

Pacific Journal of Mathematics

EXTENSIONS OF PSEUDO-VALUATIONS

JAMES A. HUCKABA

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Let w be a pseudo-valuation defined on a commutative ring R and let S be an overring of R . This paper investigates conditions needed to imply that w can be extended to S . These conditions are given in terms of a particular sequence of ideals $\{A_i\}_{i=0}^{\infty}$ in R which is called the best filtration for w . The main theorem states that if w is a pseudo-valuation on R with best filtration $\{A_i\}$ and each A_i is a contracted ideal with respect to S , then w can be extended to S . The converse of this result is then proved.

By using our main theorem and some recent results by Gilmer [1], we show in several important cases that if S is an overring of R and w is any pseudo-valuation on R possessing a best filtration, then w can be extended to S . In particular, if R is a Prüfer domain with quotient field K and if S is an overring of R such that $S \cap K = R$, then w can be extended from R to S .

We begin in §1 by defining and developing properties of a best filtration and determining classes of pseudo-valuations which have best filtrations. The main results and applications are then proved in §2.

1. Filtrations. All rings are commutative, associative, and have identity. If S is an overring of R , we assume that S and R have the same identity. A *pseudo-valuation* on the ring R is a mapping w from R into the extended real number system such that:

- (i) $w(0) = \infty, w(1) = 0$,
- (ii) $w(x - y) \geq \min \{w(x), w(y)\}$,
- (iii) $w(xy) \geq w(x) + w(y)$, for each $x, y \in R$.

w is called a *homogeneous pseudo-valuation* in case:

- (iv) $w(x^2) = 2w(x)$ for each $x \in R$.

w is called a *valuation* in case:

- (v) $w(xy) = w(x) + w(y)$ for each $x, y \in R$.

Pseudo-valuations were first introduced by Rees [3]. Rees proved in [3] that (iv) is equivalent to the condition that $w(x^n) = nw(x)$ for each positive integer n and for each $x \in R$. These functions arise quite naturally in ring theory. If A is a proper ideal of R , define $v_A(x) = n$ if $x \in A^n, x \notin A^{n+1}$ and $v_A(x) = \infty$ if $x \in A^n$ for all n . Then v_A is a pseudo-valuation. We say that v_A is *associated with the ideal* A . A sequence of ideals $\{A_i\}_{i=0}^{\infty}$ of R such that $A_0 = R, A_{i+1} \subset A_i$, and $A_i A_j \subset A_{i+j}$ for all i and j is called a *filtration* on R . Notice that the nonnegative integral powers of an ideal A of R forms a filtration,

where A° is defined to be R . Also note that any filtration $\{A_i\}$ determines a pseudo-valuation in exactly the same manner that the powers of an ideal A determines v_A . For an arbitrary pseudo-valuation w on R and a subset T of R , define $w(T) = \inf \{w(t) : t \in T\}$.

DEFINITION 1. If w is a pseudo-valuation on R , define

$$(1.1) \quad \begin{aligned} A_0 &= R \\ A_i &= \{x \in R : w(x) > w(A_{i-1})\}, \\ &\quad \text{if } w(A_{i-1}) < \infty. \\ A_i &= A_{i-1}, \text{ if } w(A_{i-1}) = \infty. \end{aligned}$$

Each member of the sequence $\{A_i\}$ is an ideal of R . The sequence defined by (1.1) has the property that $A_0 \supset A_1 \supset A_2 \supset \dots$. A_{i+1} is not necessarily a proper subset of A_i , as will be shown in Examples 1 and 2. Also note that for a given pseudo-valuation w , $\{x \in R : w(x) = \infty\} \subset \cap A_i$. The following example shows that there exists pseudo-valuations such that the sequence defined by (1.1) is not a filtration.

EXAMPLE 1. Let \mathfrak{A} be an ideal of a ring R in which $\mathfrak{A}^i \not\supseteq \mathfrak{A}^{i+1}$ for all i . Define a sequence of ideals $\{B_i\}$ as follows: $B_0 = R$, $B_1 = \mathfrak{A}^2$, $B_2 = \mathfrak{A}^3$, $B_3 = B_4 = \mathfrak{A}^5$, and $B_i = \mathfrak{A}^7 (i \geq 5)$. Then $\{B_i\}$ is a filtration in R and determines some pseudo-valuation w , where $w(x) = n$ if $x \in \varepsilon B_n$, $x \notin B_{n+1}$ and $w(x) = \infty$ if $x \in \cap B_n$. Now use Definition 1 to define A_i with respect to w . We obtain $A_i = B_i (i = 0, 1, 2, 3)$ and $A_i = B_{i+1} (i = 4, 5, \dots)$. But $\{A_i\}$ is not a filtration, since $(A_2)^2 \not\subset A_4$.

DEFINITION 2. Let w be a pseudo-valuation on R and let $\{A_i\}$ be defined by (1.1). If $\{A_i\}$ is a filtration in R such that $x \in \cap A_i$ if and only if $w(x) = \infty$, then $\{A_i\}$ is called a *best filtration* for w . Let $B(R)$ denote the class of all pseudo-valuations on R which have a best filtration.

Example 1 then implies that there are pseudo-valuations which do not have best filtrations. It is clear from the definition that if w has a best filtration, then it is unique. From now on we will talk about *the* best filtration for w .

EXAMPLE 2. Let w be a pseudo-valuation on R and let $\{A_i\}$ be the sequence defined by (1.1). It is possible for $\{A_i\}$ to be a filtration in R , yet not be the best filtration for w . Let v be a real valued nondiscrete valuation on a field K and consider v as a pseudo-valuation on its valuation ring R_v . Since the value group of v has no smallest positive element, $v(A_1) = 0$. Then $A_2 = \{x \in R : v(x) > v(A_1) = 0\} = A_1$.

By induction, we see that $A_i = A_1$ for each $i \geq 1$. Hence the sequence $\{A_i\}$ is such that $A_0 \supsetneq A_1 = A_2 = \dots$. Therefore $v \notin B(R)$. However, it is clear that $\{A_i\}$ is a filtration.

We will be interested only when the sequence defined by (1.1) is a filtration. This always happens in one important case.

REMARK 1. If v is a valuation on a ring R and if $\{A_i\}$ is the sequence of ideals defined by (1.1), then $\{A_i\}$ is a filtration in R .

Proof. It is clear that $A_i \supset A_{i+1}$ for each i . Hence, to complete the proof we need to show that $A_i A_j \subset A_{i+j}$ for all nonnegative integers i and j . We fix j and use induction on i . Clearly $A_0 A_j \subset A_{0+j}$. Assume that $A_{i-1} A_j \subset A_{i+j-1}$ for $i \geq 1$. Let $x \in A_i A_j$, then $x = \sum_{k=1}^n a_k b_k$ where $a_k \in A_i$ and $b_k \in A_j$. We may assume without loss of generality that $v(a_1) + v(b_1) = \min_{k=1}^n (v(a_k) + v(b_k))$. Then $v(x) \geq v(a_1) + v(b_1)$. Case 1: If $v(A_{i-1}) < \infty$, then $v(a_1) > v(A_{i-1})$, and thus $v(x) > v(A_{i-1}) + v(A_j) = v(A_{i-1} A_j) \geq v(A_{i+j-1})$. By Definition 1, $x \in A_{i+j}$. Case 2: If $v(A_{i-1}) = \infty$, then $v(x) = \infty$, which implies that $x \in A_{i+j}$. Therefore $A_i A_j \subset A_{i+j}$.

LEMMA 1. Let $w \in B(R)$ and let $\{A_i\}$ be the best filtration for w . Then:

- (1) $A_i = A_{i+1}$ if and only if $w(A_i) = \infty$.
- (2) Let $x \in A_i$ and $x \notin A_{i+1}$. Then $y \in A_i$ and $y \notin A_{i+1}$ if and only if $w(x) = w(y)$. In fact, $w(x) = w(A_i)$.
- (3) If $y \in A_i$ and $z \notin A_i$, then $w(y) > w(z)$.
- (4) If $w(x) < \infty$, then there exists an integer i such that $x \in A_i$ and $x \notin A_{i+1}$.

Proof. (1) Suppose $A_i = A_{i+1}$. By induction we see that $A_i = A_{i+t}$ for each positive integer t . If $w(A_i) < \infty$, then there is an element $x \in A_i$ such that $w(x) < \infty$. But $x \in \bigcap A_i$ which implies that $w(x) = \infty$, a contradiction. Conversely, if $w(A_i) = \infty$, then $A_i = A_{i+1}$ by definition of the best filtration.

(2) First note that $x \in A_i, x \notin A_{i+1}$ implies that $w(x) = w(A_i)$. If $y \in A_i, y \notin A_{i+1}$, then clearly $w(x) = w(y)$. Conversely, assume $w(x) = w(y)$. If $i = 0$, then $w(y) \leq w(A_1)$ and hence y is in A_0 , but not in A_1 . If $i > 0$, then $w(A_{i-1}) \leq w(A_i)$. If equality holds, then $A_i = A_{i+1}$, which implies that $x \in A_{i+1}$. Therefore $w(A_{i-1}) < w(A_i)$, which implies that $y \in A_i$. Also $y \notin A_{i+1}$, for if so, then $w(y) > w(A_i)$.

(3) and (4) are clear.

The converse of the above result is also true.

LEMMA 2. *Let w be a pseudo-valuation on R and let $\{B_i\}$ be a filtration in R satisfying properties (1)–(4). Then $\{B_i\}$ is the best filtration for w .*

Proof. Clearly $x \in \bigcap B_i$ if and only if $w(x) = \infty$. Suppose that $w(B_{i-1}) < \infty$. By properties (2) and (3) $B_i = \{x \in R: w(x) \geq w(B_i)\}$. Thus $B_i \subset \{x \in R: w(x) > w(B_{i-1})\}$. On the other hand, suppose that $w(x) > w(B_{i-1})$. If $w(x) = \infty$, then $x \in \bigcap B_j$ and hence $x \in B_i$. If $w(x) < \infty$, choose k such that $x \in B_k$ and $x \notin B_{k+1}$. Suppose that $k \leq i-1$, then $B_k \supset B_{i-1}$, so $w(x) = w(B_k) \leq w(B_{i-1})$, a contradiction. So we must have $k > i-1$ and hence $x \in B_i$. Therefore $B_i = \{x \in R: w(x) > w(B_{i-1})\}$.

By (1), if $w(B_{i-1}) = \infty$, then $B_{i-1} = B_i$.

We assume from now on that all pseudo-valuations w which are considered have the property that there exists at least one x such that $0 < w(x) < \infty$.

LEMMA 3. (a) *If w is a homogeneous pseudo-valuation on R and if $\{A_i\}$ is the sequence of ideals defined by (1.1), then $w(A_i) < \infty$ for each i .*

(b) *If w is a pseudo-valuation on a ring R and if $\{A_i\}$ is the sequence of ideals defined by (1.1) such that each A_i is finitely generated, then $w(A_{i-1}) < \infty$ implies that $w(A_i) > w(A_{i-1})$.*

Proof. (a) Suppose, to the contrary, that i is the smallest positive integer such that $w(A_i) = \infty$. Since w is nontrivial, $i \geq 2$. Choose $x \in A_{i-1}$, $x \notin A_i$. Then $0 < w(x) < \infty$, and $w(x^2) > w(x) \geq w(A_{i-1})$, so $x^2 \in A_i$. But, $w(A_i) \leq w(x^2) = 2w(x) < \infty$, contradicting the assumption that $w(A_i) = \infty$.

(b) Let a_1, \dots, a_n be a basis for A_i . Choose a_k such that $w(a_k) = \min\{w(a_1), \dots, w(a_n)\}$. Then $w(A_i) = w(a_k)$. Since $a_k \in A_i$, $w(a_k) > w(A_{i-1})$ and therefore $w(A_i) > w(A_{i-1})$.

The following theorem shows that there are many pseudo-valuations with best filtrations.

THEOREM 1. (1) *Any pseudo-valuation associated with an ideal is in $B(R)$. More generally, any pseudo-valuation determined by a filtration $\{B_i\}$, where $B_i = B_{i+1}$ implies that $B_i = B_{i+k}$ for each positive integer k , is in $B(R)$.*

(2) *If the sequence $\{A_i\}$ of ideals defined by (1.1) is a filtration and if $\lim_{i \rightarrow \infty} w(A_i) = \infty$, then $w \in B(R)$. Both of these conditions are satisfied if w is a valuation on R and R is noetherian.*

(3) A pseudo-valuation w on R such that the range of w is equal to the set of all multiples of some positive real number $t > 0$ is in $B(R)$. This includes all integrally valued homogeneous pseudo-valuations w such that there is an $x \in R$ for which $w(x) = 1$.

(4) All integrally valued pseudo-valuations and pseudo-valuations on a noetherian ring such that (1.1) forms a filtrations are in $B(R)$.

Proof. (1) Clear.

(2) Let w and $\{A_i\}$ satisfy the hypothesis of the first statement of (2). Clearly $w(x) = \infty$ implies that $x \in \bigcap A_i$. Let $x \in \bigcap A_i$, then $w(x) \geq w(A_{i-1})$ for each i . Since $\lim_{i \rightarrow \infty} w(A_i) = \infty$, $w(x) = \infty$.

We will now prove the second statement of (2). Let v be a valuation on a noetherian ring R . By Remark 1, the sequence of ideals $\{A_i\}$ defined by (1.1) is a filtration in R . We need to show that $\lim_{i \rightarrow \infty} v(A_i) = \infty$. Consider a basis $\{y_1, \dots, y_r\}$ for the ideal A_1 . Let $v(y_1) = \min \{v(y_1), \dots, v(y_r)\}$. Then y_1 is an element of R with the property that $v(y_1) = \varepsilon$ is a minimal positive element in the range of v . Assume that $\lim_{i \rightarrow \infty} v(A_i) = t < \infty$. By Lemma 3 (b), $v(A_i) > v(A_{i-1})$ for each i . Thus we can choose a sequence $\{x_j\} \in R$ so that $v(x_j) = a_j$ where $(t - \varepsilon) < a_1 < a_2 < \dots$, and each $a_j < t$. Let B be the ideal generated by $\{x_j\}$. Since R is noetherian there exists a positive integer n so that $\{x_1, \dots, x_n\}$ is a basis of B . Let $p > n$, then $x_p \in B$ and so $x_p = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \in R$. Then $v(x_p) \geq \min \{v(\alpha_1 x_1), \dots, v(\alpha_n x_n)\}$. Let $v(\alpha_j x_j)$ be this minimum. Case 1: If $v(\alpha_j) \neq 0$, then $v(x_p) \geq v(\alpha_j) + a_j \geq \varepsilon + a_j \geq t$, which is a contradiction. Case 2: If $v(\alpha_j) = 0$, then $v(x_p) = a_p > a_j = v(x_j) = v(\alpha_j x_j)$. By properties of a valuation, $v(\alpha_j x_j) = v(\alpha_k x_k)$ for some $k \leq n$, $k \neq j$. Since $v(x_j) \neq v(x_k)$, $v(\alpha_k) \neq 0$. Hence, $v(x_p) \geq v(\alpha_k) + a_k \geq t$, a contradiction. This proves that $\lim_{i \rightarrow \infty} v(A_i) = \infty$.

(3) Define $B_0 = R$ and inductively, $B_i = \{x \in R: w(x) \geq i \cdot t\}$. The sequence $\{B_i\}$ satisfies the hypothesis of Lemma 2 and is a best filtration for w .

(4) The first part is clear. For the second part use the same technique as in (2).

2. The main results. The following notation will be used in this section. Let S be an overring of R . If A is an ideal of R , then the extension of A to S , $A \cdot S$, will be denoted by A^e . If B is an ideal of S , then the contraction of B to R , $B \cap R$, will be denoted by B^e .

THEOREM 2. Suppose that S is an overring of R , $w_0 \in B(R)$, and $\{A_i\}$ is the best filtration for w_0 . If each A_i is a contracted ideal with respect to S , then w_0 can be extended to S .

Proof. Define $B_i = A_i^e$ for each i . Then $\{B_i\}$ is a filtration on S .

Define a mapping w on S as follows: $w(x) = w_0(A_i)$ if $x \in B_i, x \notin B_{i+1}$ and $w(x) = \infty$ if $x \in \cap B_i$. We will show that w is a pseudo-valuation on S which extends w_0 to S . Property (i) of the definition of pseudo-valuation is obviously satisfied. Suppose that $x \in B_i, x \notin B_{i+1}$ and $y \in B_j, y \notin B_{j+1}$. Without loss of generality, assume that $i \leq j$. Then $x - y \in B_i$ and hence, $w(x - y) \geq w_0(A_i) = \min\{w_0(A_i), w_0(A_j)\} = \min\{w(x), w(y)\}$. Similarly if either $x \in B_i$ for all i or $y \in B_j$ for all j , then $w(x - y) \geq \min\{w(x), w(y)\}$. This proves property (ii).

Finally, we wish to show $w(xy) \geq w(x) + w(y)$. Again let $x \in B_i, x \notin B_{i+1}$ and $y \in B_j, y \notin B_{j+1}$. Then $xy \in B_i B_j \subset B_{i+j}$, so that

$$w(xy) \geq w_0(A_{i+j}) .$$

If $w_0(A_i) + w_0(A_j) \leq w_0(A_{i+j})$, then $w(xy) \geq w(x) + w(y)$. On the other hand, if $w_0(A_i) + w_0(A_j) > w_0(A_{i+j})$, there are two cases to consider. Case 1: Suppose there is a positive integer t such that

$$w_0(A_i) + w_0(A_j) \leq w_0(A_{i+j+t}) ,$$

but $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t-1})$. Then $A_i A_j \subset A_{i+j+t}$, and hence $B_i B_j \subset B_{i+j+t}$. Since $xy \in B_i B_j \subset B_{i+j+t}$, $w(xy) \geq w(x) + w(y)$. Case 2: Suppose that $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t})$ for all t . Then $w_0(A_i A_j) > w_0(A_k)$ for all k , which implies that $A_i A_j \subset A_k$ for all k . Hence,

$$xy \in A_i^e A_j^e \subset (\bigcap_{k=1}^{\infty} A_k)^e \subset \bigcap_{k=1}^{\infty} (A_k^e) = \bigcap_{k=1}^{\infty} B_k .$$

Therefore $w(xy) = \infty$ and $w(xy) \geq w(x) + w(y)$. When either $x \in \cap B_k$ or $y \in \cap B_k$, clearly $w(xy) = w(x) + w(y) = \infty$. This proves property (iii), showing that w is a pseudo-valuation on S .

It is easy to see that w extends w_0 . Take $z \in R$. If $z \in A_i, z \notin A_{i+1}$ then by Lemma 1, $w_0(z) = w_0(A_i)$. Clearly $z \in B_i$. Suppose $z \in B_{i+1}$, since z is also in $R, z \in A_{i+1}^{e^c} = A_{i+1}$ a contradiction. Therefore $z \notin B_{i+1}$, and hence $w(z) = w_0(A_i)$. If $z \in \cap A_i$, then $z \in \cap B_i$ which implies that $w_0(z) = w(z) = \infty$.

A subring R of a ring S is said to have *property C* with respect to S in case each ideal of R is a contraction of an ideal in S . In [1], Gilmer shows that in several cases, if S is an overring of R which is integrally dependent on R , then R has property *C* with respect to S . Using Gilmer's theory we obtain several applications of Theorem 2, which are listed in the corollaries below. A *Prüfer domain* is a domain R with identity in which each finitely generated ideal is invertible, or equivalently, in which R_P is a valuation ring for each prime ideal P in R . An ideal A of a ring R is called a *valuation ideal* in case there exists a valuation ring R_v containing R and an ideal B of R_v such that $B \cap R = A$.

COROLLARY 1. *Suppose that R is a Prüfer domain with quotient field K and that R is a subdomain of R_1 . If $R_1 \cap K = R$, then every $w \in B(R)$ can be extended to R_1 .*

Proof. By [1; p. 563, Corollary 2], R has property C with respect to R_1 . Then each ideal in a best filtration for w is a contracted ideal with respect to R_1 . By Theorem 2, w can be extended to R_1 .

COROLLARY 2. *Let R be a domain, let $w \in B(R)$, suppose that R_1 is integral over R , and let $\{A_i\}$ be the best filtration for w . If each A_i is an intersection of valuation ideals of R , then w can be extended to R_1 .*

Proof. Apply [1; p. 564, Th. 2] and Theorem 2.

It is known that if R is an integrally closed domain, A is a complete ideal in R if and only if A is the intersection of valuation ideals. Now let R be an integrally closed domain with quotient field K , L a finite algebraic extension of K , and R' the integral closure of R in L . By [1; p. 569, Th. 6] and Theorem 2, we have:

COROLLARY 3. *If R' has an integral basis over R and if $w \in B(R)$, then w can be extended to R' .*

THEOREM 3. *Suppose that R is a subring of the ring S and suppose that w_0 is a pseudo-valuation on R which has an extension to a pseudo-valuation w on S . If α belongs to the set of extended reals, then the ideals $A_\alpha = \{x \in R: w_0(x) > \alpha\}$ and $B_\alpha = \{x \in R: w_0(x) \geq \alpha\}$ are contractions of ideals of S .*

Proof. A_α is the contraction of $A'_\alpha = \{x \in S: w(x) > \alpha\}$ and B_α is the contraction of $B'_\alpha = \{x \in S: w(x) \geq \alpha\}$.

The converse of Theorem 2 is also true.

THEOREM 4. *Let S be an overring of R , let $w_0 \in B(R)$, and let $\{A_i\}$ be the best filtration for w_0 . If w_0 can be extended to S , then each A_i is a contracted ideal with respect to S .*

Proof. Apply Theorem 3.

COROLLARY 4. *Suppose $w_0 \in B(R)$ can be extended to some w on R_v , where R_v is a valuation ring. Then each ideal in the best filtration of w_0 is a valuation ideal.*

REMARK 2. Let R be a domain with quotient field K and w_0 a

nonnegative pseudo-valuation on R . (Nonnegative pseudo-valuations were the most important types of pseudo-valuations studied in [2] and [3]). w_0 can always be extended to a nonnegative pseudo-valuation w on a subring R' , where $R \subset R' \subset K$, in the following way. Let M be the set of $y \in R$ such that $w_0(y) < \infty$ and $w_0(xy) = w_0(x) + w_0(y)$ for all $x \in R$. Then M is a multiplicative subset of R not containing zero. Hence we can form the quotient ring of R with respect to M , R_M . A function w' can be defined on R_M by $w'(x/y) = w_0(x) - w_0(y)$. w' is not necessarily nonnegative. However, if $R' = \{z \in R_M: w'(z) \geq 0\}$ and w is the restriction of w' to R' , then R' is a ring and w is an extension of w_0 to R' . R' is called the *natural domain* of w_0 . This type of extension was discussed and used in [2].

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REFERENCES

1. Robert W. Gilmer, Jr., *Contracted ideals with respect to integral extensions* Duke Math. J. **34** (1967), 561-571.
2. James A. Huckaba, *Some results on pseudo-valuations* (to appear in Duke Math. J.)
3. D. Rees, *Valuations associated with a local ring I* , Proc. London Math. Soc. (3) **5** (1955), 107-128.

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