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FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS. II

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In a previous paper one of the authors considered groups G with r. b. n (representation bound n) and $n < p^2$ for some prime p . Here we continue this study. We first offer a new proof of the fact that if $n = p - 1$ then G has a normal Sylow p -subgroup. Next we show that if $n = p^{3/2}$ then $p^2 \nmid |G/O_p(G)|$. Finally we consider $n = 2p - 1$ and with the help of the modular theory we obtain a fairly precise description of the structure of G . In particular we show that its composition factors are either p -solvable or isomorphic to $PSL(2, p)$, $PSL(2, p - 1)$ for p a Fermat prime or $PSL(2, p + 1)$ for p a Mersenne prime.

Now the irreducible characters of $PSL(2, p)$ have degrees (see [10] p. 128) $1, p, p \pm 1, (p \pm 1)/2$ for p odd and those of $PSL(2, 2^a)$ have degrees (see [10] p. 134) $1, 2^a, 2^a \pm 1$. Thus for $p > 2$ the linear groups of the preceding paragraph do in fact have r.b. $(2p - 1)$.

The notation here is standard. In addition, if χ is a character of G we let $\det \chi$ denote the linear character which is the determinant of the representation associated with χ . Also $n_p(G)$ denotes the number of Sylow p -subgroups of G .

LEMMA 1. *Let G be a group with r.b.n. and let $N \neq G$ be a subgroup. Suppose $G = \bigcup_{i=0}^t Nx_iN$ is the (N, N) -double coset decomposition of G with $x_0 = 1$. Set $a_i = |Nx_iN|/|N| = [N : N \cap N^{x_i}]$. Then $n \geq (a_1 + a_2 + \dots + a_t)/t$.*

Proof. Let $\theta = (1_N)^G$ be the character of the permutation representation of G on the cosets of N . Then $\theta(1) = [G : N]$, $[\theta, 1_G] = 1$ and $\|\theta\|^2 = 1 + t$. Since $[\theta, 1_G] = 1$ we can write

$$\theta = 1_G + b_1\chi_1 + \dots + b_s\chi_s$$

where the χ_i are distinct nonprincipal irreducible characters of G . Thus since G has r.b. n we have

$$\begin{aligned} 1 + nt &= 1 + n(\|\theta\|^2 - 1) = 1 + n(b_1^2 + \dots + b_s^2) \\ &\geq 1 + n(b_1 + \dots + b_s) \geq 1 + b_1\chi_1(1) + \dots + b_s\chi_s(1) \\ &= \theta(1) = 1 + (a_1 + a_2 + \dots + a_t) \end{aligned}$$

and the result follows.

LEMMA 2. *Let G be a group with r.b.n.*

(i) *Let $N \neq G$ be a subgroup. Then*

$$n \geq \min \{[N: N \cap N^x] \mid x \in G - N\}.$$

(ii) *Let π be a set of primes and let H be a maximal π -subgroup of G . Then either $H \trianglelefteq G$ or $n \geq \min \{[H: H \cap H^x] \mid x \in G - N(H)\}$.*

Proof. (i) follows immediately from Lemma 1. Now let H be as in (ii) and suppose H is not normal in G . Set $N = N(H) \neq G$. Since H is a maximal π -subgroup it follows that $H = O_\pi(N)$. Thus if $x \in G$ then $H^x = O_\pi(N^x)$ so $H \cap N^x = H \cap H^x$ and

$$[N: N \cap N^x] \geq [H: H \cap N^x] = [H: H \cap H^x].$$

Thus the result follows from (i).

Applying Lemma 2(ii) with $\pi = \{p\}$ and H a Sylow p -subgroup of G yields

THEOREM 3. *Let p be a prime and let G be a group with r.b. $(p-1)$. Then $n_p(G) = 1$.*

This result was originally proved in [7] (Theorem E) in a much more complicated way.

LEMMA 4. *Let G have r.b. (p^2-1) and let Q_1 and Q_2 be p -subgroups of G with $\langle Q_1, Q_2 \rangle$ not a p -group. Then $n_p(C(Q_1) \cap C(Q_2)) = 1$. If further the Sylow p -subgroups of G are abelian, then*

$$n_p(N(Q_1) \cap N(Q_2)) = 1$$

Proof. Set $W = \langle Q_1, Q_2 \rangle$. Since W is not a p -group we see that $n_p(W) > 1$. We assume now that $n_p(C) > 1$ where $C = C(Q_1) \cap C(Q_2) = C(W)$ and derive a contradiction. Set $Z = W \cap C$ so that Z is central in W and C and let $\bar{W} = W/Z$, $\bar{C} = C/Z$. Since Z is central we have easily $n_p(\bar{W}) > 1$, $n_p(\bar{C}) > 1$ and $(WC)/Z = \bar{W} \times \bar{C}$. By Theorem 3 both \bar{W} and \bar{C} have irreducible characters of degree $\geq p$ and hence $\bar{W} \times \bar{C}$ has an irreducible character of degree $\geq p^2$. This is a contradiction since G has r.b. (p^2-1) and this property is inherited by subgroups and quotient groups. If the Sylow p -subgroups of G are abelian then any p -group normalizing Q_i centralizes it. Thus the second result follows from the first.

THEOREM 5. *Let p be a prime and let G be a group with r.b. $p^{3/2}$. Then $p^2 \nmid |G/O_p(G)|$.*

Proof. If $p = 2$ then G has r.b.2 and the result follows from Theorem C of [7]. Thus we can assume that $p \geq 3$ and clearly also that $O_p(G) = \langle 1 \rangle$. Since $p^2 - p - 1 \geq [p^{3/2}]$ for $p \geq 3$, Proposition 1.3 of [6] implies that a Sylow p -subgroup P of G is abelian. We assume that $|P| \geq p^2$ and derive a contradiction. Set $n = p^{3/2}$.

Let $N = N(P)$ so that $N \neq G$. By Lemma 2(i) there exists $w \in G - N$ with $n \geq [N : N \cap N^w] \geq [P : P \cap P^w]$. Set $Q = P \cap P^w$ so since $p^2 > n$ and $w \notin N$ we see that $[P : Q] = p$ and hence $Q \neq \langle 1 \rangle$. Let $M = N(Q)$. Since $P \trianglelefteq N$, $P^w \trianglelefteq N^w$ we have $Q \trianglelefteq (N \cap N^w)$. Also $Q \trianglelefteq P$ and $P \not\subseteq N^w$. Hence $M \cap N \cong \langle P, N \cap N^w \rangle$ so $[N : N \cap M] \leq n/p = p^{1/2}$.

We now make the following crucial observation. If $[M : M \cap M^x] < p^2$ for some $x \in M$ then Q and Q^x commute elementwise and $x \in MNM$. To see this suppose that Q and Q^x do not commute. Then since the Sylow p -subgroups of G are abelian, $\langle Q, Q^x \rangle$ is not a p -group. By Lemma 4, $n_p(M \cap M^x) = 1$ so if $U = O_p(M \cap M^x)$ then U is also a Sylow p -subgroup of $M \cap M^x$. Now $p^2 \nmid [M : M \cap M^x]$ and $Q \not\subseteq M^x$ clearly so QU is a Sylow p -subgroup of M . Since $N_M(QU) \cong \langle Q, M \cap M^x \rangle$ we have $[M : N_M(QU)] < p$ and hence by Sylow's theorem $QU \trianglelefteq M$ and $n_p(M) = 1$. This is a contradiction since $Q = P \cap P^w$ and $P, P^w \not\subseteq M$. Thus Q and Q^x commute and since $Q \neq Q^x$ and $[P : Q] = p$ it follows that $QQ^x = P^{y^{-1}}$ is a Sylow p -subgroup of G . Thus Q, Q^y and Q^{xy} are all contained in P . By Burnside's lemma these three groups are conjugate in N . Thus $Q^y = Q^h$, $Q^{xy} = Q^k$ for some $h, k \in N$. This yields $yh^{-1} \in M$, $xyk^{-1} \in M$ so

$$x = (xyk^{-1})kh^{-1}(yh^{-1})^{-1} \in MNM.$$

Since $Q \neq \langle 1 \rangle$ we have $M \neq G$. Let $G = \bigcup_{i=0}^t Mx_iM$ be the (M, M) -double coset decomposition of G with $x_0 = 1$. Set $a_i = |Mx_iM|/|M|$ and suppose that there are precisely r such $i \neq 0$ with $a_i < p^2$ and s with $a_i \geq p^2$. Then by Lemma 1, $p^{3/2} = n \geq (r + p^2s)/(r + s)$. Clearly $r \neq 0$ here so

$$p^{3/2} \geq (r + p^2s)/(r + s) > p^2/(1 + r/s).$$

If $s = 0$ then by the preceding paragraph Q commutes with all its conjugates. This implies that $\langle Q^x \mid x \in G \rangle$ is a nontrivial normal p -subgroup of G , a contradiction. Thus $s \geq 1$. Also if $a_i < p^2$ then $Mx_iM \subseteq MNM$ by the above. Since we have seen that $[N : N \cap M] \leq p^{1/2}$ we have $r \leq p^{1/2} - 1$ since the double coset M itself is not counted. Thus $r/s \leq p^{1/2} - 1$ and

$$p^{3/2} > p^2/(1 + r/s) \geq p^{3/2}$$

a contradiction and the result follows.

We now turn to the main result of this paper.

THEOREM 6. *Let p be a prime and let G be a nonabelian simple group with r.b.($2p - 1$). Then $p > 2$ and we have one of the following.*

- (i) G is a p' -group.
- (ii) $G \cong \text{PSL}(2, p)$ for $p > 3$.
- (iii) $G \cong \text{PSL}(2, p - 1)$ for p a Fermat prime, $p > 3$.
- (iv) $G \cong \text{PSL}(2, p + 1)$ for p a Mersenne prime.

Proof. Since groups with r.b.3 are solvable (Corollary 6.5 of [8]) we have $p > 2$. By Theorem 5 since $p^{3/2} \geq 2p - 1$ for $p > 2$ we have $p^2 \nmid |G|$. If $p \nmid |G|$ then G satisfies (i) above. Thus we can assume now that G has a Sylow p -subgroup P of order p . Let $B_1(p)$ denote the principal p -block of G . We will use freely the structure of $B_1(p)$ and its associated tree as described in [1] and [2]. Since $B_1(p)$ contains a nonprincipal irreducible character χ and $\chi(1) < 2p$, Lemma 1 of [3] implies that P is self centralizing. Let $N = N(P)$ and let $e = |N/P|$ so that $e \mid (p - 1)$. By Burnside's transfer theorem $e > 1$. We assume now that $G \not\cong \text{PSL}(2, p)$ and $G \not\cong \text{PSL}(2, p - 1)$ for p a Fermat prime.

Step 1. Let θ denote an exceptional character in $B_1(p)$. Then the tree of $B_1(p)$ must be one of the following.

- (1) $e = 2$ $1_G \circ \frac{\theta}{p+2} \frac{\chi_1}{p+1}$
- (2) $e = 2$ $1_G \circ \frac{\chi_1}{2p-1} \frac{\theta}{2p-2}$
- (3) $e = 3$ $1_G \circ \frac{\chi_1}{p-1} \frac{\theta}{2p-3} \frac{\chi_2}{p-1}$
- (4) $e = 4$ $1_G \circ \frac{\chi_1}{p-1} \frac{\theta}{2p-4} \frac{\chi_2}{2p-1} \frac{\chi_3}{p+1}$
- (5) $e = 4$ $1_G \circ \frac{\chi_2}{2p-1} \frac{\theta}{2p-4} \frac{\chi_1}{p-1}$
 $\quad \quad \quad \searrow$
 $\quad \quad \quad \chi_3$
 $\quad \quad \quad p+1$
- (6) $e = 5$ $1_G \circ \frac{\chi_1}{2p-1} \frac{\theta}{2p-5} \frac{\chi_2}{2p-1} \frac{\chi_3}{p+1}$
 $\quad \quad \quad \searrow$
 $\quad \quad \quad \chi_4$
 $\quad \quad \quad p+1$

Here the degree of the character is written below the character designation.

Since G is nonabelian and simple it follows that every nonprincipal irreducible representation of G is faithful. Thus by Theorem 2 of [5] the degrees of the nonprincipal ordinary irreducible characters of G are all $\geq p - 1$ and by Theorem 1 of [4] the degrees of the nonprincipal

irreducible Brauer characters of G are $\geq 2(p-1)/3$. Now G has r.b. $(2p-1)$ and the degrees of the ordinary irreducible characters of $B_1(p)$ satisfy $\chi(1) \equiv \pm 1(p)$ if χ is nonexceptional and $\theta(1) \equiv \pm e(p)$. This yields easily $\chi(1) = p-1, p+1, 2p-1$ or $\chi = 1_G$ and $\theta(1) = 2p-e, p+e$ or $\theta(1) = e$ if $e = p-1$.

Suppose first that $p = 3$. Since $e \mid (p-1)$ we have $e = 2$ and the tree is a line with three vertices. Now the center degree is maximal and the principal character must occur so we have clearly

$$1_G \circ \text{---} \overset{\chi_1}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi_2}{\underset{\circ}{\text{---}}} \text{---} \frac{\chi_2}{4}.$$

This is either tree (1) or tree (2) according to which of χ_1 or χ_2 we consider exceptional.

Now assume that $p \geq 5$. Let $\chi \in B_1(p)$ with $\chi(1) = p-1$ and suppose that χ is modular reducible. We can denote this latter fact graphically by

$$\overset{\chi_1}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi_2}{\underset{\circ}{\text{---}}} \text{---} \frac{\chi_2}{s},$$

where r and s are degrees of modular constituents. Since $r+s \leq p-1$ we cannot have both $r, s \geq 2(p-1)/3$. This implies that one of the modular constituents is the principal character and the other has degree $p-2$.

Now let $\chi \in B_1(p)$ with $\chi(1) = p+1$ and suppose χ is modular reducible. Say we have

$$\overset{\chi_1}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi_2}{\underset{\circ}{\text{---}}} \text{---} \frac{\chi_2}{s}.$$

Here by the alternating nature of the tree $\chi_1, \chi_2 \neq 1_G$ so $r, s \geq 2(p-1)/3$. Since $r+s \leq p+1$ this yields $4(p-1)/3 \leq p+1$ so $p = 5$ or 7 . In fact either $p = 5, r = s = 3$ or $p = 7, r = s = 4$. We consider these in turn. Let $p = 5$. Since $e \mid (p-1)$ we have $e = 2$ or 4 . Certainly $e \neq 2$ here since $B_1(p)$ contains $e+1$ ordinary irreducible characters including 1_G and hence $e = 4$. Thus the degrees of the nonprincipal ordinary irreducible characters of $B_1(p)$ are $4, 6$ or 9 and the nonprincipal Brauer characters have degrees ≥ 3 . The former fact implies that χ_1 and χ_2 are modular reducible so the tree having five vertices is a straight line. This yields easily

$$1_G \circ \text{---} \overset{\chi_1}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi_2}{\underset{\circ}{\text{---}}} \text{---} \overset{\chi_3}{\underset{\circ}{\text{---}}} \text{---} \frac{\chi_3}{6}$$

and we obtain tree (4). Now let $p = 7$ so that $e = 2, 3$ or 6 . Clearly neither χ_1 nor χ_2 can be modular irreducible so the tree has at least five vertices and hence $e = 6$. Thus the degrees of the nonprincipal

ordinary irreducible characters of $B_1(p)$ are 6, 8 or 13 and the non-principal Brauer characters have degrees ≥ 4 . This latter fact implies that neither χ_1 nor χ_2 can have degree 6 and hence $\chi_1(1) = \chi_2(1) = 13$. Now at least one of χ_1 or χ_2 , say χ_1 , is not adjacent to 1_G . Hence all characters adjacent to χ_1 have degree $p + 1$ and as we have seen these are either modular irreducible or have constituents of degree 4. This shows that all modular constituents of χ_1 have degree divisible by 4, a contradiction since $\chi_1(1) = 13$. Thus this case does not occur. It now clearly suffices to assume for the remainder of this step that all $\chi \in B_1(p)$ with $\chi(1) = p + 1$ are modular irreducible.

Let θ denote an exceptional character of $B_1(p)$ and we consider the possible branches leaving the vertex associated with θ . Suppose first that $\theta(1) \equiv +e(p)$. The above implies easily that we can only have

$$\begin{array}{c} \theta \\ \circ \text{---} 1_G \\ \theta \\ \circ \text{---} \underset{p+1}{\underset{\circ}{\chi}} \end{array} .$$

Now the first branch must occur precisely once and let the second branch occur a times. Since the tree has e edges we have

$$\begin{aligned} 1 + a &= e \\ 1 + a(p + 1) &= \theta(1) . \end{aligned}$$

Now $e \geq 2$ so $a \geq 1$ and hence $\theta(1) > p$. Thus $\theta(1) = p + e$ and we obtain $a = 1, e = 2$ and this is tree (1).

Now let $\theta(1) \equiv -e(p)$ so that $\theta(1) = 2p - e$. Using the above information and the alternating nature of the tree we see easily that the only possible branches leaving the vertex associated with θ are

$$\begin{array}{c} \theta \\ \circ \text{---} \underset{p-1}{\underset{\circ}{\chi}} \\ \theta \\ \circ \text{---} \underset{p-2}{\underset{\circ}{\chi}} \text{---} \underset{p-1}{\underset{\circ}{\chi}} \text{---} 1_G \\ \theta \\ \circ \text{---} \underset{p-2}{\underset{\circ}{\chi}} \text{---} \underset{2p-1}{\underset{\circ}{\chi}} \text{---} \underset{p+1}{\underset{\circ}{\chi'}} \\ \theta \\ \circ \text{---} \underset{p-3}{\underset{\circ}{\chi}} \text{---} \underset{2p-1}{\underset{\circ}{\chi}} \text{---} \underset{p+1}{\underset{\circ}{\chi'}} \\ \qquad \qquad \qquad \searrow \\ \qquad \qquad \qquad \circ \\ \qquad \qquad \qquad 1_G \\ \theta \\ \circ \text{---} \underset{2p-2}{\underset{\circ}{\chi}} \text{---} \underset{2p-1}{\underset{\circ}{\chi}} \text{---} 1_G . \end{array}$$

If the last branch occurs then since $2p - e = \theta(1) \geq 2p - 2$ we have $e = 2$ and this is tree (2). Thus we can assume that only the first

four branches occur say with multiplicities a, b, c, d respectively. Since there are precisely e edges in the tree we have

$$a + 2b + 2c + 3d = e$$

$$a(p-1) + b(p-2) + c(p-2) + d(p-3) = \theta(1) = 2p - e.$$

Adding these two and dividing by p yields

$$a + b + c + d = 2.$$

In addition the vertex of 1_G occurs precisely once so $b + d = 1$. Thus $a + c = 1$ and there are four possibilities which are easily seen to be trees (3), (4), (5) and (6).

Step 2. Let $N = N(P)$. We consider the restriction of the ordinary irreducible characters of G to N .

Now $N = PE$ is a Frobenius group of order pe with $E = \langle x \rangle$ cyclic of order e . N has precisely e linear characters, namely those of $N/P \cong E$, and the remaining irreducible characters have degree e . Let Δ denote any sum of irreducible characters of N of degree e . Clearly $\Delta_E = \Delta(1)/e \cdot \rho_E$ where ρ_E is the regular character of E . This yields easily

$$\Delta(x) = 0$$

$$(1) \quad \det \Delta(x) = \begin{cases} 1 & \text{if } e \text{ is odd} \\ (-1)^{d(1)/e} & \text{if } e \text{ is even.} \end{cases}$$

If e is even let δ denote the linear character of N given by $\delta(x) = -1$.

Let ψ be an ordinary irreducible character of G with $\psi \in B_1(p)$. Since P is self centralizing it follows that $B_1(p)$ is the unique p -block of positive defect and hence ψ belongs to a block of defect 0. Thus $p \mid \psi(1)$. Since G has r.b. $(2p-1)$ this yields $\psi(1) = p$ and clearly $\psi_P = \rho_P$. Thus $\psi_N = \lambda + \Delta$ where λ is linear and $\Delta(1) = p-1$. Now G is simple so the linear character $\det \psi$ must be principal and hence $1 = \det \psi(x) = \lambda(x) \det \Delta(x)$. This yields by the above

$$(2) \quad \psi_N = \begin{cases} 1_N + \Delta & \text{if } e \text{ is odd} \\ 1_N + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is even} \\ \delta + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is odd.} \end{cases}$$

Now let $\chi \in B_1(p)$ and let $m(\chi)$ denote the number of linear characters counting multiplicities which occur in χ_N . Obviously $m(\chi)$ is the multiplicity of 1_P in χ_P . Suppose first that χ_i, χ_j are nonexceptional ordinary irreducible characters of $B_1(p)$ which are adjacent in the tree. Then $\chi_i + \chi_j = \Phi$ is a projective character and since $\Phi_P = \Phi(1)/p \cdot \rho_P$ we have easily

$$m(\chi_i) + m(\chi_j) = [\chi_i(1) + \chi_j(1)]/p.$$

Now suppose that χ_i is adjacent to the vertex of the exceptional characters $\{\theta_k\}$. Clearly $m(\theta_r) = m(\theta_s)$ for all r, s so since $\chi_i + \sum_{k=1}^{(p-1)/e} \theta_k = \Phi$ we have with $\theta = \theta_1$

$$m(\chi_i) + (p-1)/e \cdot m(\theta) = [\chi_i(1) + (p-1)/e \cdot \theta(1)]/p.$$

Using the above two equations, the fact that the tree of $B_1(p)$ is connected and the obvious fact that $m(1_G) = 1$ we obtain easily for irreducible χ

$$(3) \quad m(\chi) = \begin{cases} k-1 & \text{if } \chi(1) = kp-1 \\ k & \text{if } \chi(1) = kp \pm e \\ k+1 & \text{if } \chi(1) = kp+1. \end{cases}$$

There is of course additional information available, for example the fact that $\det \chi(x) = 1$ and the position of χ in the tree, which further limits the structure of χ_N .

Step 3. If tree (1) occurs in step 1, then p is a Mersenne prime and $G \cong PSL(2, p+1)$.

By assumption $e = 2$ and the tree of $B_1(p)$ has the form

$$1_G \circ \text{---} \frac{1_G}{p+2} \text{---} \frac{\theta_i}{p+2} \text{---} \frac{\zeta}{p+1} \text{---} \frac{\chi}{p+1}$$

where θ_i represents the $(p-1)/2$ exceptional characters with $\theta = \theta_1$. Let $\{\psi_j \mid j = 1, 2, \dots, k\}$ denote the set of irreducible characters of G not in $B_1(p)$.

Let α denote the unique nonprincipal linear character of N . By (1) and (2) we have

$$(4) \quad \begin{aligned} \psi_{jN} &= 1_N + \Delta'_j, & \psi_j(x) &= 1 & \text{for } p \equiv 1 \pmod{4} \\ \psi_{jN} &= \alpha + \Delta'_j, & \psi_j(x) &= -1 & \text{for } p \equiv 3 \pmod{4}. \end{aligned}$$

By (3), $\theta_{iN} = \lambda + \Delta_i$ and since the θ_i are all algebraically conjugate λ is the same for all i . Thus $\det \theta_i(x) = 1$ and equation (1) yield

$$(5) \quad \begin{aligned} \theta_{iN} &= \alpha + \Delta_i, & \theta_i(x) &= -1 & \text{for } p \equiv 1 \pmod{4} \\ \theta_{iN} &= 1_N + \Delta_i, & \theta_i(x) &= 1 & \text{for } p \equiv 3 \pmod{4}. \end{aligned}$$

Now $\chi_N = a1_N + b\alpha + \Delta$ with $a + b = 2$ by (3) and since x is a p' -element $a - b = \chi(x) = \zeta(x) = \theta(x) - 1$. Thus (5) yields

$$(6) \quad \begin{aligned} \chi_N &= 2\alpha + \Delta, & \chi(x) &= -2 & \text{for } p \equiv 1 \pmod{4} \\ \chi_N &= 1_N + \alpha + \Delta, & \chi(x) &= 0 & \text{for } p \equiv 3 \pmod{4}. \end{aligned}$$

Equations (4), (5) and (6) and Frobenius reciprocity now yield

$$\begin{aligned}
 (7) \quad (1_N)^G &= 1_G + \sum_1^k \psi_j, & \alpha^G &= 2\chi + \sum_1^{(p-1)/2} \theta_i \quad \text{for } p \equiv 1 \pmod{4} \\
 (1_N)^G &= 1_G + \chi + \sum_1^{(p-1)/2} \theta_i, & \alpha^G &= \chi + \sum_1^k \psi_j \quad \text{for } p \equiv 3 \pmod{4}.
 \end{aligned}$$

Thus since $(1_N)^G(1) = \alpha^G(1) = [G:N]$ and $|N| = 2p$ we obtain easily

$$\begin{aligned}
 (8) \quad |G| &= p(p^2 + 5p + 2), & k &= (p+5)/2 \quad \text{for } p \equiv 1 \pmod{4} \\
 |G| &= p(p+1)(p+2), & k &= (p+1)/2 \quad \text{for } p \equiv 3 \pmod{4}.
 \end{aligned}$$

Now using $|C(x)| = \sum \eta(x)\overline{\eta(x)}$, where η runs over all ordinary irreducible characters of G , along with equations (4), (5), (6) and (8) we have

$$\begin{aligned}
 (9) \quad |C(x)| &= p+7 \quad \text{for } p \equiv 1 \pmod{4} \\
 |C(x)| &= p+1 \quad \text{for } p \equiv 3 \pmod{4}.
 \end{aligned}$$

Since $(p+7) \nmid p(p^2 + 5p + 2)$ the case $p \equiv 1 \pmod{4}$ is eliminated. Thus $p \equiv 3 \pmod{4}$.

Set $S = C(x)$ so that $|S| = p+1$ and $[G:S] = p(p+2)$. We consider $(1_S)^G$. Since this character is rational and the θ_i are algebraically conjugate we have

$$(1_S)^G = 1_G + a \sum_1^{(p-1)/2} \theta_i + b\chi + \sum_1^k c_j \psi_j.$$

Set $c = \sum_1^k c_j$. By considering degrees we have

$$p(p+2) = 1 + a(p+2)(p-1)/2 + b(p+1) + cp$$

and evaluating at x yields

$$0 < (1_S)^G(x) = 1 + a(p-1)/2 - c$$

by (4), (5), (6) and the fact that $x \in S$. Certainly $a \leq 2$. Also $b \leq \chi(1)/2 = (p+1)/2$ by Frobenius reciprocity and the fact that $\chi(x) = 0$. Thus $a = 0$ yields a contradiction. If $a = 1$ then $b \equiv 0 \pmod{p}$ so $b = 0$ and $c = (p+3)/2 > 1 + a(p-1)/2$ again a contradiction. Thus $a = 2$ and we have easily

$$(10) \quad (1_S)^G = 1_G + 2 \sum_1^{(p-1)/2} \theta_i + \chi$$

so $(1_S)^G(x) = p$ by (5) and (6). By definition of induced character and the fact that $S = C(x)$ this implies that S contains precisely p distinct conjugates of x . Since $|S| = p+1$ this shows that S is an elementary abelian 2-group and therefore that S is a Sylow 2-subgroup of G and p is a Mersenne prime. By Burnside's lemma the nonidentity elements of S are all conjugate in $N(S)$ so $N(S) > S$.

Set $H = N(S) > S$. Then $(1_H)^G$ is a national constituent of $(1_S)^G$ and $(1_H)^G(1) \leq p(p+2)/3$. Thus by (10) we have easily

$$(11) \quad (1_H)^G = 1_G + \chi.$$

Therefore G is a doubly transitive permutation group on the set Ω where $H = G_\infty$ for some point $\infty \in \Omega$. By (10) χ_s contains 1_s with multiplicity one so (11) implies that S has two orbits on Ω . Hence since $|S| = |\Omega| - 1$, S is in fact a regular normal subgroup of G_∞ . Now $|H| = p(p+1)$ so if \tilde{P} is a Sylow p -subgroup of H , then since \tilde{P} is self centralizing and $|\tilde{P}| = |S| - 1$ we see that G is sharply 3-transitive.

With the structure of H as given above we can clearly identify Ω with $GF(p+1) \cup \{\infty\}$ in such a way that S is the set of translations $\left\{ \begin{pmatrix} z \\ z+r \end{pmatrix} \middle| r \in GF(p+1) \right\}$ and \tilde{P} is the set $\left\{ \begin{pmatrix} z \\ sz \end{pmatrix} \middle| s \in GF(p+1), s \neq 0 \right\}$. Let $\tilde{x} \in G$ with $\tilde{x} = (0 \infty)(1) \dots$. Then \tilde{x} has order 2 and normalizes $\tilde{P} = G_{\infty_0}$ so \tilde{x} acts in a dihedral manner on \tilde{P} . If $\tilde{x} = \begin{pmatrix} z \\ f(z) \end{pmatrix}$ then for all $s \in GF(p+1)$, $s \neq 0$

$$\begin{pmatrix} z \\ sz \end{pmatrix} \begin{pmatrix} z \\ f(z) \end{pmatrix} = \begin{pmatrix} z \\ f(z) \end{pmatrix} \begin{pmatrix} z \\ s^{-1}z \end{pmatrix}$$

so $f(sz) = s^{-1}f(z)$. Setting $z = 1$ yields $f(s) = s^{-1}$. Thus $\tilde{x} = \begin{pmatrix} z \\ 1/z \end{pmatrix}$ and since $G = \langle H, \tilde{x} \rangle$ we have clearly $G \subseteq PSL(2, p+1)$. By (8) we have in fact $G \cong PSL(2, p+1)$ and this step follows.

Step 4. Completion of the proof.

We now consider the remaining trees in turn. Let $\{\psi_j \mid j = 1, 2, \dots, k\}$ denote the set of ordinary irreducible characters of G not in $B_1(p)$.

Suppose first that we have tree (2). If $p = 3$ this is the same as tree (1) so we assume that $p > 3$. From

$$1_G \circ \frac{1_G}{2p-1} \underset{\circ}{\chi} \underset{\circ}{\zeta} \underset{\circ}{\theta_i} \underset{2p-2}{\theta_i}$$

and (3) and $\det \chi(x) = 1$ we have $\chi_N = 1_N + \Delta$. Let α be the unique nonprincipal linear character of N so that we have by (3) $\theta_{iN} = a1_N + b\alpha + \Delta_i$ with $a + b = 2$. Since x is a p' -element $a - b = \theta_i(x) = \zeta(x) = \chi(x) - 1 = 0$ so $\theta_{iN} = 1_N + \alpha + \Delta_i$. Now by (2) all the ψ_j occur in either $(1_N)^G$ or α^G depending on the parity of $(p-1)/2$. Since $(1_N)^G(1) = \alpha^G(1)$, the above and Frobenius reciprocity imply that the ψ_j occur in α^G and hence

$$(1_N)^G = 1_G + \chi + \sum_1^{(p-1)/2} \theta_i.$$

Since $|N| = 2p$ this yields $|G| = 2p(p^2 + 1)$. Now $\theta_i(1) \mid |G|$ so $(p-1) \mid (p^2 + 1)$ and this is easily seen to be a contradiction for $p > 3$.

Now consider tree (3)

$$1_G \circ \frac{1_G}{p-1} \frac{\chi_1}{\circ} \frac{\zeta_1}{2p-3} \frac{\theta_2}{\circ} \frac{\zeta_2}{p-1} \frac{\chi_2}{\circ}.$$

By (3), $\chi_{iN} = \Delta_i$ and $\chi_i(x) = 0$ for $i = 1, 2$. This implies that $\zeta_i(x) = -1$, $\zeta_2(x) = 0$ so $\theta_i(x) = -1$. Now by (3), $\theta_{iN} = a1_N + b\alpha + c\alpha^2 + \Delta'_i$ where α is a nonprincipal linear character and $a + b + c = 2$. Since $\theta_i(x) = -1$ we have easily $\theta_{iN} = \alpha + \alpha^2 + \Delta'_i$. Applying Frobenius reciprocity to the above and (2) we have

$$(1_N)^G = 1_G + \sum_1^k \psi_j, \quad \alpha^G = \sum_1^{(p-1)/3} \theta_i$$

and this yields easily

$$|G| = p(p-1)(2p-3), \quad k = (2p-5)/3.$$

Using $|C(x)| = \sum \bar{\gamma}(x)\gamma(x)$ along with the above and (2) we obtain $|C(x)| = p-1$. Now clearly x is a real element so $|C^*(x)| = 2(p-1)$ where $C^*(x) = \{g \in G \mid x^g = x \text{ or } x^{-1}\}$. Since $2(p-1)$ does not divide $|G|$ as given above, it follows that this tree does not occur.

Suppose tree (4) or (5) occurred. Since $\chi_i(1) = p-1$ and $\det \chi_i(x) = 1$, (1) and (3) imply that $(p-1)/4$ is even. Hence by (2), $\psi_{jN} = 1_N + \Delta_j$. Now there are four linear characters of N and at most two occur in θ_N so choose $\alpha \neq 1_N$ such that α does not occur in θ_N . Thus α can occur only in χ_{2N} or χ_{3N} with multiplicity at most two. Hence

$$[G:N] = \alpha^G(1) \leq 2\chi_2(1) + 2\chi_3(1) = 6p.$$

Now choose β so that β occurs in θ_N . Then

$$[G:N] = \beta^G(1) \geq \sum_1^{(p-1)/4} \theta_i(1) = (2p-4)(p-1)/4.$$

Since $(p-1)/4$ is even and $(p-1)/4 \neq 2$ we have $(p-1)/4 \geq 4$, $p \geq 17$ and

$$6p \geq \alpha^G(1) = \beta^G(1) \geq 4(2p-4),$$

a contradiction.

Finally consider tree (6). By (1), (2) and (3) we have easily $\chi_{1N} = 1_N + \Delta$, $\chi_{2N} = 1_N \Delta'$, and $\psi_{jN} = 1_N + \Delta_j$. Now since $e = 5$ and $m(\theta) = 2$ we can choose a linear character α of N with $\alpha \neq 1_N$ and such that α does not occur in θ_N . Hence by the above and the fact that $m(\chi_3) = m(\chi_4) = 2$ we have $\alpha^G = a\chi_3 + b\chi_4$ with $a, b \leq 2$. Thus since

$$[G:N] = \alpha^G(1) = a\chi_3(1) + b\chi_4(1) = (a+b)(p+1)$$

and $[G : N] \equiv 1 \pmod{p}$ we have $[G : N] = p + 1$. Now choose β so that β occurs in θ_N . Then

$$p + 1 = [G : N] = \beta^G(1) \geq \sum_1^{(p-1)/5} \theta_i(1) = (2p - 5)(p - 1)/5,$$

a contradiction since $5 \mid (p - 1)$ implies that $p \geq 11$. This therefore completes the proof of the theorem.

Finally we consider the remaining groups with r.b. $(2p - 1)$.

THEOREM 7. *Let p be a prime and let G be a group with r.b. $(2p - 1)$. Then we have one of the following.*

- (i) G has a normal abelian Sylow p -subgroup.
- (ii) G is solvable and has p -length 1.
- (iii) $G/Z(G) \cong \text{PSL}(2, p)$ or $\text{PGL}(2, p)$ for $p > 3$.
- (iv) $G/Z(G) \cong \text{PSL}(2, p - 1)$ for p a Fermat prime, $p > 3$.
- (v) $G/Z(G) \cong \text{PSL}(2, p + 1)$ for p a Mersenne prime.
- (vi) $G/Z(G) \cong \text{Sym}(4)$ for $p = 2$.

Proof. If $p = 2$ then G has r.b.3. Thus by Corollary 6.5 of [8], G satisfies (ii) or (vi) above. Now let $p > 2$. Since $2p - 1 \leq p^{3/2}$, Theorem 5 implies that $p^2 \nmid |G/O_p(G)|$. With this additional fact it is easy to see that the proof of the main theorem of [6] applies also to groups with r.b. $(2p - 1)$ with $p > 2$ yielding the same conclusion. (The $p > 2$ assumption is used crucially in the last paragraph of the proof of Proposition 3.1 of [6].) Thus either G satisfies (i) or (ii) above or $G = P_1 \times G_1$ where P_1 is an abelian p -group and $p^2 \nmid |G_1|$. Clearly G_1 has r.b. $(2p - 1)$ and if G_1 satisfies any of the above then so does G . Therefore it suffices to assume that $G = G_1$ or equivalently that $p^2 \nmid |G|$. We assume now that G does not satisfy (i). This of course implies that $p \mid |G|$.

Let $K = O_{p'}(G)$ and let H/K be a minimal normal subgroup of G/K . Then $p \mid |H/K|$ and since $p^2 \nmid |G/K|$ this implies that H/K is the unique minimal normal subgroup. Now H/K is a product of isomorphic simple groups and $p^2 \nmid |H/K|$ so H/K is simple. If $|H/K| = p$ then G is p -solvable of p -length 1. Thus since G does not have a normal Sylow p -subgroup, Proposition 2.3 of [6] implies that G is solvable and G satisfies (ii). Hence it suffices to assume that $\bar{H} = H/K$ is a nonabelian simple group. It is convenient to first consider the possibility $p \geq 5$.

Since \bar{H} is the unique minimal normal subgroup of $\bar{G} = G/K$ we have $C_{\bar{G}}(\bar{H}) = \langle 1 \rangle$ and thus $\bar{G} \subseteq \text{Aut } \bar{H}$. Suppose \bar{T} is a subgroup of \bar{H} with $1 < [\bar{H} : \bar{T}] < 2p$. Since \bar{H} is simple and $p \mid |\bar{H}|$ we cannot have $[\bar{H} : \bar{T}] < p$. Thus $p \leq [\bar{H} : \bar{T}] < 2p$ and \bar{T} is maximal in \bar{H} and hence self normalizing. If \bar{T} were abelian it would follow easily that

\bar{T} is a T.I. set and then \bar{H} is a simple Frobenius group, a contradiction. Thus \bar{T} is nonabelian.

Let ψ be an irreducible character of K and let χ be an irreducible constituent of ψ^H . If $e = [\chi_K, \psi]_K$ then $\chi(1) = et\psi(1)$ where $t = [H:T]$ and T is the inertial group of ψ in H . Suppose $T < H$ and set $\bar{T} = T/K$. Since $\chi(1) < 2p$ we have $t < 2p$ and thus by the remarks of the preceding paragraph $t \geq p$ and \bar{T} is nonabelian. Thus we have $2p > \chi(1) = et\psi(1) \geq ep\psi(1)$ so $e = \psi(1) = 1$. Now there exists an irreducible character η of T with $\eta^H = \chi$ and $\eta_K = e\psi = \psi$. Since \bar{T} is nonabelian we can choose a nonlinear irreducible character β of T containing K in its kernel. Thus since η is linear, $\eta_0 = \eta\beta$ is also an irreducible character of T . Let χ_0 be an irreducible constituent of η_0^H . Then $\beta(1)\psi = \eta_{0K}$ occurs in χ_{0K} and therefore $[\chi_{0K}, \psi] \geq \beta(1) > 1$. The above reasoning applied to χ_0 now yields a contradiction. Thus $H = T$ and H fixes all irreducible characters of K . By Brauer's lemma, H fixes all conjugacy classes of K . Let P be a Sylow p -subgroup of H . Then P fixes each class of K and since K is a p' -group, P centralizes K . Thus if $C = C_H(K)$ then $KC > K$ and since H/K is simple we have $H = KC$. Now $C/(C \cap K) \cong \bar{H}$ and $Z(C) \cong C \cap K$ so $Z(C) = C \cap K$.

Let D denote the last term in the derived series of C . Then clearly $D = D'$, $D/Z(D) \cong \bar{H}$ and $Z = Z(D) = D \cap K$. Thus Z is a homomorphic image of the Schur multiplier of \bar{H} . By Theorem 6, $\bar{H} \cong PSL(2, p)$, $PSL(2, p-1)$ for p a Fermat prime or $PSL(2, p+1)$ for p a Mersenne prime. We have by assumption $p \geq 5$. Also for $p = 5$, $PSL(2, p) \cong PSL(2, p-1)$ and we will view this group as $PSL(2, p)$. By [10] (Satz IX, p. 119) either $Z = \langle 1 \rangle$ or $\bar{H} \cong PSL(2, p)$, $D \cong SL(2, p)$ and $|Z| = 2$.

We show now that K is central. Suppose first that $Z = \langle 1 \rangle$ so that $H \cong D \times K$. Let χ be a fixed irreducible character of D with $\chi(1) = p$ and let λ be an irreducible character of K . Then $\chi\lambda$ is an irreducible character of H so $2p > \chi(1)\lambda(1) = p\lambda(1)$ and $\lambda(1) = 1$. Thus K is abelian and central in H . If K is not central in G , then some linear character λ of K is not fixed by G . This implies easily that if θ is a constituent of $(\chi\lambda)^G$ then $\theta(1) \geq 2\chi(1) = 2p$, a contradiction. Thus K is central in G in this case. Now let $Z \neq \langle 1 \rangle$ so that $|Z| = 2$ and $D \cong SL(2, p)$. We have an epimorphism $D \times K \rightarrow DK = H$ where the kernel is the third subgroup W of order 2 in the group generated by the copies of Z in D and K . Let λ be an irreducible character of K . Since Z is central in K and $|Z| = 2$ it is easy to see from the character table of $SL(2, p)$ ([10], p. 128) that there exists an irreducible character χ of D with $\chi(1) \geq p$ and with W in the kernel of $\chi\lambda$, an irreducible character of $D \times K$. Thus $\chi\lambda$ is a character of H . The preceding argument now shows first that K is abelian and then that K is central. We have therefore shown that $G/Z(G) \cong \bar{G}$ and it remains

to identify \bar{G} .

Now $\bar{G} \cong \text{Aut } \bar{H}$ and \bar{H} is a 2-dimensional projective group so the possibilities for \bar{G} are given by Satz 1 of [9]. Suppose first that $\bar{H} \cong PSL(2, p)$. Then either $\bar{G} \cong PSL(2, p)$ or $\bar{G} \cong PGL(2, p)$ and we have (iii). Note the fact that $PGL(2, p)$ has r.b.($2p - 1$) can be seen from the character table on page 136 of [10]. We consider the remaining two cases. Thus $\bar{H} \cong PSL(2, s)$ with $2^n = s = p \pm 1$ and \bar{G}/\bar{H} is isomorphic to a subgroup of the Galois group of $GF(2^n)/GF(2)$, a cyclic group of order n . Suppose $\bar{G} > \bar{H}$ and let $t \in \bar{G}$ correspond to a nontrivial field automorphism $x \rightarrow x^j$. Then in the notation of page 134 of [10], but replacing upper case by lower case letters, we have $t^{-1}at = a^j \neq a$. Since $s > 4$ by our assumption for $p = 5$ it follows easily that $a^j \neq a^{-1}$ so a^j is not conjugate to a in \bar{H} . From the character table of \bar{H} we now see easily that t moves some irreducible character of \bar{H} of degree $s + 1$ and thus \bar{G} has an irreducible character of degree at least $2(s + 1) \geq 2p$, a contradiction. Hence $\bar{G} = \bar{H}$ and G satisfies (iv) or (v). This completes the proof of the theorem for $p \geq 5$.

Finally let $p = 3$. Since \bar{H} is a nonabelian simple group with r.b.($2p - 1$), $\bar{H} \cong PSL(2, 4) \cong PSL(2, 5)$ by Theorem 6. Certainly G is not 5-solvable and G has r.b.($2 \cdot 5 - 1$). Thus by the prime 5 case already proved, $G/Z(G) \cong PSL(2, 5)$ or $PGL(2, 5)$. Since the latter group has an irreducible character of degree $6 > 2p - 1$ we have $G/Z(G) \cong PSL(2, 5) \cong PSL(2, p + 1)$ and G satisfies (v). Thus the result follows.

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