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## **MULTIPLIER ALGEBRAS OF BIORTHOGONAL SYSTEMS**

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Let  $\{e_i, E_i\}$  be a total biorthogonal system in a linear topological space  $X$ . The multiplier algebra of  $X$  with respect to  $\{e_i, E_i\}$  written  $M(X)$  is the set of all scalar sequences  $(t^{(i)})$  such that for each  $x \in X$  there is  $y \in X$  with

$$E_i(y) = t^{(i)} E_i(x).$$

The form of  $M(X)$  is determined when  $\{e_i, E_i\}$  is a norming complete biorthogonal system in a Banach space or a basis in a complete barreled space. It is shown that a sequence space is the multiplier algebra for a basis in a Banach space if and only if it is a  $\gamma$ -perfect  $BK$ -algebra.

A biorthogonal system is a double sequence  $\{e_i, E_i\}$  with each  $e_i$  in a locally convex space  $X$  and each  $E_i$  a continuous linear functional on  $X$  (i.e., in  $X^*$ ) which satisfies the relationship

$$E_i(e_j) = \delta_{ij} \text{ (Kronecker } \delta) \quad i, j = 1, 2, \dots$$

The biorthogonal system is *total* if  $\{E_i\}$  is total on  $X$ ; that is,  $E_i(x) = 0$  for each  $i$  implies  $x = 0$ . If  $\{e_i, E_i\}$  is a total biorthogonal system then the space  $X$  can be identified with the space of all sequences  $(E_i(x))$  by means of the natural correspondence  $x$  to  $(E_i(x))$ . Under this correspondence  $e_i$  becomes the  $i$ th coordinate vector, the sequence which has a one in the  $i$ th coordinate and 0's elsewhere and  $E_i$  becomes the  $i$ th coordinate functional, the functional whose value on the sequence  $(x^{(1)}, x^{(2)}, \dots)$  is  $x^{(i)}$ . This identification will be assumed whenever a total biorthogonal system is considered.

DEFINITION 1.1. Let  $\{e_i, E_i\}$  be a total biorthogonal system in a locally convex space  $X$ . A scalar sequence

$$t = (t^{(1)}, t^{(2)}, \dots)$$

is a *multiplier of  $X$  with respect to  $\{e_i, E_i\}$*  if for each  $x \in X$  there is  $y \in X$  for which

$$E_i(y) = t^{(i)} E_i(x) \quad i = 1, 2, \dots$$

The set of all such  $t$  is written  $M(X; e_i, E_i)$  or simply  $M(X)$  and called the *multiplier algebra of  $X$  (with respect to  $\{e_i, E_i\}$ )*.

In other words  $M(X)$  is the set of all  $t$  such that

$$tx \in X \text{ whenever } x \in X$$

where  $X$  is now considered a sequence space and multiplication of sequences is defined coordinatewise. It is now obvious that  $M(X)$  forms a linear algebra of operators from  $X$  into  $X$ ; namely the operators which are diagonal with respect to  $\{e_i, E_i\}$ . Multiplication in this algebra is defined coordinatewise. The purpose of this paper is to study the properties of the space  $M(X)$  and the possible forms which it can assume with varying hypotheses on  $X$  or on  $\{e_i, E_i\}$ . Results of this type were obtained by Yamazaki in [11] and [12] for  $\{e_i\}$  a basis in a Banach space. The concept of imultiplier space is implicitly treated in [4].

Throughout this paper it is immaterial whether the scalar field considered consists of the real or complex numbers.

2. Sequence spaces: notation and basic facts. A set of scalar sequences which is closed under coordinatewise addition and scalar multiplication is a *sequence space*; if it is closed under coordinatewise multiplication as well it will be called a *sequence algebra*. The  $i$ th coordinate vector is written  $e_i$ ; the  $i$ th coordinate functional,  $E_i$ . If each  $E_i$  is continuous on a locally convex sequence space (algebra)  $X$  and  $e_i \in X$  for each  $i$  then  $X$  is called a *K-space* (algebra). If in addition  $X$  is an  $F$ -space (complete metric linear space)  $X$  will be called an *FK-space* or *FK-algebra* as the case may be. If  $X$  is a Banach space (algebra),  $X$  will be called a *BK-space* (algebra). Note that in an *FK-algebra*  $X$  the functions  $x \rightarrow tx$  and  $x \rightarrow xt$  are continuous in  $x$  for fixed  $t$  by the continuity of the coordinate functionals and the closed graph theorem. This is enough to conclude that a *BK-algebra* is a Banach algebra without identity. See p. 860 and 861 of [3].

The following are well known sequence spaces. For additional discussion see Chapter IV of [2], [5] or p. 289 of [10]:

$\omega$  sometimes called  $s$  is the set of all scalar sequences. Endowed with the topology of coordinatewise convergence it is an *FK-algebra*.

$\varphi$  is the linear span of  $\{e_i\}$  in  $\omega$ , i.e., the space of all finitely nonzero sequences.

$l^1$  is the set of all sequences  $t$  such that

$$\|t\| = \sum_{i=1}^{\infty} |t^{(i)}| < \infty$$

which is a *BK-space* with this norm.

$m$  is the set of all sequences  $t$  such that

$$\|t\| = \sup_i |t^{(i)}| < \infty$$

which is a *BK algebra* with this norm.

$bs$  is the set of all sequences  $t$  such that

$$\|t\| = \sup_n \left| \sum_{i=1}^n t^{(i)} \right| < \infty$$

which is a  $BK$  space with this norm.

$cs$  is the closed linear span of  $\{e_i\}$  in  $bs$ ; it consists of all sequences  $t$  such that  $\sum_{i=1}^{\infty} t^{(i)}$  converges.

$bv$  is the set of all sequences  $t$  such that

$$\|t\| = \lim_n |t_n| + \sum_{i=1}^{\infty} |t_i - t_{i+1}| < \infty$$

which is a  $BK$ -algebra with this norm. See §3 of [4] or p. 3 of [11]. Yamazaki denoted  $bv$  by  $w$ .

NOTATION 2.1. (a) Let  $t, u \in \omega$  be such that  $ut \in cs$ ; the sum  $\sum_{i=1}^{\infty} u^{(i)} t^{(i)}$  is denoted by  $(u, t)$ .

(b) For  $A \subseteq \omega$

$A^\alpha$ , the  $\alpha$ -dual of  $A$  is  $\{t: ut \in l^1, u \in A\}$

$A^\beta$ , the  $\beta$ -dual of  $A$  is  $\{t: ut \in cs, u \in A\}$

$A^\gamma$ , the  $\gamma$ -dual of  $A$  is  $\{t: ut \in bs, u \in A\}$ .

(c) For  $A$  and  $B \subseteq \omega$

$AB = \{uv: u \in A, v \in B\}$ .

(d) For  $t \in \omega$  and  $A \subseteq \omega$

$t^{-1}A = \{u \in \omega: tu \in A\}$ .

(e) For  $A \subseteq \omega$

$A^c = \{t \in \varphi: |(t, u)| \leq 1, u \in A\}$ .

(f) For  $A \subseteq \varphi$

$A^\omega = \{t \in \omega: |(t, u)| \leq 1, u \in A\}$ .

(g) For  $X$  a  $K$ -space,  $X^\diamond$  is the space of all sequences  $(f(e_i))$  as  $f$  ranges over  $X^*$ . Note that for  $t \in X^\diamond$  with  $t^{(i)} = f(e_i)$ ,  $E_i(t) = t^{(i)} = f(e_i)$ .

Gamma-perfect  $BK$ -spaces can be constructed by means of sequential norms. A sequential norm (s.n.) is a function  $P$  from  $\omega$  into  $R^*$  which is an extended norm and in addition satisfies the condition

$$P(x) = \sup_n P\left(\sum_{i=1}^n x^{(i)} e_i\right) x \in \omega.$$

If

$$0 < \inf_n P(e_n) \leq \sup_n P(e_n) < \infty$$

$P$  is a proper sequential norm (p.s.n.). For  $P$  a s.n.,  $S_P$  is the set of all  $x \in \omega$  for which  $P(x) < \infty$  endowed with the topology determined by  $P\varepsilon$ . The closed linear span of  $\{e_1, e_2, \dots\}$  in  $S_P$  is denoted by  $S_P^0$ . The

following proposition contains information about sequential norms which was derived in [7] and [8] and which we shall use in §6.

PROPOSITION 2.2. (a) *If  $P$  is a s.n.  $S_P$  is a  $\gamma$ -perfect BK-space. If  $X$  is a  $\gamma$ -perfect BK-space there is an s.n.  $P$  such that  $X = S_P$ .*

(b) *If  $P$  is a s.n. the function  $P'$  given by*

$$P'(x) = \sup \left\{ \sup_n \left| \sum_{i=1}^n x^{(i)} y^{(i)} \right| : P(y) \leq 1 \right\}$$

*is a s.n. and  $P'' = P$ . If  $P$  is a p.s.n. so is  $P'$ .*

(c)  $(S_P^0)^\delta = S_P$ , and  $(S_{P'}^0)^\delta = S_P$ .

(d) *An s.n.  $P$  is a p.s.n. if and only if  $l \subseteq S_P \subseteq m$ .*

### 3. Preliminary results.

PROPOSITION 3.1. *If  $\{e_i, E_i\}$  is a total biorthogonal system in  $X$*

$$(3-1) \quad M(X) = \cap \{y^{-1}X : y \in S\}$$

*where  $S$  is any absorbing subset of  $X$ .*

*Proof.* Let  $R$  denote the set on the right of (3-1). If  $t \in R$  and  $x \in X$  there is  $a > 0$  such that  $ax \in S$ . Thus  $atx \in X$  so that  $tx \in X$  which implies  $t \in M(X)$ . If  $t \in M(X)$  then  $tx \in X$  for every  $x \in X$  so in particular for every  $x \in S$ .

A complete biorthogonal system is a total biorthogonal system  $\{e_i, E_i\}$  on  $X$  such that  $\text{sp}\{e_i\} (= \varphi)$ , the linear span of  $\{e_i\}$  is dense in  $X$ .

PROPOSITION 3.2. *Let  $\{e_i, E_i\}$  be a complete biorthogonal system in  $X$ .*

(a) *For each  $t \in M(X)$ , the mapping  $x \rightarrow tx$  is a closed linear operator from  $X$  into  $X$ .*

(b) *The set  $M_c(X)$  of all  $t \in M(X)$  for which  $x \rightarrow tx$  is continuous is a closed sub-algebra of  $\mathcal{L}(X)$  where  $\mathcal{L}(X)$  has any topology containing the topology of simple convergence. Here  $\mathcal{L}(X)$  denotes the space of continuous operators from  $X$  into  $X$ .*

*Proof.* (a) Obvious.

(b) Define  $E_i \otimes e_j$  on  $\mathcal{L}(X)$  by

$$E_i \otimes e_j(T) = E_i(Te_j).$$

Then  $E_i \otimes e_j$  is a continuous linear functional on  $\mathcal{L}(X)$  given the topology of simple convergence. Therefore

$$S = \cap \{[E_i \otimes e_j]^{-1}(0) : i \neq j\}$$

is closed in this topology and every topology containing it.

The following statement generalizes Theorem 1 of [11].

**COROLLARY 3.3.** *If  $\{e_i, E_i\}$  is a complete biorthogonal system in a Banach space  $X$  then there is a topology on  $M(X)$  which makes it a BK algebra.*

*Proof.* In this case  $M_c(X) = M(X)$  since the mapping  $x \rightarrow tx$  is closed. Thus by 3.2  $M(X)$  is a Banach algebra with the norm

$$\|t\| = \sup\{\|tx\| : \|x\| \leq 1\}.$$

Each  $E_i$  is continuous on  $M(X)$  since

$$E_i(T) = E_i \otimes e_i(T_i)$$

where  $T_i(x) = tx$ ; and  $E_i \otimes e_i$  is continuous on  $\mathcal{L}(X)$ .

**PROPOSITION 3.4.** *If  $\{e_i, E_i\}$  is a total biorthogonal system in a linear topological space  $X$  then*

$$M_c(X; e_i, E_i) \subseteq M_c(X_0; e_i, E_i)$$

where  $X_0$  is the closed linear span of  $\{e_i\}$  in  $X$ .

*Proof.* If  $t \in M_c(X)$  then  $tx \in \varphi$  for  $x \in \varphi$ . Since  $\varphi$  is dense in  $X_0$  and  $t$  is continuous  $tx \in X_0$  for  $x \in X_0$  so that  $t \in M_c(X_0)$ .

If  $\{e_i, E_i\}$  is a complete biorthogonal system on  $X$ ,  $X^*$  is isomorphic to  $X^\circ$  under the correspondence of  $f$  in  $X^*$  to  $(f(e_i))$  in  $X^\circ$  and  $\{e_i, E_i\}$  is a total biorthogonal system on  $X^\circ$ .

**PROPOSITION 3.5.** *If  $\{e_i, E_i\}$  is a complete biorthogonal system in a locally convex space  $X$  then*

$$M_c(X; e_i, E_i) \subseteq M(X^\circ; e_i, E_i).$$

*Proof.* If  $t \in M_c$  for  $f \in X^*$  let  $f_i(x) = f(tx)$ ,  $x \in X$ . Then  $f_i \in X^*$  and  $f_i(e_i) = t^{(i)}f(e_i)$  for each  $i$  so that  $ty \in X^\circ$  for  $y \in X^\circ$ .

**4. Multiplier algebras of a norming biorthogonal system in a Banach space.** A biorthogonal system  $\{e_i, E_i\}$  in a normed space  $X$  is called *norming* if the topology of  $X$  is determined by a norm of the type

$$\|x\| = \sup\{|f(x)| : f \in A\}$$

where  $A$  is a subset of the linear span of  $\{E_i\}$  in  $X^*$ . An equivalent condition is that the above norm be given by

$$(4-1) \quad \|x\| = \sup \{|(x, t)| : t \in A\}$$

where  $A$  is a subset of  $\varphi$ .

If  $\{e_i, E_i\}$  is a complete biorthogonal system which is norming on  $X$  and the norm of  $X$  is given by (4-1) it may be assumed that  $A$  consists of all sequences  $t$  in  $\varphi$  for which

$$(4-2) \quad |(t, x)| \leq \|x\| \quad x \in X.$$

Denote by  $\hat{X}$  the space of all  $x \in \omega$  for which

$$(4-3) \quad \|x\| = \sup \{|(x, t)| : t \in A\} < \infty.$$

The function defined in (4-3) is a norm since  $a_n^{-1} e^n \in A$  for  $n = 1, 2, \dots$  where  $a_n$  is the norm of  $E_n$  as a member of  $X^*$ . With this norm  $\hat{X}$  is a  $BK$ -space in which  $X$  is the closed linear span of  $\{e_i\}$ .

**PROPOSITION 4.1.** *The space  $X^\delta$  consists of all  $y \in \omega$  for which*

$$\|y\|' = \sup \{|(y, x)| : x \in A^\varphi\} < \infty.$$

*The correspondence*

$$(4-4) \quad f \text{ to } (f(e_i)) \quad f \in X^*$$

*is an isometry from  $X^*$  onto  $(X^\delta, \|\cdot\|')$ . The correspondence*

$$(4-5) \quad g \text{ to } (g(e_i)) \quad g \in (X_0^\delta)^*$$

*is an isometry from  $(X_0^\delta)^*$  onto  $\hat{X}$ , where  $X_0^\delta$  denotes the closed linear span of  $\{e_i\}$  in  $X^\delta$ .*

*Proof.* The correspondence in (4-4) is clearly well defined and linear.

If  $f \in X^*$  and  $x \in A^\varphi$  then  $x \in X$  and  $\|x\| \leq 1$ . Thus

$$\begin{aligned} |(f(e_i), x)| &= \left| \sum_i x^{(i)} f(e_i) \right| \\ &= |f(x)| \leq \|f\| \end{aligned}$$

so that

$$\|(f(e_i))\|' \leq \|f\|.$$

If  $y \in X^\delta$  define  $f$  on  $\{\varphi, \|\cdot\|\}$  by

$$f(x) = (y, x).$$

Then  $f$  is a bounded linear functional on  $\{\varphi, |||\}$  for which

$$|f(x)| \leq ||y||' x \in \varphi, ||x|| \leq 1$$

because

$$A^\circ = \{x \in \varphi: ||x|| \leq 1\}.$$

Since  $\varphi$  is dense in  $X$ ,  $f$  can be continuously extended to  $X$  with

$$||f|| \leq ||y||'.$$

For this extended  $f$

$$(f(e_i)) = (y, e_i) = y_i \quad i = 1, 2, \dots.$$

Therefore, the correspondence in (4-4) is an isometry.

That (4-5) is an isometry from  $(X_0^*)^*$  onto  $\hat{X}$  will follow from an analogous argument if it is shown that

$$A^{\circ\circ} = A.$$

When  $A$  has the form given by (4-2).

That  $A^{\circ\circ} \supseteq A$  is clear, if  $z \in A^{\circ\circ}$  then

$$|(z, x)| \leq 1 \quad x \in A^\circ$$

but  $A^\circ$  is dense in the unit ball of  $X$  so that

$$|(z, x)| \leq 1 \quad x \in X, ||x|| \leq 1.$$

Thus if

$$f(x) = (z, x) \quad x \in X$$

we have  $||f|| \leq 1$  and

$$f(e_i) = z_i \quad i = 1, 2, \dots$$

so that  $z \in A$ . It is here that the assumption that  $\{e_i, E_i\}$  is norming was used.

**THEOREM 4.2.** *If  $\{e_i, E_i\}$  is a norming complete biorthogonal system in the Banach space  $X$  then  $M(X)$  is of the form*

$$(4-6) \quad \bigcup_{n=1}^{\infty} n(AA^{\circ})^{\circ}$$

where  $A$  is a coordinatewise bounded subset of  $\varphi$  which contains a multiple of  $e_i$  for each  $i$ .

*Proof.* Let  $A$  be given by (4-2) and let  $Z$  denote the sequence space (4-6).



By 3.4, 3.5 and the fact that  $M_c(Y) = M(Y)$  for  $Y$  a Banach space we have

$$M(X) \subseteq M(X^\circ) \subseteq M(X^\circ_\circ) \subseteq M(\hat{X}) \subseteq M(X)$$

so that  $M(X)$  and  $M(\hat{X})$  are equal. It will be shown that  $M(\hat{X}) = Z$ . Suppose  $t \in M(\hat{X})$ , then there is  $k$  such that

$$\|tx\| \leq k\|x\| \quad x \in \hat{X}.$$

If  $s \in A$  and  $x \in A^\omega$

$$|(sx, t)| = |(s, tx)| \leq k$$

so that

$$t \in k(AA^\omega)^\omega \subseteq Z.$$

If  $z \in Z$  and  $x \in \hat{X}$ ,  $x/\|x\| \in A^\omega$  so if  $n$  is such that  $z \in n(AA^\omega)^\omega$

$$|(t, zx)| = |(tx, z)| \leq n\|x\|$$

for  $t \in A$ . Therefore  $zx \in \hat{X}$ .

*Question.* If  $\{e_i, E_i\}$  is a complete biorthogonal system in the Banach space  $X$  and  $M(X)$  has the form (4-6) is  $\{e_i, E_i\}$  norming?

**5. Multiplier algebras of bases.** A biorthogonal system  $\{e_i, E_i\}$  in a linear topological space  $X$  is a (Schauder) *basis* for  $X$  if

$$(5-1) \quad x = \sum_{i=1}^{\infty} E_i(x)e_i \quad x \in X.$$

It is an *unconditional basis* if the convergence in (5-1) is unconditional. If  $X$  is a l.c.s.  $\{e_i, E_i\}$  is an *absolute basis* if the convergence in (5-1) is absolute, i.e., if

$$\sum_{i=1}^{\infty} |E_i(x)| p(e_i) < \infty$$

for each  $x \in X$  and each continuous seminorm  $p$  on  $X$ . It is clear that an absolute basis is unconditional.

The proofs of 5.1, 5.2, and 5.3 are omitted since these statements are essentially known. See p. 205 of [10] and Propositions 4 and 5 of [1].

**LEMMA 5.1.** *For  $\{e_i, E_i\}$  a complete biorthogonal system in a barreled l.c.s.  $X$  it is always true that  $X^\circ \subseteq X^\circ_\circ$ .*

**LEMMA 5.2.** *For  $\{e_i, E_i\}$  a basis in a locally convex space  $X$ ,  $X^\circ \subseteq X^\circ_\circ$ .*

PROPOSITION 5.3. *For  $\{e_i, E_i\}$  a complete biorthogonal system in a barreled locally convex space  $X$  the following are equivalent.*

- (a)  $\{e_i, E_i\}$  is a basis for  $X$ .
- (b)  $X^\delta = X^\beta$ .
- (c)  $X^\delta = X^\gamma$ .

THEOREM 5.4. *If  $\{e_i, E_i\}$  is a basis of a complete barreled space  $X$  then*

$$M_c(X) = M(X) = (XX^\beta)^\beta = (XX^\gamma)^\gamma.$$

*Proof.* Suppose  $t \in M(X)$ ,  $x \in X$  and  $y \in X^\beta$ . Then  $tx \in X$  so that  $txy \in cs$  by 5.1 which implies  $t \in (XX^\beta)^\beta$ .

Let  $P$  denote the family of all continuous seminorms on  $X$ . Let  $\hat{X}$  be the linear space of all  $x \in \omega$  such that

$$p'(x) = \sup_n p\left(\sum_{i=1}^n x^{(i)} e_i\right) < \infty, \quad p \in P.$$

Since  $X$  is barreled,  $p'$  restricted to  $X$  is continuous and since  $\varphi$  is dense in  $X$ ,  $p'(x) \geq p(x)$  for  $x \in X$ . Thus  $X$  is the closed linear span of  $\{e_i\}$  in the space  $\hat{X}$  with the topology determined by the seminorms  $\{p': p \in P\}$ .

For  $t \in (XX^\gamma)^\gamma$  define

$$p_t(x) = \sup_n p\left(\sum_{i=1}^n t^{(i)} x^{(i)} e_i\right).$$

Since  $t \in (XX^\gamma)^\gamma$ ,  $\{\sum_{i=1}^n t^{(i)} x^{(i)} e_i; n = 1, 2, \dots\}$  is a weakly bounded, thus a strongly bounded subset of  $X$  so that

$$P_t(x) < \infty \quad x \in X,$$

and  $p_t$  is a continuous seminorm on  $X$ . If  $x \in \hat{X}$ ,  $tx \in \hat{X}$  and

$$p'(tx) = p'_t(x)$$

so that  $t \in M_c(\hat{X})$ . By 3.4,  $(XX^\gamma)^\gamma \subseteq M_c(X)$ . Thus

$$M_c(X) \subseteq M(X) \subseteq (XX^\gamma)^\beta \subseteq (XX^\gamma)^\gamma \subseteq M_c(X)$$

which establishes the result.

COROLLARY 5.5. *If  $\{e_i, E_i\}$  is an unconditional basis of a complete barreled space  $X$  then*

$$M(X) = (XX^\alpha)^\alpha.$$

*Proof.* Since  $\{e_i, E_i\}$  is an unconditional basis  $X^\alpha = X^\delta$ . If  $t \in (XX^\gamma)^\gamma$

and  $u \in XX^r$  let  $u = xy$  with  $x \in X, y \in X^r = X^\delta = X^\alpha$ . Let  $v^{(i)} = \text{sgn } t^{(i)}u^{(i)}$  then  $vy \in X^\alpha$  so that  $vu \in XX^r$ . Hence,  $vut \in bs$  so that  $ut \in l^1$ . Therefore,  $(XX^r)^r \subseteq (XX^\alpha)^\alpha$  from which the conclusion follows.

**THEOREM 5.6.** *Let  $\{e_i, E_i\}$  be an absolute basis of a sequentially complete locally convex space  $X$ . If  $P$  is any family of continuous seminorms which determines the topology of  $X$  then*

$$M(X) = (AA^\alpha)^\alpha$$

where

$$A = \{(p(e_i)): p \in P\}.$$

*Proof.* The hypotheses of this theorem imply that  $X = A^\alpha$ . Now  $t \in M(X)$  if and only if  $tx \in X$  whenever  $x \in X$  which happens if and only if  $txy \in l^1$  whenever  $x \in A^\alpha = X$  and  $y \in A$ . This will hold if and only if  $t \in (AA^\alpha)^\alpha$ .

**EXAMPLES.**  $M(\omega) = M(\varphi) = \omega$ ;  $M(cs) = M(bv_0) = bv$ ;  $M(c_0) = M(l^1) = m$ .

Let  $X$  be the space of all real sequences  $x$  for which

$$p_k(x) = \sum_n |x_n| k^n < \infty \quad k = 1, 2, \dots$$

Then  $X$  with the seminorms  $p_1, p_2, \dots$  is a nuclear  $F$ -space which is equivalent to the space of all infinitely differentiable real functions of period  $2\pi$ . See §5 of [6]. If  $A = \{k^n: k = 1, 2, \dots\}$  it is clear that  $X = A^\alpha$ . If  $x \in A^\alpha, x = xe \in AA^\alpha$  and if  $x \in AA^\alpha, x = y(k^n)$  for some  $k$  and  $y \in A^\alpha$ . But

$$\sum_{n=1}^{\infty} |t_n| k^n h^n = \sum_{n=1}^{\infty} |t_n| (kh)^n$$

so  $x \in A^\alpha$ . Thus  $M(X) = A^{\alpha\alpha} = X^\delta$ .

The following theorem is a version of Theorem 3 and Corollary 2 of [4]. For the definition of  $B_r$ -complete see p. 162 of [9].

**THEOREM 5.7.** *A complete biorthogonal system  $\{e_i, E_i\}$  in a space  $X$  which is barreled and  $B_r$  complete is a basis for  $X$  if and only if  $M(X) \supseteq bv$ . It is an unconditional basis for  $X$  if and only if  $M(X) \supseteq m$ .*

**6. Gamma-perfect BK-algebras.** A proper sequential norm  $P$  which satisfies the inequality

$$(6-1) \quad P(xy) \leq P(x)P(y)$$

will be called an algebraic p.s.n. (a.p.s.n.).

**THEOREM 6.1.** *The following statements are equivalent for a sequence space  $M$ :*

- (a)  $M$  is a multiplier algebra for a basis in a Banach space;
- (b)  $M$  is a  $\gamma$ -perfect  $BK$ -algebra containing  $e$ ;
- (c)  $M = S_P$  for  $P$  an a.p.s.n. with  $P(e) < \infty$ .

*Proof.* (a)  $\Rightarrow$  (b). If  $M$  is a multiplier algebra for a basis in a Banach space  $X$  it is a  $BK$ -algebra containing  $e$  by 3.3 and  $\gamma$ -perfect since it is the  $\gamma$ -dual of  $XX^\gamma$  by 5.4.

(b)  $\Rightarrow$  (c). Suppose  $M$  is a  $\gamma$ -perfect  $BK$ -algebra containing  $e$ . By 2.2 (a) there is a sequential norm  $Q$  such that  $M = S_Q$ . It is routine to verify that  $P$  given by

$$P(x) = \sup \{Q(xy): Q(y) \leq 1\}$$

is a s.n. equivalent to  $Q$  (i.e.,  $S_P = S_Q$ ) which satisfies (6-1). It remains to show

$$(6-2) \quad 0 < \inf_n P(e_n) \leq \sup_n P(e_n) < \infty.$$

Since  $P(e_n) = P(e_n e_n) \leq P(e_n)^2$ , the left inequality of (6-2) is valid. Since  $e \in S_P$ ,  $bs \supseteq S_P^\gamma$  so that  $bv \subseteq S_P$ . That  $S_P$  is  $\gamma$ -perfect follows from 2.2 (a). The identity map from  $bv$  into  $S_P$  is continuous and  $\{e_n: n = 1, 2, \dots\}$  is bounded in  $S_P$ . Hence the right inequality in (6-2) is true. Therefore,  $P$  is an a.p.s.n.

(c)  $\Rightarrow$  (a). If  $P$  is an a.p.s.n. with  $P(e) < \infty$ ,  $S_P$  is a  $BK$ -algebra with identity so  $M(S_P) = S_P$ . But since  $S_P = (S_P^0)^\delta$  and  $S_{P'} = (S_P^0)^\delta$ ,  $M(S_P^0) = M$  and  $\{e_i, E_i\}$  is a basis for  $S_P^0$  (2.2).

The following theorem gives a means of constructing  $\gamma$ -perfect  $BK$ -algebras with identity which are distinct from  $bv$  and  $m$ . Let  $N$  denote the sequence of positive integers and  $N_k$  a subsequence of the form

$$(6-3) \quad N_k = \{k(1) < k(2) < \dots\}.$$

**THEOREM 6.2.** (a) *Let  $N_1, N_2, \dots, N_r$  be a partition of  $N$  with each  $N_k$  given by (6-3). For each  $k$  let  $P_k$  be an a.p.s.n. for which  $P_k(e) < \infty$ . Define  $P$  by*

$$P(x) = \max \{P_k(x^{k(1)}, x^{k(2)}, \dots): k = 1, 2, \dots, r\}.$$

*Then  $P$  is an a.p.s.n. and  $S_P$  is a  $\gamma$ -perfect  $BK$ -algebra containing  $e$ .*

(b) *Let  $N_1, N_2, \dots$  be an infinite partition of  $N$  with each  $N_k$  given by (6-3). For each  $k$  let  $P_k$  be an a.p.s.n. for which*

$$(6-4) \quad \sup_k P_k(e) < \infty .$$

Define  $P$  by

$$P(x) = \sup_k \{P_k(x^{k(1)}, x^{k(2)}, \dots)\} .$$

Then  $P$  is an a.p.s.n. and  $S_P$  is a  $\gamma$ -perfect  $BK$ -algebra containing  $e$ .

*Proof.* The proof of (a) is omitted since it is similar to but less difficult than that of (b).

(b) It is straightforward to verify that  $P$  is a norm. That  $P$  is a.s.n. follows from the equalities:

$$\begin{aligned} \sup_n P\left(\sum_{i=1}^n x^{(i)} e_i\right) &= \sup_n \sup_k \left\{P_k\left(\sum_{k(i) \leq n} x^{k(i)} e_i\right)\right\} \\ &= \sup_k \left\{\sup_n P_k\left(\sum_{k(i) \leq n} x^{k(i)} e_i\right)\right\} \\ &= \sup_k \{P_k(x^{k(1)}, x^{k(2)}, \dots)\} \end{aligned}$$

since each  $P_k$  is an s.n. That  $S_P$  is an algebra follows since

$$\begin{aligned} P(xy) &= \sup_k \{P_k(x^{k(i)} y^{k(i)})\} \\ &\leq \sup_k \{P_k(x^{k(i)}) P_k(y^{k(i)})\} \\ &\leq P(x) P(y) . \end{aligned}$$

Therefore,  $S_P$  is a  $\gamma$ -perfect  $BK$ -algebra; it contains  $e$  because of (6-4).

EXAMPLE. Let  $M$  be the set of all sequences  $x$  such that

$$P(x) = \sup_k \left\{ \sum_{i=1}^{\infty} |x^{k(i)} - x^{k(i+1)}| + \lim_{k(i)} |x^{k(i)}| \right\} < \infty$$

for  $N_1, N_2, \dots$  a partition of the integers with each  $N_k$  given by (6-3). Then  $M$  is a  $\gamma$ -perfect  $BK$ -algebra containing  $e$  but is neither  $m$  nor  $bv$ . The sequence  $y$  with

$$y^{(i)} = \begin{cases} 1 & \text{for } i = k(1) \text{ for each } k \\ 0 & \text{otherwise} \end{cases}$$

is in  $M$  but not in  $bv$ . The sequence  $z$  with

$$z^{(i)} = \begin{cases} 1 & \text{for } i = k(2j-1) \text{ for each } j < k \text{ and all } k \\ 0 & \text{otherwise} \end{cases}$$

is in  $m$  but not  $M$ .

*Question.* Are there  $\gamma$ -perfect  $BK$ -algebras other than those in

the smallest class of  $BK$ -algebras which contain  $bv.$ , and  $m$  and are closed under the operations described in Theorem 6.2.?

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Bruce Langworthy Chalmers, <i>On boundary behavior of the Bergman kernel function and related domain functionals</i> .....	243
William Eugene Coppage, <i>Peirce decomposition in simple Lie-admissible power-associative rings</i> .....	251
Edwin Duda, <i>Compactness of mappings</i> .....	259
Earl F. Ecklund Jr., <i>On prime divisors of the binomial coefficient</i> .....	267
Don E. Edmondson, <i>A modular topological lattice</i> .....	271
Phillip Alan Griffith, <i>A note on a theorem of Hill</i> .....	279
Marcel Herzog, <i>On finite groups with independent cyclic Sylow subgroups</i> .....	285
James A. Huckaba, <i>Extensions of pseudo-valuations</i> .....	295
S. A. Huq, <i>Semivarieties and subfunctors of the identity functor</i> .....	303
I. Martin (Irving) Isaacs and Donald Steven Passman, <i>Finite groups with small character degrees and large prime divisors. II</i> .....	311
Carl Kallina, <i>A Green's function approach to perturbations of periodic solutions</i> .....	325
Al (Allen Frederick) Kelley, Jr., <i>Analytic two-dimensional subcenter manifolds for systems with an integral</i> .....	335
Alistair H. Lachlan, <i>Initial segments of one-one degrees</i> .....	351
Marion-Josephine Lim, <i>Rank k Grassmann products</i> .....	367
Raymond J. McGivney and William Henry Ruckle, <i>Multiplier algebras of biorthogonal systems</i> .....	375
J. K. Oddson, <i>On the rate of decay of solutions of parabolic differential equations</i> .....	389
Helmut R. Salzmann, <i>Geometries on surfaces</i> .....	397
Annemarie Schlette, <i>Artinian, almost abelian groups and their groups of automorphisms</i> .....	403
Edgar Lee Stout, <i>Additional results on modules over polydisc algebras</i> .....	427
Lajos Tamássy, <i>A characteristic property of the sphere</i> .....	439
Mark Lawrence Teply, <i>Some aspects of Goldie's torsion theory</i> .....	447
Freddie Eugene Tidmore, <i>Extremal structure of star-shaped sets</i> .....	461
Leon Jarome Weill, <i>Unconditional and shrinking bases in locally convex spaces</i> .....	467