intersecting K. Therefore M is connected.

Note. It is clear that \wedge will be continuous as an operation on

M since it is continuous on L. Thus when D is a topological chain M is a closed topological sub- \wedge -semilattice of L. In order to study the operation ∇ relative to continuity, it is necessary to restrict S to being finite, in view of the following lemma.

LEMMA 6. If D is a topological chain and $2 \leq \text{card } D$ and S is infinite, then ∇ is not continuous.

Let c < d in D and define $k: S \to D$ by k(s) = c for every $s \in S$. Then $k \bigtriangledown k = k$. Let $r \in S$ and define $W_r = \{f \in M \mid f(r) < d\}$; then W_r is an open subset of M containing k. Let U be any open set of M containing k, then there exist s_1, s_2, \dots, s_n distinct elements of S and U_1, U_2, \dots, U_n open sets of D such that if $W = \{f \in M \mid f(s_i) \in U_i \text{ for } i = 1, 2, \dots, n\}, k \in W \subset D$. Now $k \in W$ implies

$$c\in \ \cap \left\{ U_{i}\,|\,i=1,\,2,\,\cdots,\,n
ight\}$$
 .

Since S is infinite there exist $a, b \in S - \{s_1, s_2, \dots, s_n\}$ such that $a \neq b$. Define h and g by h(a) = d and h(x) = c if $x \in S - \{a\}$, and g(b) = dand g(x) = c if $x \in S - \{b\}$. Therefore $h, g \in W$ and $h \bigtriangledown g \notin W_r$ since $h \bigtriangledown g$ is the constant function defined by d. Therefore $U \bigtriangledown U \not\subset W_r$ and ∇ is not continuous.

DEFINITION. For S finite and $2 \leq \text{card } S$, denote max

$$f = \max \{ f(s) \mid s \in S \}$$
, $I(f) = \{ s \in S \mid f(s) = \max f \}$.

Then define $f^{-}: S \to D$ by

(1) if I(f) is not a unit set, $f^{-}(s) = \max f$ for every $s \in S$, and (2) if I(f) is a unit set, $f^{-}(s) = \max f$ for $s \in I(f)$, and

$$f^{-}(s) = \max \{ f(t) \mid t \in S - I(f) \}$$
 for $s \in S - I(f)$.

LEMMA 7. If S is finite and $2 \leq \text{card } S$, then (1) $f \in L$ implies f^-M and $f \in M$ if and only if $f = f^-$, (2) $f \leq f^-$, $f^{--} = f^-$, and $f \leq g$ implies $f^- \leq g^-$, (3) $f, g \in M$ implies $f \nabla g = (f \vee g)^-$.

The lemma is a straight forward catalog of the properties following from the definition directly.

LEMMA 8. If S is finite, $2 \leq \text{card } S$ and D is a topological chain, then the function $J: L \to M$ defined by $J(f) = f^-$ is a retraction of L onto M. From Lemma 7 it suffices to show that J is continuous. This is done by letting U be an open set in D and $r \in S$, defining $W = \{f \in M \mid f(r) \in U\}$ and showing that $J^{-1}(W)$ is open. It is shown to be open by case argument. Let $g \in L$ and $g^- \in W$. If I(g) is not a unit set, then g^- is constant and max $g \in U$. Define $V_1 = \{f \in L \mid f(s) \in U$ for every $s \in S\}$. Since S is finite, V_1 is open and contains g. Further $h \in V_1$ implies $h^-(r) \in U$. If $I(g) = \{r\}$, let b be an element of $S - \{r\}$. Define $V_2 = \{f \in L \mid f(r) \in U_1, \text{ and } f(s) \in U_2 \text{ for } s \in S - \{r\}\}$ where $U_1 = U \cap \{x \in D \mid z < x\}$ and $U_2 = \{x \in D \mid x < z\}$ if there exist $z \in D$ such that $g^-(b) < z < g^-(r)$, and if there does not exist such an element z, $U_1 = U \cap \{x \in D \mid g^-(b) < x\}$ and $U_2 = \{x \in D \mid x < g^-(r)\}$. Then $g \in V_2$, V_2 is open and $f \in V_2$ implies $f^-(r) \in U$. The other case is handled in a similar fashion.

THEOREM 3. If D is a topological chain and S is finite, then (M, \leq) is a modular topological lattice which is nondistributive if card S > 2 and card D > 1.

If card S = 1, M = L and therefore (M, \leq) is a topological distributive lattice. If card $S \geq 2$, then Lemma 7 and 8 establish that $\overline{\vee}$ is continuous since it is the composition of continuous maps. Therefore (M, \leq) is a topological lattice since \wedge is continuous for every S. Theorem 1 establishes the modularity of M while its corollary the nondistributive nature of M when $3 \leq \text{card } S$ and $2 \leq \text{card } D$.

DEFINITION. Let n be a positive integer and $3 \leq n$, then let M_n denote the lattice constructed as the M above in the case where $S = \{1, 2, \dots, n\}$ and $D = \{x \in R \mid 0 \leq x \leq 1\}$ with its usual order and the operations of D being $x \lor y = \max\{x, y\}$ and $x \land y = \min\{x, y\}$. When S is the set of positive integers and D as previously described let M_{∞} denote the lattice M constructed. Then the following results are immediate.

THEOREM 4. For each positive integer $n \ge 3$, M_n is a compact connected topological lattice which is modular and not distributive.

COROLLARY 1. M_3 is a compact connected topological lattice, modular and not distributive, which is a two dimensional subset of R^3 that cannot be embedded in R^2 .

COROLLARY 2. M_{∞} is a compact connected topological semilattice, which is a modular lattice, and not a topological lattice.

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