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EXCEPTIONAL $3/2$ -TRANSITIVE PERMUTATION GROUPS

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Solvable 3/2-transitive permutation groups were previously classified to within a finite number of exceptions. In this paper the exceptional groups are determined. They have degrees 3^2 , 5^2 , 7^2 , 11^2 , 17^2 and 3^4 . In addition, these groups are shown to have no transitive extensions.

There are three families of groups which play a special role here. Let q be a prime. We let $\mathcal{S}(q^n)$ denote the set of all semilinear transformations on the finite field $GF(q^n)$. Thus $\mathcal{S}(q^n)$ consists of all transformations

$$x \rightarrow ax^\sigma + b$$

with $a, b \in GF(q^n)$, $a \neq 0$ and σ a field automorphism. Clearly this is a solvable group, doubly transitive on $GF(q^n)$.

We let $\mathcal{S}_0(q^n)$ be the group acting on a 2-dimensional space over $GF(q^n)$ which contains the transformations

$$(x, y) \rightarrow (x, y) \begin{pmatrix} a & 0 \\ 0 & \pm a^{-1} \end{pmatrix} + (b, c)$$

and

$$(x, y) \rightarrow (x, y) \begin{pmatrix} 0 & a \\ \pm a^{-1} & 0 \end{pmatrix} + (b, c)$$

with $a, b, c \in GF(q^n)$ and $a \neq 0$. We see easily that $\mathcal{S}_0(q^n)$ is solvable and if $q \neq 2$ then it acts 3/2-transitively on the 2-dimensional space.

Finally we let $\Gamma(q^n)$ denote the set of all functions of the form

$$x \rightarrow \frac{ax^\sigma + b}{cx^\sigma + d}$$

with $a, b, c, d \in GF(q^n)$, $ad - bc \neq 0$ and σ a field automorphism. These functions permute the set $GF(q^n) \cup \{\infty\}$ and $\Gamma(q^n)$ is triply transitive. Clearly $\Gamma(q^n)_\infty = \mathcal{S}(q^n)$ is solvable. Let $\bar{\Gamma}(q^n)$ denote the subgroup of $\Gamma(q^n)$ consisting of these functions of the form

$$x \rightarrow \frac{ax + b}{cx + d}$$

with $ad - bc$ a nonzero square in $GF(q^n)$.

The following results are proved here.

THEOREM A. *Let \mathfrak{G} be a linear group acting on vector space \mathfrak{V} of order q^n . Suppose that \mathfrak{G} acts half-transitively but not semi-regularly on \mathfrak{V}^* . If \mathfrak{G} is primitive as a linear group then*

- (i) $\mathcal{O}_p(\mathfrak{G})$ is cyclic for $p > 2$.
- (ii) The Frattini subgroup $\Phi(\mathcal{O}_2(\mathfrak{G}))$ is cyclic and

$$[\mathcal{O}_2(\mathfrak{G}) : \Phi(\mathcal{O}_2(\mathfrak{G}))] \leq 2^8.$$

THEOREM B. *Let \mathfrak{G} be a solvable 3/2-transitive permutation group. Then with suitable identification, \mathfrak{G} satisfies one of the following.*

- (i) \mathfrak{G} is a Frobenius group.
- (ii) $\mathfrak{G} \cong \mathcal{S}(q^n)$
- (iii) $\mathfrak{G} = \mathcal{S}_0(q^n)$ or
- (iv) \mathfrak{G} has degree $3^2, 5^2, 7^2, 11^2, 17^2$ or 3^4 .

The exceptions of (iv) above do in fact exist. If $\deg \mathfrak{G} \neq 17^2$ then we can take \mathfrak{G} to be an exceptional solvable doubly transitive group, while if $\deg \mathfrak{G} = 17^2$ then we construct this group explicitly and show that it has order $96 \cdot 17^2$.

THEOREM C. *Let \mathfrak{G} be a 5/2-transitive permutation group and suppose that the stabilizer of a point is solvable. Then with suitable identification we have one of the following*

- (i) \mathfrak{G} is a Zassenhaus group or
- (ii) $\Gamma(q^n) \cong \mathfrak{G} > \bar{\Gamma}(q^n)$.

The main result here is Theorem B. Theorem A isolates that part of the proof in which solvability is not assumed. Theorem C follows immediately from the results of [8] and the fact that these exceptional groups have no transitive extensions.

1. Preliminaries. We will be concerned here with linear groups \mathfrak{G} which act half-transitively but not semi-regularly on the set \mathfrak{V}^* of nonzero vectors. This implies (see [11], Th. 10.4) that \mathfrak{G} acts irreducibly on \mathfrak{V} . There are thus two possibilities according to whether \mathfrak{G} is primitive or imprimitive as a linear group. The latter case is completely classified in Theorem 4.2 of [7] which we restate below for convenience.

THEOREM 1.1. *Let \mathfrak{G} act faithfully on vector space \mathfrak{V} over $GF(q)$ and let \mathfrak{G} act half-transitively but not semi-regularly on \mathfrak{V}^* . If \mathfrak{G} is imprimitive as a linear group, then \mathfrak{G} satisfies one of the following*

- (i) $\mathfrak{G} = \mathcal{T}_0(q^n)$ with $q \neq 2$ and n an integer.

- (ii) $|\mathfrak{B}| = 3^4$ and \mathfrak{G} is isomorphic to a central product of the dihedral and quaternion groups of order 8.
- (iii) $|\mathfrak{B}| = 2^6$ and \mathfrak{G} is isomorphic to the dihedral group of order 18 with cyclic Sylow 3-subgroup.

Here $\mathcal{S}_0(q^n)$ is the stabilizer in $\mathcal{S}_0(q^n)$ of the zero vector and hence we know all these groups explicitly. Thus we need only consider the primitive case here.

Let \mathfrak{G} be a primitive linear group and let \mathfrak{B} be a normal p -subgroup of \mathfrak{G} . Since every normal abelian subgroup of \mathfrak{G} is cyclic (see for example [9], Lemma 1) it follows that every characteristic abelian subgroup of \mathfrak{B} is cyclic. Hence by definition \mathfrak{B} is a group of symplectic type. A characterization of these groups can be found in [1]. In particular for $p > 2$, \mathfrak{B} is a central product of one cyclic p -group and any number of nonabelian groups of order p^3 and period p . If $p = 2$, then \mathfrak{B} is a central product of either a cyclic 2-group or a 2-group of maximal class (that is, a dihedral, semidihedral or quaternion group) and any number of nonabelian groups of order 8. A special case of these are groups of type $E(p, m)$.

We say \mathfrak{G} is a group of type $E(p, m)$ with $m \neq 0$ if \mathfrak{G} has the following structure. If $p > 2$, then \mathfrak{G} is a central product of m nonabelian groups of order p^3 and period p . If $p = 2$, then \mathfrak{G} is a central product of a cyclic group of order 2 or 4, and m nonabelian groups of order 8. Thus in both cases $|\mathfrak{G}'| = p$, $Z(\mathfrak{G})$, the center of \mathfrak{G} , is cyclic and $[\mathfrak{G} : Z(\mathfrak{G})] = p^{2m}$. Moreover $|Z(\mathfrak{G})| = p$ for $p > 2$ and $|Z(\mathfrak{G})| = 2$ or 4 for $p = 2$. We call m the width of \mathfrak{G} .

Again let \mathfrak{B} be of symplectic type. If $p > 2$, then $\Omega_1(\mathfrak{B})$, the subgroup generated by all elements of order p , is either cyclic (if \mathfrak{B} is) or of type $E(p, m)$. If $p = 2$, then the Frattini subgroup $\Phi(\mathfrak{B})$ is cyclic, and $\Omega_2(C_{\mathfrak{B}}\Phi(\mathfrak{B}))$ is either cyclic or of type $E(2, m)$. The latter group is cyclic only if \mathfrak{B} is cyclic or $|\mathfrak{B}| \geq 16$ and \mathfrak{B} is maximal class. Thus modulo the above mentioned exceptions \mathfrak{B} contains a characteristic subgroup \mathfrak{G} of type $E(p, m)$ with $m \neq 0$.

If $p > 2$, then for each (p, m) there is precisely one group of type $E(p, m)$. On the other hand, if $p = 2$, then for each m there are three isomorphism classes for $E(2, m)$ and we describe these now. For convenience we will use the following notation throughout this paper: \mathfrak{D} denotes the dihedral group of order 8, \mathfrak{Q} denotes the quaternion group of order 8, and \mathfrak{Z} denotes a cyclic group of order 4. Furthermore any product of these written as \mathfrak{DQ} , \mathfrak{ZDQ} , etc. will indicate a central product. Now we have easily $\mathfrak{DD} \cong \mathfrak{QD}$ and $\mathfrak{ZD} \cong \mathfrak{ZQ}$. Hence if \mathfrak{G} is type $E(2, m)$ then \mathfrak{G} is isomorphic to one of the following three groups.

$$\begin{aligned}
 \text{iso I: } \mathfrak{G} &\cong \underbrace{\mathfrak{D}\mathfrak{D} \cdots \mathfrak{D}}_{m-1} \\
 \text{iso II: } \mathfrak{G} &\cong \mathfrak{D}\mathfrak{D} \cdots \mathfrak{D} \\
 &\quad \underbrace{\hspace{1.5cm}}_{m-1} \\
 \text{iso III: } \mathfrak{G} &\cong \mathfrak{B}\mathfrak{D}\mathfrak{D} \cdots \mathfrak{D} . \\
 &\quad \underbrace{\hspace{1.5cm}}_{m-1}
 \end{aligned}$$

We will see below that these three groups are nonisomorphic.

For any group \mathfrak{G} we let $I(\mathfrak{G})$ denote the number of its noncentral involutions.

LEMMA 1.2. *Let \mathfrak{G} be a group of type $E(2, m)$. Then*

$$\begin{aligned}
 I(\mathfrak{G}) &= 2^{2m} + (-2)^m - 2 \quad \text{if } \mathfrak{G} = \text{iso I} \\
 &= 2^{2m} - (-2)^m - 2 \quad \text{if } \mathfrak{G} = \text{iso II} \\
 &= 2^{2m+1} - 2 \quad \text{if } \mathfrak{G} = \text{iso III} .
 \end{aligned}$$

In particular these three groups are nonisomorphic. Moreover with the exception of $\mathfrak{G} = \mathfrak{D}$, \mathfrak{G} is generated by all its noncentral involutions.

Proof. Let $I^*(\mathfrak{G})$ denote the number of elements $G \in \mathfrak{G}$ with $G^2 = 1$. Then $I(\mathfrak{G}) = I^*(\mathfrak{G}) - 2$. Suppose \mathfrak{G} is iso I or II and write $\mathfrak{G} = \mathfrak{G}_1\mathfrak{D}$ where \mathfrak{G}_1 is type $E(2, m-1)$. Clearly

$$I^*(\mathfrak{G}) = 3(|\mathfrak{G}_1| - I^*(\mathfrak{G}_1)) + I^*(\mathfrak{G}_1) .$$

Thus if $I^*(\mathfrak{G}_1) = 2^{2(m-1)} + \delta(-2)^{m-1}$ then $I^*(\mathfrak{G}) = 2^{2m} + \delta(-2)^m$. Hence the first two results follow easily. If $\mathfrak{G} = \text{iso III}$, let $Z(\mathfrak{G}) = \langle Z \rangle$. Then the map $X \rightarrow XZ$ yields a one to one correspondence between the elements of \mathfrak{G} with square 1 and those of order 4. Hence clearly $I^*(\mathfrak{G}) = 1/2 |\mathfrak{G}| = 2^{2m+1}$.

Now any such \mathfrak{G} can be written as $\mathfrak{G}_1\mathfrak{D}\mathfrak{D} \cdots \mathfrak{D}$ and of course \mathfrak{D} is generated by its noncentral involutions. Since the same is easily seen to be true for $\mathfrak{G}_1 = \mathfrak{D}, \mathfrak{D}\mathfrak{D}$ or $\mathfrak{B}\mathfrak{D}$, the result follows.

Let \mathfrak{G} be type $E(p, m)$ and let $\mathfrak{W} = \mathfrak{G}/Z(\mathfrak{G})$. Then \mathfrak{W} is elementary abelian of order p^{2m} and we view this additively as a $2m$ -dimensional vector space over $GF(p)$. If $p = 2$ we say $W \in \mathfrak{W}$ is an involution vector if the coset corresponding to W in \mathfrak{G} contains an involution of \mathfrak{G} . Here we let $i(\mathfrak{W})$ denote the number of such involution vectors.

LEMMA 1.3. *Let \mathfrak{S} be a group of automorphisms of group \mathfrak{G} of type $E(p, m)$ which centralizes $Z(\mathfrak{G})$ and let \mathfrak{R} be the subgroup of \mathfrak{S} consisting of those elements which centralize \mathfrak{W} . Then*

(i) \mathfrak{R} is isomorphic to a subgroup of the direct product of

$Z(\mathfrak{G})$ taken $2m$ times.

(ii) The commutator map $(,)$ of \mathfrak{G} induces a nonsingular skew-symmetric bilinear form on \mathfrak{B} . As such $\mathfrak{G}/\mathfrak{R}$ is contained isomorphically in the symplectic group $Sp(2m, p)$.

(iii) If $p = 2$, then in addition $\mathfrak{G}/\mathfrak{R}$ permutes the $i(\mathfrak{B})$ involution vectors of \mathfrak{B} . Here

$$\begin{aligned} i(\mathfrak{B}) &= 2^{2m-1} - (-2)^{m-1} - 1 & \text{if } \mathfrak{G} = \text{iso I} \\ &= 2^{2m-1} + (-2)^{m-1} - 1 & \text{if } \mathfrak{G} = \text{iso II} \\ &= 2^{2m} - 1 & \text{if } \mathfrak{G} = \text{iso III.} \end{aligned}$$

Proof. (i) Let E_1, \dots, E_{2m} be a set of coset representatives of $Z(\mathfrak{G})$ in \mathfrak{G} . We define $\theta: \mathfrak{R} \rightarrow \prod Z(\mathfrak{G})$ ($2m$ times) by $\theta(K) = \prod E_i^K E_i^{-1}$. This is easily seen to be a monomorphism.

(ii) and (iii) If W is an involution vector then we see easily that the coset of W contains precisely two noncentral involutions of \mathfrak{G} . Hence $i(\mathfrak{B}) = 1/2I(\mathfrak{G})$. The result now follows easily.

We now consider the action of \mathfrak{G} on a vector space \mathfrak{B} .

LEMMA 1.4. Let group \mathfrak{G} of type $E(p, m)$ act on vector space \mathfrak{B} of order q^n . Suppose further that \mathfrak{G}' acts without fixed points on \mathfrak{B}^* . Then

(i) $sp^m \mid n$ where s is the smallest positive integer with $|Z(\mathfrak{G})| \mid q^s - 1$.

(ii) If $T \in \mathfrak{G} - Z(\mathfrak{G})$ has order p then $|C_{\mathfrak{B}}(T)| = q^{n/p}$.

(iii) If $x \in \mathfrak{B}^*$, then \mathfrak{G}_x , the stabilizer of x in \mathfrak{G} is elementary abelian.

Proof. (i) Since \mathfrak{G}' acts without fixed points $q \neq p$. By complete reducibility we can assume that \mathfrak{G} acts irreducibly on \mathfrak{B} . Let χ be the character of an absolutely irreducible constituent of \mathfrak{G} . From the representation of \mathfrak{G} as a homomorphic image of a direct product of nonabelian group of order p^3 (and possibly a cyclic group of order 4 if $p = 2$) we see easily that $\deg \chi = p^m$ and χ vanishes off $Z(\mathfrak{G})$. Hence by definition of s , $GF(q)(\chi) = GF(q^s)$ and \mathfrak{B} contains as absolutely irreducible constituents the s algebraic conjugates of the representation affording χ . Thus (i) follows.

(ii) We wish to show here that $\dim C_{\mathfrak{B}}(T) = n/p$. This dimension is clearly invariant under field extension so by complete reducibility we can assume \mathfrak{B} is absolutely irreducible. If θ is the corresponding complex character then $\theta(T)$ is a sum of p th roots of unity (including 1) and $\theta(T) \neq 0$. Hence all eigenvalues occur with the same multiplicity n/p and (ii) follows.

(iii) This is clear since $\Phi(\mathfrak{G})$ acts semiregularly on $\mathfrak{B}^\#$.

LEMMA 1.5. *Let group \mathfrak{G} of type $E(p, m)$ act on vector space \mathfrak{B} of order q^n and let $T \in \mathfrak{G} - Z(\mathfrak{G})$ have order p . Suppose further that \mathfrak{G} acts without fixed points on \mathfrak{B} . Then*

(i) *There exists $x \in \mathfrak{B}^\#$ with $\mathfrak{G}_x = \langle 1 \rangle$ with the following exceptions which occur for $p = 2$: (a) $q^n = 3^2$, $\mathfrak{G} = \mathfrak{D}$, (b) $q^n = 5^2$, $\mathfrak{G} = \mathfrak{B}\mathfrak{D}$, (c) $q^n = 3^4$, $\mathfrak{G} = \mathfrak{D}\mathfrak{D}$. In each of these exceptions $|\mathfrak{G}_x| = 2$ for all $x \in \mathfrak{B}^\#$.*

(ii) *There exists $x \in \mathfrak{B}^\#$ with $\mathfrak{G}_x = \langle T \rangle$ with the following exceptions which occur for $p = 2$: (a) $q^n = 3^4$, $\mathfrak{G} = \mathfrak{D}\mathfrak{D}$, (b) $q^n = 5^4$, $\mathfrak{G} = \mathfrak{B}\mathfrak{D}\mathfrak{D}$, (c) $q^n = 3^8$, $\mathfrak{G} = \mathfrak{D}\mathfrak{D}\mathfrak{D}$. In each of these exceptions $|\mathfrak{G}_x| = 4$ or 1 for all $x \in \mathfrak{B}^\#$.*

Proof. (i) We first note that by [4] Theorem II (a), (b) and (c) are in fact exceptions. Suppose now that $\mathfrak{G}_x \neq \langle 1 \rangle$ for all $x \in \mathfrak{B}^\#$. Then every element of $\mathfrak{B}^\#$ is centralized by a noncentral element $P \in \mathfrak{G}$ of order p . Thus

$$\mathfrak{B} = \bigcup_P C_{\mathfrak{B}}(P)$$

where the union is over representatives of the N noncentral subgroups of \mathfrak{G} of order p . By Lemma 1.4 we have

$$q^n = |\mathfrak{B}| \leq Nq^{n/p}$$

and $q^{n(1-1/p)} \leq N$.

Let $p > 2$. Then $N < p^{2m+1}/(p-1)$ and $n \geq sp^m$. Furthermore $p \mid q^s - 1$ so $q^s \geq p+1$. Thus

$$\begin{aligned} p^{p^{m-p}m-1} &< (p+1)^{p^{m-p}m-1} \leq q^{s(p^{m-p}m-1)} \\ &\leq q^{n(1-1/p)} \leq N < p^{2m+1}/(p-1) < p^{2m+1}. \end{aligned}$$

This yields $p^{m-1}(p-1) < 2m+1$ and since $p > 2$ we have $p=3$, $m=1$ here. However with $p=3$, $m=1$ the equation

$$(p+1)^{p^{m-p}m-1} < p^{2m+1}/(p-1)$$

is not satisfied so $p > 2$ cannot occur here.

Now let $p=2$ so that $N = I(\mathfrak{G})$. Suppose first that $|Z(\mathfrak{G})| = 4$. Then $4 \mid q^s - 1$ and $I(\mathfrak{G}) < 2^{2m+1}$. Thus

$$5^{2^{m-1}} \leq q^{s2^{m-1}} \leq q^{n(1-1/p)} \leq N \leq 2^{2m+1}.$$

This yields $5^{2^{m-1}} < 2^{2m+1}$ so $m=1$ or 2 . If $m=1$, then $q^{n/2} < 8$ and $4 \mid q^s - 1$ yields $q^n = 5^2$ and we have exception (b). If $m=2$, then $q^{n/2} < 32$, $4 \mid q^s - 1$ and $4s \mid n$ yields $q^n = 5^4$. We show now that this possibility does not occur. Let $x \in \mathfrak{B}^\#$ and suppose that $\mathfrak{G}_x \neq \langle 1 \rangle$.

Choose $P \in \mathfrak{E}_x^*$. Since \mathfrak{E}_x is abelian $\mathfrak{E}_x \cong C_{\mathfrak{E}}(P) = \langle P \rangle \times \bar{\mathfrak{E}}$ where $\bar{\mathfrak{E}} \cong 3\mathfrak{D}$. Now $x \in C_{\mathfrak{B}}(P)$, $|C_{\mathfrak{B}}(P)| = 5^2$ and $\bar{\mathfrak{E}}$ acts on this subspace. Since this action yields the exceptional case (b) we have $|\bar{\mathfrak{E}}_x| = 2$ and hence $|\mathfrak{E}_x| = 4$. Thus for all $x \in \mathfrak{B}^*$, $|\mathfrak{E}_x| = 1$ or 4 . This, by the way, is the exceptional case (b) of part (ii). If $\mathfrak{B} = \bigcup C_{\mathfrak{B}}(P)$ then since each \mathfrak{E}_x is elementary abelian, we see that this union covers \mathfrak{B} three times. Thus

$$5^4 - 1 = |\mathfrak{B}^*| \leq \frac{1}{3}I(\mathfrak{E}) \cdot (5^2 - 1) < \frac{1}{3} \cdot 2^5(5^2 - 1)$$

a contradiction.

Now let $|Z(\mathfrak{E})| = 2$ so $I(\mathfrak{E}) \leq 2^{2m} + 2^m - 2$. Since $q^s \geq 3$ we have

$$3^{2m-1} \leq q^{s2m-1} \leq q^{n(1-1/p)} \leq N \leq 2^{2m} + 2^m - 2.$$

This yields $3^{2m-1} < 2^{2m} + 2^m$ so $m = 1$ or 2 . If $m = 1$ then $q^{n/2} \leq 4$ so $q^n = 3^2$. Clearly $\mathfrak{E} \neq \mathfrak{D}$ so we have exception (a) here. If $m = 2$, then $4 \mid n$ and $q^{n/2} \leq 18$ yields $q^n = 3^4$. If $\mathfrak{E} \cong \mathfrak{D}\mathfrak{D}$ we have exception (c). We show finally that $\mathfrak{E} \neq \mathfrak{D}\mathfrak{D}$. Let $x \in \mathfrak{B}^*$ and suppose $\mathfrak{E}_x \neq \langle 1 \rangle$. Choose $P \in \mathfrak{E}_x^*$ and let $C_{\mathfrak{E}}(P) = \langle P \rangle \times \bar{\mathfrak{E}}$. Here $\bar{\mathfrak{E}}$ is nonabelian of order 8. Since $C_{\mathfrak{E}}(\bar{\mathfrak{E}})$ contains P we see that $C_{\mathfrak{E}}(\bar{\mathfrak{E}}) \cong \mathfrak{D}$ and hence $\mathfrak{E} \cong \mathfrak{D}\bar{\mathfrak{E}}$. Thus $\bar{\mathfrak{E}} \cong \mathfrak{D}$. This implies as above that $|\bar{\mathfrak{E}}_x| = 2$ and $|\mathfrak{E}_x| = 4$, thereby yielding exception (a) of part (ii). Again if $\mathfrak{B} = \bigcup C_{\mathfrak{B}}(P)$, then \mathfrak{B} is triply covered so

$$3^4 - 1 = |\mathfrak{B}^*| \leq \frac{1}{3}I(\mathfrak{E})(3^2 - 1) < \frac{1}{3}20(3^2 - 1),$$

a contradiction. This completes the proof of (i).

(ii) If $m = 1$, then any abelian subgroup of \mathfrak{E} of order 4 meets $Z(\mathfrak{E})$. Since $Z(\mathfrak{E})$ acts semiregularly, we conclude that for all $x \in \mathfrak{B}^*$, $|\mathfrak{E}_x| = 1$ or 2 . Thus the result follows here.

Let $m \geq 2$. Then $C_{\mathfrak{E}}(T) = \langle T \rangle \times \bar{\mathfrak{E}}$ where $\bar{\mathfrak{E}}$ is type $E(p, m-1)$. Note that $\mathfrak{E} = \bar{\mathfrak{E}}C_{\mathfrak{E}}(\bar{\mathfrak{E}})$ and $T \in C_{\mathfrak{E}}(\bar{\mathfrak{E}})$. Thus if $p = 2$ then $C_{\mathfrak{E}}(\bar{\mathfrak{E}}) \cong \mathfrak{D}$ and the isomorphism class of $\bar{\mathfrak{E}}$ is uniquely determined by $\mathfrak{E} \cong \bar{\mathfrak{E}}\mathfrak{D}$. Now $\bar{\mathfrak{E}}$ acts on $C_{\mathfrak{B}}(T)$ a subspace of size $q^{n/2}$ and hence if this is not one of the exceptions of part (i), then there exists $x \in C_{\mathfrak{B}}(T)^*$ with $\bar{\mathfrak{E}}_x = \langle 1 \rangle$. Since $T \in \mathfrak{E}_x$ and \mathfrak{E}_x is abelian, it then follows that $\mathfrak{E}_x = \langle T \rangle$. The result now clearly follows.

We now turn to a variant of an argument used in [2] (§ 2.5).

LEMMA 1.6. *Let $\mathfrak{G} = \mathfrak{E}\mathfrak{J}$ where \mathfrak{E} is type $E(p, m)$, $\mathfrak{E} \triangle \mathfrak{G}$ and $\mathfrak{J} = \langle J \rangle$ is cyclic of order j . Suppose \mathfrak{G} acts on F -vector space \mathfrak{B} in such a way that the restriction to \mathfrak{E} is faithful and absolutely*

irreducible. If further the characteristic of F is prime to $|\mathfrak{G}|$, then there exists nonnegative integers a_0, a_1, \dots, a_{j-1} satisfying

- (i) $a_0 + a_1 + \dots + a_{j-1} = p^m$
- (ii) $a_0^2 + a_1^2 + \dots + a_{j-1}^2 \leq N$ and
- (iii) $a_0 = \dim_F C_{\mathfrak{B}}(J)$

where N is the number of orbits in $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$ under the action of \mathfrak{S} .

Proof. Since $\dim_F C_{\mathfrak{B}}(J)$ is clearly invariant under field extension, we can assume F is algebraically closed. Let $\varepsilon \in F$ be a primitive j th root of unity and suppose that ε^i occurs as an eigenvalue of J with multiplicity a_i for $i = 0, 1, \dots, j-1$. If Σ denotes the enveloping algebra of this representation then clearly

$$\begin{aligned} a_0 + a_1 + \dots + a_{j-1} &= \dim_F \mathfrak{B} \\ a_0^2 + a_1^2 + \dots + a_{j-1}^2 &= \dim_F C_{\Sigma}(J) \\ a_0 &= \dim_F C_{\mathfrak{B}}(J). \end{aligned}$$

Now \mathfrak{B} is a faithful absolutely irreducible \mathfrak{G} -module so $\dim_F \mathfrak{B} = p^m$. Hence (i) and (iii) follow. In addition the group ring $F(\mathfrak{G})$ maps onto Σ in the obvious manner. Under this map $Z(\mathfrak{G})$ is sent into the field of scalars so the image of $F[\mathfrak{G}]$ is spanned by p^{2m} coset representatives of $Z(\mathfrak{G})$ in \mathfrak{G} . But $\dim_F \Sigma = p^{2m}$ so these must in fact form a basis of Σ . With this choice of basis we see clearly that $\dim_F C_{\Sigma}(J)$ is at most equal to the number of orbits of \mathfrak{S} on $\mathfrak{G}/Z(\mathfrak{G})$ so the result follows.

The following two results enable us to use inductive methods in our study of half-transitive linear groups.

LEMMA 1.7. *Let \mathfrak{G} be a half-transitive permutation group and let $\mathfrak{N} \trianglelefteq \mathfrak{G}$. Suppose that either $\mathfrak{N} = \langle 1 \rangle$ or \mathfrak{N} acts half-transitively. Let $\mathfrak{G} \cong \mathfrak{S} \rtimes \mathfrak{N}$ where $\mathfrak{S}/\mathfrak{N}$ is a normal Hall subgroup of $\mathfrak{G}/\mathfrak{N}$. Then \mathfrak{S} acts half-transitively.*

Proof. See Lemma 2.1 of [5].

LEMMA 1.8. (*Reduction Lemma*). *Let \mathfrak{G} be a linear group on $GF(q)$ -vector space \mathfrak{B} and suppose that \mathfrak{G} acts half-transitively but not semiregularly on \mathfrak{B}^* . Let \mathfrak{E} be a group of type $E(p, m)$ with $\mathfrak{E} \trianglelefteq \mathfrak{G}$. Then there exists a linear group $\bar{\mathfrak{G}}$ acting on $GF(q)$ -vector space \mathfrak{U} and a normal subgroup $\bar{\mathfrak{E}}$ of $\bar{\mathfrak{G}}$ satisfying*

- (i) $\bar{\mathfrak{G}}$ acts half-transitively on \mathfrak{U} .
- (ii) $\bar{\mathfrak{E}} \cong \mathfrak{E}$ and $\bar{\mathfrak{E}}$ acts irreducibly on \mathfrak{U} .
- (iii) If \mathfrak{G} is solvable so is $\bar{\mathfrak{G}}$.

(iv) If $\mathfrak{G} \neq \mathfrak{Q}$, then $\bar{\mathfrak{G}}$ does not act semiregularly on \mathfrak{U} .

(v) Suppose that either $p > 2$ or $p = 2$ and $m \geq 2$. Then either $\bar{\mathfrak{G}} = \bar{\mathfrak{G}} \cong \mathfrak{DQ}$ with $q = 3$ or $\bar{\mathfrak{G}}$ is primitive as a linear group.

Proof. Since \mathfrak{G} does not act semiregularly, it acts irreducibly on \mathfrak{B} . By Clifford's theorem all irreducible \mathfrak{G} constituents of \mathfrak{B} are conjugate and hence \mathfrak{G} acts faithfully on each. Let \mathfrak{U} be an irreducible \mathfrak{G} -submodule of \mathfrak{B} and let $\mathfrak{N} = \{G \in \mathfrak{G} \mid \mathfrak{U}G = \mathfrak{U}\}$. Suppose $x \in \mathfrak{U}^*$. Since $\mathfrak{G} \trianglelefteq \bar{\mathfrak{G}}$

$$(x\mathfrak{G})\mathfrak{G}_x = (x\mathfrak{G}_x)\mathfrak{G} = x\mathfrak{G}$$

and hence \mathfrak{G}_x normalizes $x\mathfrak{G}$. Moreover \mathfrak{G} acts irreducibly on \mathfrak{U} so \mathfrak{U} is the linear span of $x\mathfrak{G}$ and hence $\mathfrak{G}_x \subseteq \mathfrak{N}$. If \mathfrak{R} is the kernel of the action of \mathfrak{N} on \mathfrak{U} , then clearly $\bar{\mathfrak{G}} = \mathfrak{N}/\mathfrak{R}$ acts semiregularly on \mathfrak{U}^* . Since \mathfrak{G} acts faithfully on \mathfrak{U} , $\bar{\mathfrak{G}} = \mathfrak{G}\mathfrak{R}/\mathfrak{R} \cong \mathfrak{G}$. Also $\bar{\mathfrak{G}} \trianglelefteq \bar{\mathfrak{G}}$ and $\bar{\mathfrak{G}}$ acts irreducibly on \mathfrak{U} so (i), (ii) and (iii) follow.

We have $\bar{\mathfrak{G}} \cong \mathfrak{G}$. Thus if $\mathfrak{G} \neq \mathfrak{Q}$ then $\bar{\mathfrak{G}}$, and hence $\bar{\mathfrak{G}}$, cannot act semiregularly. This yields (iv). Finally suppose that either $p > 2$ or $p = 2$ and $m \geq 2$. Then $\mathfrak{G} \neq \mathfrak{Q}$ so $\bar{\mathfrak{G}}$ does not act semiregularly. Hence if $\bar{\mathfrak{G}}$ is imprimitive as a linear group, then the structure of $\bar{\mathfrak{G}}$ is given in Theorem 1.1. In both (i) and (iii) of that theorem $\bar{\mathfrak{G}}$ has a normal abelian subgroup of index 2 and hence $\bar{\mathfrak{G}}$ could not possibly contain $\bar{\mathfrak{G}}$. Thus only (ii) of that theorem can occur here and since $m \geq 2$ this yields $\bar{\mathfrak{G}} = \bar{\mathfrak{G}} \cong \mathfrak{DQ}$ and $|\mathfrak{U}| = 3^4$. This completes the proof of the lemma.

We close this section by offering a precise statement of Lemma 6 of [4]. The proof is the same and will not be repeated.

LEMMA 1.9. *Let \mathfrak{G} act faithfully on vector space \mathfrak{B} and half-transitively on \mathfrak{B}^* . Suppose that for all $x \in \mathfrak{B}^*$, $|\mathfrak{G}_x| = 2$. If \mathfrak{G} has a central involution, then $|\mathfrak{B}| = q^{2r}$ with $q \neq 2$ and $q^r + 1 = I(\bar{\mathfrak{G}})$.*

2. Theorem A. The following assumptions hold throughout this section.

ASSUMPTIONS. Group \mathfrak{G} acts faithfully on vector space \mathfrak{B} of order q^n and half-transitively but not semiregularly on \mathfrak{B}^* . \mathfrak{G} is a group of type $E(p, m)$ with $\mathfrak{G} \trianglelefteq \bar{\mathfrak{G}}$. In addition \mathfrak{G} acts irreducibly on \mathfrak{B} and $\bar{\mathfrak{G}}$ is primitive as a linear group.

It is convenient to keep track of four separate possibilities.

DEFINITION. We define the type of \mathfrak{G} as follows.

- type I: $p > 2$
 type II: $p = 2, |Z(\mathfrak{G})| = 2$
 type III: $p = 2, |Z(\mathfrak{G})| = 4, Z(\mathfrak{G}) \subseteq Z(\mathfrak{G})$
 type IV: $p = 2, |Z(\mathfrak{G})| = 4, Z(\mathfrak{G}) \not\subseteq Z(\mathfrak{G})$.

LEMMA 2.1. *Let $s \geq 1$ be minimal with $|Z(\mathfrak{G})| \mid q^s - 1$. Let \mathfrak{M} be any subgroup of \mathfrak{G} with $\mathfrak{G} \subseteq \mathfrak{M} \subseteq C_{\mathfrak{G}}(Z(\mathfrak{G}))$. Then $\mathfrak{M} \subseteq GL(p^m, q^s)$ and this representation of \mathfrak{M} is absolutely irreducible. Furthermore $n = sp^m$ and we have the following*

type I: $s \mid (p - 1)$

type II: $s = 1$

type III: $s = 1$ or 2

type IV: $s = 2$, and if $\bar{\mathfrak{M}}$ is a q' -subgroup of \mathfrak{G} with $\mathfrak{G} \subseteq \bar{\mathfrak{M}}$ and $\bar{\mathfrak{M}} \not\subseteq C_{\mathfrak{G}}(Z(\mathfrak{G}))$, then $\bar{\mathfrak{M}} \subseteq GL(p^{m+1}, q)$ and this is an absolutely irreducible representation.

Proof. If s is defined as above then $GF(q^s)$ is clearly the minimal splitting field of the representation of \mathfrak{G} . Hence $n = sp^m$ since we are dealing with finite fields here and since the absolutely irreducible constituents of \mathfrak{G} have degree p^m .

Now \mathfrak{G} is primitive as a linear group so by Lemma 1.1 of [5], $C_{\mathfrak{G}}(Z(\mathfrak{G})) \subseteq GL(p^m, q^s)$. Let \mathfrak{M} be a subgroup of \mathfrak{G} with

$$\mathfrak{G} \subseteq \mathfrak{M} \subseteq C_{\mathfrak{G}}(z(\mathfrak{G}))$$

so that $\mathfrak{M} \subseteq GL(p^m, q^s)$. Since $\mathfrak{M} \supseteq \mathfrak{G}$ and the degree of this representation is p^m , the representation is clearly absolutely irreducible.

The results on the value of s for types I, II and III are clear. Let \mathfrak{G} be type IV. Then certainly $s = 1$ or 2 . If $s = 1$, then since \mathfrak{G} is primitive, $Z(\mathfrak{G})$ consists of scalar matrices and is therefore central in \mathfrak{G} , a contradiction. Thus $s = 2$. Let $\bar{\mathfrak{M}}$ be given with $\mathfrak{G} \subseteq \bar{\mathfrak{M}}$, $\bar{\mathfrak{M}} \not\subseteq C_{\mathfrak{G}}(Z(\mathfrak{G}))$. Since $s = 2$, $\bar{\mathfrak{M}} \subseteq GL(p^{m+1}, q)$. Clearly $\bar{\mathfrak{M}}$ is either absolutely irreducible or it has two absolutely irreducible constituents of degree p^m . In the latter case, $Z(\mathfrak{G})$ would be central in each such constituent and hence in $\bar{\mathfrak{M}}$, a contradiction.

LEMMA 2.2. *Let \mathfrak{M} be a p -group acting faithfully and absolutely irreducibly on F -vector space \mathfrak{V} . Let $\dim_F \mathfrak{V} = k$. Then there exists subgroups \mathfrak{N} and \mathfrak{R} of \mathfrak{M} and an \mathfrak{N} -subspace \mathfrak{U} of \mathfrak{V} with the representation of \mathfrak{M} on \mathfrak{V} induced from that of \mathfrak{N} on \mathfrak{U} . Furthermore $\mathfrak{R} = C_{\mathfrak{M}}(\mathfrak{U})$ and either*

(i) $[\mathfrak{M} : \mathfrak{N}] = k$, $\dim \mathfrak{U} = 1$ and $\mathfrak{N}/\mathfrak{R}$ is cyclic, or

(ii) $[\mathfrak{M} : \mathfrak{N}] = k/2$, $\dim \mathfrak{U} = 2$, $\mathfrak{N}/\mathfrak{R}$ is dihedral, semidihedral or quaternion and $p = 2$.

Proof. The result is trivial if $\text{char } F = p$ so assume this is not the case. Applying Roquette's theorem ([9]) repeatedly we can find \mathfrak{N} , \mathfrak{R} and \mathfrak{U} as above with $\mathfrak{N}/\mathfrak{R}$ cyclic, dihedral, semidihedral or quaternion. Since \mathfrak{M} is absolutely irreducible so is the action of $\mathfrak{N}/\mathfrak{R}$ on \mathfrak{U} . Thus $\dim \mathfrak{U} = 1$ if $\mathfrak{N}/\mathfrak{R}$ is cyclic and $\dim \mathfrak{U} = 2$ otherwise.

LEMMA 2.3. *Let w denote the period of a Sylow p -subgroup of $C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}))$. Then for all $x \in \mathfrak{B}^*$ we have*

- type I: $[\mathfrak{G} : \mathfrak{G}_x]_p \leq p^m \min \{w, |q^s - 1|_p\}$
- type II: $[\mathfrak{G} : \mathfrak{G}_x]_p \leq p^{m+1} \min \{w, |q^2 - 1|_p\}$
- type III: $[\mathfrak{G} : \mathfrak{G}_x]_p \leq p^m \min \{w, |q^2 - 1|_p\}$
- type IV: $[\mathfrak{G} : \mathfrak{G}_x]_p \leq p^{m+1} \min \{w, |q^2 - 1|_p\}$.

Proof. We consider types I, II and III first. Let \mathfrak{P} be a Sylow p -subgroup of \mathfrak{G} . Then $\mathfrak{P} \supseteq \mathfrak{G}$ and $\mathbf{Z}(\mathfrak{G})$ is central in \mathfrak{P} . By Lemma 2.1 we can view \mathfrak{P} as a subgroup of $GL(p^m, q^s)$ and this representation is absolutely irreducible. Let \mathfrak{N} , \mathfrak{R} and \mathfrak{U} be as in the preceding lemma with $\mathfrak{M} = \mathfrak{P}$. Note that for $y \in \mathfrak{U}^*$, $\mathfrak{P}_y \supseteq \mathfrak{R}$. If $\mathfrak{N}/\mathfrak{R}$ is cyclic, then $[\mathfrak{P} : \mathfrak{N}] = p^m$, $[\mathfrak{N} : \mathfrak{R}] \leq \min \{w, |q^s - 1|_p\}$ so

$$[\mathfrak{P} : \mathfrak{P}_y] \leq p^m \min \{w, |q^s - 1|_p\}.$$

Suppose that $\mathfrak{N}/\mathfrak{R}$ is not cyclic. Then $p = 2$. Now it is clear that $\mathbf{Z}(\mathfrak{G}) \subseteq \mathbf{Z}(\mathfrak{P}) \subseteq \mathfrak{N}$ and $\mathbf{Z}(\mathfrak{G}) \cap \mathfrak{R} = \langle 1 \rangle$. Thus since 2-groups of maximal class have centers of order 2, \mathfrak{G} must be type II. Here $[\mathfrak{P} : \mathfrak{N}] = p^{m-1}$ and $[\mathfrak{N} : \mathfrak{R}] \leq p \min \{w, |q^{2s} - 1|_p\}$ since $\mathfrak{N}/\mathfrak{R}$ has a cyclic subgroup of index $p = 2$ which has a faithful irreducible representation in $GF(q^{2s})$. Note that $s = 1$ here. Now by half-transitivity, for all $x \in \mathfrak{B}^*$

$$[\mathfrak{G} : \mathfrak{G}_x]_p = [\mathfrak{G} : \mathfrak{G}_y]_p \leq [\mathfrak{P} : \mathfrak{P}_y]_p.$$

Thus the first three results follow.

Now let \mathfrak{G} be type IV and again let \mathfrak{P} be a Sylow p -subgroup of \mathfrak{G} . Let $\mathfrak{M} = C_{\mathfrak{P}}(\mathbf{Z}(\mathfrak{G}))$ so that $\mathfrak{P} > \mathfrak{M} \supseteq \mathfrak{G}$ and $[\mathfrak{P} : \mathfrak{M}] = 2$. By Lemma 2.1, \mathfrak{P} is absolutely irreducible as a subgroup of $GL(p^{m+1}, q)$. We extend the field now to $GF(q^s) = GF(q^2)$. Thus we let \mathfrak{P} act on $\mathfrak{B} \otimes GF(q^2)$ and this representation is again absolutely irreducible. If the restriction to \mathfrak{M} were irreducible, then since $4 \mid q^s - 1$, $\mathbf{Z}(\mathfrak{G})$ which is central in \mathfrak{M} would consist of scalar matrices and hence it would be central in \mathfrak{P} , a contradiction. Thus the representation of \mathfrak{P} is induced from one of \mathfrak{M} . Let \mathfrak{N} , \mathfrak{R} and $\mathfrak{U} \subseteq \mathfrak{B} \otimes GF(q^2)$ be as in the preceding lemma with $\mathfrak{N} \subseteq \mathfrak{M}$. Since $\mathbf{Z}(\mathfrak{G}) \subseteq \mathfrak{N}$ and $|\mathbf{Z}(\mathfrak{G})| = 4$ we see that $\mathfrak{N}/\mathfrak{R}$ is cyclic. Hence $[\mathfrak{N} : \mathfrak{R}] \leq \min \{w, |q^s - 1|_p\}$. Moreover $[\mathfrak{P} : \mathfrak{N}] = p^{m+1}$ so

$$[\mathfrak{P} : \mathfrak{R}] \leq p^{m+1} \min \{w, |q^s - 1|_p\}.$$

Now all elements of \mathfrak{R} have a common nonzero fixed point in $\mathfrak{B} \otimes GF(q^s)$. This means that a certain set of simultaneous linear equations over $GF(q)$ has a nonzero solution over $GF(q^s)$. Thus there is a nonzero solution over $GF(q)$ and hence there exists $y \in \mathfrak{B}^*$ with $\mathfrak{P}_y \cong \mathfrak{R}$. The result now follows as above.

LEMMA 2.4. *Let $\mathfrak{A} = C_{\mathfrak{G}}(\mathfrak{E})$. Then \mathfrak{A} is a normal cyclic subgroup of \mathfrak{G} which is central in $C_{\mathfrak{G}}(Z(\mathfrak{E}))$ and acts semiregularly on \mathfrak{B}^* . Suppose that $m \geq 3$ if $p = 2$. Then there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$ and hence $|\mathfrak{G} : \mathfrak{G}_x|_p \geq |\mathfrak{A}_p| p^{2m}$ where \mathfrak{A}_p is the normal Sylow p -subgroup of \mathfrak{A} . This yields*

$$\begin{aligned} \text{type I: } & w \geq p^m |\mathfrak{A}_p|, & |q^s - 1|_p & \geq p^{m+1} \\ \text{type II: } & w \geq p^{m-1} |\mathfrak{A}_p|, & |q^2 - 1|_p & \geq p^m \\ \text{type III: } & w \geq p^m |\mathfrak{A}_p|, & |q^2 - 1|_p & \geq p^{m+2} \\ \text{type IV: } & w \geq p^{m-1} |\mathfrak{A}_p|, & |q^2 - 1|_p & \geq p^{m+1}. \end{aligned}$$

Proof. Since \mathfrak{E} is irreducible, Schur's lemma guarantees that \mathfrak{A} is cyclic and acts semiregularly. Clearly $\mathfrak{A} \subseteq C_{\mathfrak{G}}(Z(\mathfrak{E}))$. By Lemma 2.1, $\mathfrak{E} \subseteq C_{\mathfrak{G}}(Z(\mathfrak{E})) \subseteq GL(p^m, q^s)$ and this is an absolutely irreducible representation of \mathfrak{E} . Since \mathfrak{A} centralizes \mathfrak{E} , \mathfrak{A} consists of scalar matrices here and hence \mathfrak{A} is central in $C_{\mathfrak{G}}(Z(\mathfrak{E}))$.

If $p > 2$ set $\mathfrak{E}^* = \mathfrak{E}$ while if $p = 2$ we set $\mathfrak{E}^* = \mathfrak{A}^*\mathfrak{E}$ where $\mathfrak{A}^* = \{A \in \mathfrak{A} \mid A^4 = 1\}$. Then \mathfrak{E}^* is also of type $E(p, m)$ and every subgroup of $\mathfrak{A}\mathfrak{E}$ of order p is in \mathfrak{E}^* . With the additional assumption that $m \geq 3$ if $p = 2$, Lemma 1.5 applied to \mathfrak{E}^* guarantees the existence of a point $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{E}^* = \langle 1 \rangle$. This clearly yields $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$.

Now $\mathfrak{A}_p\mathfrak{E} \trianglelefteq \mathfrak{G}$ and $|\mathfrak{A}_p\mathfrak{E}| = |\mathfrak{A}_p| p^{2m}$. If x is as above then

$$|\mathfrak{G}|_p \geq |\mathfrak{G}_x \mathfrak{A}_p\mathfrak{E}|_p = |\mathfrak{G}_x|_p |\mathfrak{A}_p\mathfrak{E}| = |\mathfrak{G}_x|_p |\mathfrak{A}_p| p^{2m}$$

and hence $|\mathfrak{G} : \mathfrak{G}_x|_p \geq |\mathfrak{A}_p| p^{2m}$. By half-transitivity this holds for all $x \in \mathfrak{B}^*$. Combining this with the results of Lemma 2.3 and noting that $|\mathfrak{A}_p| \geq p$ for type I and II groups and $|\mathfrak{A}_p| \geq p^2$ for type III and IV groups, we clearly obtain our result.

LEMMA 2.5. *Let $\mathfrak{S} = C_{\mathfrak{G}}(Z(\mathfrak{E}))$. Then \mathfrak{G} has the following structure.*

- (i) $\mathfrak{G}/\mathfrak{S}$ is cyclic
- (ii) $\mathfrak{S}/\mathfrak{A}\mathfrak{E}$ acts faithfully on $\mathfrak{B} = \mathfrak{E}/Z(\mathfrak{E})$ and as a linear group on \mathfrak{B} we have $\mathfrak{S}/\mathfrak{A}\mathfrak{E} \subseteq Sp(2m, p)$
- (iii) $\mathfrak{A}\mathfrak{E}/\mathfrak{A}$ is elementary abelian of order p^{2m}
- (iv) \mathfrak{A} is cyclic.

Proof. All results but (ii) are clear. Let $\mathfrak{B} = C_{\mathfrak{S}}(\mathfrak{B})$. Clearly $\mathfrak{B} \cong \mathfrak{A}\mathfrak{E}$. The result will follow from Lemma 1.3 if we show that

$\mathfrak{B} = \mathfrak{A}\mathfrak{C}$.

Suppose first that $|Z(\mathfrak{C})| = p$ so \mathfrak{C} is type I or II. By Lemma 1.3, $\mathfrak{B}/\mathfrak{A} \subseteq Z(\mathfrak{C}) \times Z(\mathfrak{C}) \times \cdots \times Z(\mathfrak{C})$ ($2m$ times). Hence $[\mathfrak{B}:\mathfrak{A}] \leq p^{2m}$. Since $[\mathfrak{A}\mathfrak{C}:\mathfrak{A}] = p^{2m}$ we have $\mathfrak{B} = \mathfrak{A}\mathfrak{C}$ here. Now let $|Z(\mathfrak{C})| = p^2$ so $p = 2$ and \mathfrak{C} is type III or IV. As above $\mathfrak{B}/\mathfrak{A} \subseteq Z(\mathfrak{C}) \times Z(\mathfrak{C}) \times \cdots \times Z(\mathfrak{C})$ ($2m$ times) so $\mathfrak{B}/\mathfrak{A}$ is a 2-group. Since \mathfrak{A} is central in \mathfrak{G} , \mathfrak{B} is nilpotent with Sylow 2-subgroup \mathfrak{B}_2 . Now \mathfrak{G} is primitive and $\mathfrak{B}_2 \triangle \mathfrak{G}$ so \mathfrak{B}_2 is of symplectic type. Clearly $Z(\mathfrak{B}_2) = \mathfrak{A}_2$ and $|\mathfrak{A}_2| \geq 4$ here. Hence \mathfrak{B}_2 is the central product of \mathfrak{A}_2 and a group of type $E(2, r)$. Thus $\mathfrak{B}_2/\mathfrak{A}_2$ has period 2 and we can conclude again that $[\mathfrak{B}:\mathfrak{A}] \leq p^{2m}$. The result follows.

LEMMA 2.6. *We must have one of the following.*

type I: $p = 3, m \leq 2$

type II: $p = 2, m \leq 6$

type III: $p = 2, m \leq 3$

type IV: $p = 2, m \leq 5$.

Proof. We first show the following.

type I: $w \leq p(2m - 1) |\mathfrak{A}_p|$

$w \leq |\mathfrak{A}_p|$ for $m = 1, p > 3$

type II: $w \leq p^2(2m - 1) |\mathfrak{A}_p|$

type III: $w \leq p(2m - 1) |\mathfrak{A}_p|$

type IV: $w \leq p(2m - 1) |\mathfrak{A}_p|$.

Now the p -period of $Sp(2m, p)$ is clearly at most $(2m - 1)p$. If \mathfrak{C} is type I, III or IV, then the period of $\mathfrak{A}_p\mathfrak{C}$ is $|\mathfrak{A}_p|$. If \mathfrak{C} is type II, then the period of $\mathfrak{A}_p\mathfrak{C}$ is at most $p |\mathfrak{A}_p|$. Combining these facts with the structure given in the preceding lemma yields all the above facts except for the one concerning $p > 3, m = 1$.

Now let $m = 1$ and $p > 3$. Let \mathfrak{P} be a Sylow p -subgroup of $C_{\mathfrak{G}}(Z(\mathfrak{C}))$ and hence a Sylow p -subgroup of \mathfrak{G} . Thus $\mathfrak{P} \supseteq \mathfrak{A}_p\mathfrak{C}$ and since $|Sp(2, p)|_p = p$ we have $[\mathfrak{P}:\mathfrak{A}_p\mathfrak{C}] \leq p$. Since \mathfrak{C} does not act semiregularly we have $p \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{B}^*$. As we have seen, there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{A}_p\mathfrak{C} = \langle 1 \rangle$. Let $\bar{\mathfrak{P}}$ be a subgroup of \mathfrak{G}_x of order p . By taking a suitable conjugate of \mathfrak{P} if necessary we can assume that $\bar{\mathfrak{P}} \subseteq \mathfrak{P}$. Then $\mathfrak{P} = \mathfrak{A}_p(\mathfrak{C}\bar{\mathfrak{P}})$. Now $|\mathfrak{C}\bar{\mathfrak{P}}| = p^4$ and this group is generated by elements of order p . Hence if $p > 3$, then $\mathfrak{C}\bar{\mathfrak{P}}$ has period p . Since \mathfrak{A}_p is central in \mathfrak{P} we see that \mathfrak{P} has period $|\mathfrak{A}_p|$ and the above follows.

Combining the above with the lower bound for w given in Lemma 2.4 yields the following equations.

type I: $p^m |\mathfrak{A}_p| \leq p(2m - 1) |\mathfrak{A}_p|$

$p |\mathfrak{A}_p| \leq |\mathfrak{A}_p|$ for $m = 1, p > 3$

type II: $p^{m-1} |\mathfrak{A}_p| \leq p^2(2m - 1) |\mathfrak{A}_p|$

type III: $p^m |\mathfrak{A}_p| \leq p(2m - 1) |\mathfrak{A}_p|$

type IV: $p^{m-1} | \mathfrak{U}_p | \leq p(2m-1) | \mathfrak{U}_p |$.

Note that the equations for types II, III and IV hold only for $m \geq 3$. The result now follows easily.

We note that the above yields a stronger result than Proposition 2.1 of [6] and the proof is considerably less computational. We now strengthen the above argument to eliminate additional cases. We first eliminate $p = 3$.

LEMMA 2.7. $p = 3$, $m = 1$ does not occur.

Proof. Suppose $p = 3$ and $m = 1$. Then \mathfrak{G} has the structure described in Lemma 2.5. In addition, $[\mathfrak{G} : C_{\mathfrak{G}}(Z(\mathfrak{G}))] = 1$ or 2 and $Sp(2m, p) = SL(2, p)$. By Lemma 2.4, \mathfrak{A} is central in $C_{\mathfrak{G}}(Z(\mathfrak{G})) = \mathfrak{Z}$.

Suppose that $\mathfrak{Z}/\mathfrak{A}\mathfrak{G}$ has a normal Sylow 3-subgroup $\mathfrak{B}/\mathfrak{A}\mathfrak{G}$. Then $\mathfrak{B}/\mathfrak{A}$ is a normal Sylow 3-subgroup of $\mathfrak{G}/\mathfrak{A}$. Now both \mathfrak{G} and \mathfrak{A} act half-transitively so by Lemma 1.7 \mathfrak{B} acts half-transitively on \mathfrak{B}^* . Since \mathfrak{A} is central in \mathfrak{B} , \mathfrak{B} is nilpotent and hence its normal Sylow 3-subgroup \mathfrak{B}_3 acts half-transitively. By Theorem II of [4], \mathfrak{B}_3 is cyclic, a contradiction since $\mathfrak{B}_3 \supseteq \mathfrak{G}$. Hence $\mathfrak{Z}/\mathfrak{A}\mathfrak{G}$ is a subgroup of $SL(2, 3)$ which does not have a normal Sylow 3-subgroup. This implies that $\mathfrak{Z}/\mathfrak{A}\mathfrak{G} \cong SL(2, 3)$, a group of order 24.

We show now that we cannot have $8 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{B}^*$. Assume by way of contradiction that this is the case. Let \mathfrak{P} be a subgroup of \mathfrak{G} of order 3 having a fixed point $y \neq 0$. Since $\mathfrak{U}_y = \langle 1 \rangle$ we see that $8 \mid |\mathfrak{U}_{\mathfrak{G}_y}/\mathfrak{U}|$ so $4 \mid |\mathfrak{U}_{\mathfrak{G}_y}/\mathfrak{U}|$. Now a Sylow 2-subgroup of $\mathfrak{Z}/\mathfrak{A}$ is quaternion of order 8 so \mathfrak{Z}_y has an element B of order 4. Since $B^2 \notin \mathfrak{U}\mathfrak{G}$, B does not normalize $\mathfrak{P}Z(\mathfrak{G})/Z(\mathfrak{G})$. Thus $\mathfrak{G} = \langle \mathfrak{P}, \mathfrak{P}^B \rangle \subseteq \mathfrak{G}_y$, a contradiction since $Z(\mathfrak{G})$ acts semiregularly.

Let \mathfrak{P} be a subgroup of \mathfrak{G} of order 3. We show that $\dim C_{\mathfrak{B}}(\mathfrak{P}) = 0$ or s . Since $\mathfrak{P} \subseteq GL(3, q^s)$ we see that $\dim C_{\mathfrak{B}}(\mathfrak{P}) = 0, s$ or $2s$. Suppose the dimension is $2s$. By Lemma 1.4, $\mathfrak{P} \not\subseteq \mathfrak{U}\mathfrak{G}$. Since $\mathfrak{Z}/\mathfrak{A}\mathfrak{G} \cong SL(2, 3)$ there exists $G \in \mathfrak{G}$ such that \mathfrak{P} and \mathfrak{P}^G generate this quotient. Now \mathfrak{B} is 3-dimensional over $GF(q^s)$ and $C_{\mathfrak{B}}(\mathfrak{P})$ and $C_{\mathfrak{B}}(\mathfrak{P}^G)$ are 2-dimensional subspaces. Thus there exists $x \in \mathfrak{B}^*$ with $\mathfrak{P}, \mathfrak{P}^G \subseteq \mathfrak{G}_x$. This implies that $24 \mid |\mathfrak{G}_x|$ and this contradicts the comments of the preceding paragraph.

We now proceed to count. The group $\mathfrak{G}/\mathfrak{A}$ is easily seen to contain at most 40 subgroups of order 3. If \mathfrak{P} is a group of order 3 in \mathfrak{G} , then $\mathfrak{P}\mathfrak{A}$ being abelian has at most 3 subgroups of order 3 other than $Z(\mathfrak{G})$. Hence \mathfrak{G} has at most $3 \cdot 40 = 120$ subgroups of order 3 other than $Z(\mathfrak{G})$. Each such \mathfrak{P} fixes at most $q^s - 1$ points of \mathfrak{B}^* so since clearly $3 \mid |\mathfrak{G}_x|$ we have

$$120(q^s - 1) \geq |\mathfrak{B}^*| = q^{3s} - 1$$

and

$$120 \geq q^{2s} + q^s + 1.$$

Thus $q^s \leq 10$. However by Lemma 2.4, $3^2 \mid (q^s - 1)$ so q^s being a prime power is at least 19, a contradiction. Thus $p = 3$, $m = 1$ does not occur.

LEMMA 2.8. $p = 3$, $m = 2$ does not occur.

Proof. The equation obtained in the proof of Lemma 2.6 is an equality at $p = 3$, $m = 2$. Thus all inequalities used in obtaining it must also be equalities. Thus from Lemma 2.4 we must have $w = p^m |\mathfrak{U}_p|$. Furthermore if $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{U}_p \mathfrak{G} = \langle 1 \rangle$, then $|\mathfrak{G}|_p = |\mathfrak{G}_x|_p |\mathfrak{U}_p \mathfrak{G}|$.

The latter fact implies that $\mathfrak{U}_p \mathfrak{G}$ has a complement \mathfrak{Z} in \mathfrak{B} a Sylow p -subgroup of \mathfrak{G} . Since $\mathfrak{Z} \subseteq Sp(4, 3)$, \mathfrak{Z} has period at most $(2m - 1)p = 9$ and thus $\mathfrak{G}\mathfrak{Z}$ has period at most $3 \cdot 9 = 27$. Since $\mathfrak{B} = \mathfrak{U}_p(\mathfrak{G}\mathfrak{Z})$ and \mathfrak{U}_p is central here, we have clearly $w \leq \max\{|\mathfrak{U}_p|, p^3\}$. But $w = p^2 |\mathfrak{U}_p|$ so we must have $|\mathfrak{U}_p| = p$ and \mathfrak{Z} has period 9.

Let $\mathfrak{J} = \langle J \rangle$ be a subgroup of order 9 with $\mathfrak{J} \cap \mathfrak{U}_p \mathfrak{G} = \langle 1 \rangle$. We see clearly that the Jordan form of the matrix of J with respect to its action on $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$ is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus \mathfrak{J} has

$$(3^4 - 3^3)/3^2 + (3^3 - 3)/3 + 3 = 17$$

orbits on \mathfrak{B} . Note that $\mathfrak{G}\mathfrak{J} \subseteq GL(p^m, q^s)$ and the restriction to \mathfrak{G} is absolutely irreducible. Thus if $a_0 = \dim_{GF(q^s)} C_{\mathfrak{B}}(J)$ then by Lemma 1.6

$$a_0 + a_1 + \cdots + a_s = p^m = 9.$$

$$a_0^2 + a_1^2 + \cdots + a_s^2 \leq 17.$$

These yield easily $a_0 \leq 3$ and hence $\dim_{GF(q)} C_{\mathfrak{B}}(J) = sa_0 \leq 3s$.

Let \mathcal{N} denote the set of subgroups of \mathfrak{G} of order 3 together with the set of cyclic subgroup \mathfrak{J} of order 9 with $\mathfrak{J} \cap \mathfrak{U}_p \mathfrak{G} = \langle 1 \rangle$. By the above and Lemma 1.4, if $\mathfrak{N} \in \mathcal{N}$ then $\dim C_{\mathfrak{B}}(\mathfrak{N}) \leq 3s$. We have also shown above that for all $y \in \mathfrak{B}$ there exists $\mathfrak{N} \in \mathcal{N}$ with $y \in C_{\mathfrak{B}}(\mathfrak{N})$, since in that argument, if $\mathfrak{G}_x \cap \mathfrak{U}_p \mathfrak{G} = \langle 1 \rangle$ then $\mathfrak{J} = \mathfrak{Z} \subseteq \mathfrak{G}_x$. Hence $\mathfrak{B} = \bigcup \mathcal{N} C_{\mathfrak{B}}(\mathfrak{N})$. If $|\mathcal{N}| = N$, then this yields

$$q^{9s} = |\mathfrak{B}| \leq Nq^{3s}$$

or $q^{6s} \leq N$. On the other hand by Lemma 2.5,

$$\begin{aligned} |\mathfrak{G}| &\leq 2 |\mathfrak{A}| p^4 |Sp(4, p)| \\ &\leq 2 |\mathfrak{A}| p^4 p^4 (p^4 - 1)(p^2 - 1) \leq 2 |\mathfrak{A}| p^{14}. \end{aligned}$$

Since \mathfrak{A} is central in the absolutely irreducible representation $\mathfrak{A}\mathfrak{G} \subseteq GL(p^m, q^s)$ we have $|\mathfrak{A}| < q^s$. Thus

$$N \leq |\mathfrak{G}|/2 < q^s p^{14}.$$

Combining this with the lower bound we previously obtained for N yields $q^{5s} < p^{14}$. Finally by Lemma 2.4, $q^s \geq p^3$ so $p^{15} < q^{5s} < p^{14}$, a contradiction. Thus $p = 3$, $m = 2$ does not occur.

We now consider special cases with $p = 2$.

LEMMA 2.9. *The cases type II, $m = 6$, type III, $m = 3$ and type IV, $m = 5$ do not occur.*

Proof. If we consider the inequalities obtained in the proof of Lemma 2.6, we see that any of the above mentioned cases would be eliminated if a strengthening of the inequalities by a factor of $p = 2$ could be obtained. Let us suppose that one of the above occurs.

The results of Lemma 2.4 concerning $[\mathfrak{G} : \mathfrak{G}_x]_p$ and w must be equalities. In particular this implies that for given $x \in \mathfrak{X}$ with $\mathfrak{G}_x \cap \mathfrak{A}_p \mathfrak{G} = \langle 1 \rangle$ we must have $|\mathfrak{G}|_p = |\mathfrak{G}_x \mathfrak{A}_p \mathfrak{G}|_p$. Thus $\mathfrak{A}_p \mathfrak{G}$ has a complement in a Sylow p -subgroup of \mathfrak{G} and thus also in \mathfrak{P} , a Sylow p -subgroup of \mathfrak{G} . Let $\mathfrak{P} = \mathfrak{A}_p \mathfrak{G}$ where $\mathfrak{P} \cap \mathfrak{A}_p \mathfrak{G} = \langle 1 \rangle$ and let w^* denote the period of the group $\mathfrak{G}\mathfrak{P}/\mathfrak{G}'$. Since \mathfrak{A}_p is central in \mathfrak{P} we have

$$w \leq \max \{ |\mathfrak{A}_p|, 2w^* \} \leq \begin{cases} |\mathfrak{A}_p| w^* & \text{type II} \\ \frac{1}{2} |\mathfrak{A}_p| w^* & \text{types III, IV.} \end{cases}$$

We consider w^* . Let $\mathfrak{B}^* = \mathfrak{G}/\mathfrak{G}'$ so \mathfrak{B}^* is elementary abelian of order p^{2m} or p^{2m+1} . Since \mathfrak{P} acts faithfully on $\mathfrak{G}/Z(\mathfrak{G})$, it also acts faithfully on \mathfrak{B}^* . If \mathfrak{P} has period p^d , then $w^* = p^d$ or p^{d+1} . Note that $\mathfrak{P} \subseteq GL(2m+1, p)$. If $w^* = p^d$, then since $p^d \leq p(2m)$ we have $w^* \leq p(2m)$. If $w^* = p^{d+1}$, then there must exist an element $L \in \mathfrak{P}$ of order p^d whose minimal polynomial in $GL(2m+1, p)$ has degree p^d . Thus we must have $p^d \leq 2m+1$ and $w^* \leq p(2m+1)$. The latter bound being the larger of the two holds in all cases. Now w^* is a power of 2 and in the three cases we are considering neither $2m+1$ nor $2m$ is a power of 2. Hence we have $w^* \leq p(2m-1)$ and

$$\begin{aligned} w &\leq p(2m-1) |\mathfrak{A}_p| && \text{for type II} \\ w &\leq (2m-1) |\mathfrak{A}_p| && \text{for types III, IV.} \end{aligned}$$

This therefore improves the bounds on w given in the proof of Lemma 2.6 by a factor of $p = 2$ and, as we mentioned above, this yields a contradiction.

LEMMA 2.10. *The case type IV, $m = 4$ does not occur.*

Proof. We see that in the inequalities obtained in the proof of Lemma 2.6, a strengthening by a factor of $p = 2$ would eliminate this possibility. Hence if this case occurs, then we must have the following. If $x \in \mathfrak{B}^*$, then either x is fixed by a subgroup of \mathfrak{G} of order 2 or a cyclic subgroup $\mathfrak{F} \subseteq \mathfrak{G}$ of order 8 with $\mathfrak{F} \cap \mathfrak{A}\mathfrak{G} = \langle 1 \rangle$. Let \mathcal{N} denote collection of such subgroups of both types.

We show now that if $\mathfrak{F} \in \mathcal{N}$ then $\dim C_{\mathfrak{B}}(\mathfrak{F}) \leq n/2$. We know this to be the case if $\mathfrak{F} \subseteq \mathfrak{G}$ so suppose $\mathfrak{F} = \langle J \rangle$ has order 8. Then \mathfrak{F} acts faithfully on $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$. Since $|\mathfrak{F}| = 8$ we see that in its action on \mathfrak{B} , J must have one Jordan block of rank at least 5. This implies easily that \mathfrak{F} has at most

$$\frac{2^8 - 2^7}{8} + \frac{2^7 - 2^5}{4} + \frac{2^5 - 2^4}{2} + 2^4 = 2^6$$

orbits on \mathfrak{B} . We apply Lemma 1.6 to each of the two absolutely irreducible constituents of $\mathfrak{G}\mathfrak{F}$ on $\mathfrak{B} \otimes GF(q^2)$. Hence

$$a_0^2 + a_1^2 + \cdots + a_7^2 \leq 2^6.$$

Thus $a_0 \leq 8$ and since $\dim C_{\mathfrak{B}}(\mathfrak{F})$ is invariant under field extension we have $\dim C_{\mathfrak{B}}(\mathfrak{F}) \leq 2a_0 \leq n/2$. Now

$$\mathfrak{B} = \bigcup_{\mathfrak{F} \in \mathcal{N}} C_{\mathfrak{B}}(\mathfrak{F})$$

and if $N = |\mathcal{N}|$, then $q^n = |\mathfrak{B}| \leq Nq^{n/2}$ and $q^{n/2} \leq N$. By Lemma 2.5

$$|\mathfrak{G}| \leq |\mathfrak{A}| 2^{2m} |Sp(2m, 2)|.$$

Since $|\mathfrak{A}| \leq q^s$ and $|Sp(8, 2)| \leq 2^{36}$ we have $N \leq |\mathfrak{G}| \leq q^s \cdot 2^{44}$. With $n = 2^m s = 16s$ this yields

$$q^{8s} = q^{n/2} \leq N \leq q^s \cdot 2^{44}$$

or $q^{7s} \leq 2^{44}$. Now $s = 2$ and by Lemma 2.4, $2^5 = 2^{m+1}$ divides $q^2 - 1 = q^s - 1$. Since $s = 2$ and $q^s > 9$ it follows (see for example Lemma 4 of [4]) that $q^s - 1$ cannot be a power of 2. Hence $q^s > q^s - 1 \geq 3 \cdot 2^5$ so $q^{7s} > 3^7 \cdot 2^{35}$. Combining this with the above yields $3^7 \cdot 2^{35} < q^{7s} \leq 2^{44}$ or $3^7 < 2^9$, a contradiction. Therefore this case does not occur.

LEMMA 2.11. *The case type II, $m = 5$ does not occur.*

Proof. In the inequality in the proof of Lemma 2.6 for type II, $m = 5$ we see that a strengthening by a factor of $p^2 = 4$ will yield a contradiction. Hence if $x \in \mathfrak{B}^*$ is such that $\mathfrak{G}_x \cap \mathfrak{U}\mathfrak{G} = \langle 1 \rangle$ and if \mathfrak{P} is a Sylow 2-subgroup of \mathfrak{G} extending one of \mathfrak{G}_x , then either (a) $\mathfrak{P}_x\mathfrak{G} = \mathfrak{P}$ and $w \geq 32$ or (b) $[\mathfrak{P} : \mathfrak{P}_x\mathfrak{G}] = 2$ and $w \geq 64$. In the latter case $\mathfrak{P}_x\mathfrak{G} \trianglelefteq \mathfrak{P}$ so in both cases $\mathfrak{P}_x\mathfrak{G}$ has period ≥ 32 and \mathfrak{P}_x has period ≥ 8 . Note $|\mathfrak{U}_2| = 2$ here by Lemma 2.6.

Let $\mathfrak{J} = \langle J \rangle$ be a cyclic subgroup of \mathfrak{P}_x of order 8 and let $a_0 = \dim_{GF(q)} C_{\mathfrak{B}}(J)$. Since \mathfrak{J} acts faithfully on $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$ and $|\mathfrak{J}| = 8$ we see that J must have one Jordan block of rank at least 5. This implies easily that \mathfrak{J} has at most

$$\frac{2^{10} - 2^9}{8} + \frac{2^9 - 2^7}{4} + \frac{2^7 - 2^6}{2} + 2^6 = 2^8$$

orbits on \mathfrak{B} . Hence by Lemma 1.6

$$\begin{aligned} a_0 + a_1 + \cdots + a_7 &= p^m = 32 \\ a_0^2 + a_1^2 + \cdots + a_7^2 &\leq 2^8. \end{aligned}$$

Thus $a_0 < 2^4 = 16$ and $|C_{\mathfrak{B}}(J)| = q^{a_0} \leq q^{15}$.

Now if \mathfrak{Z} is a subgroup of $\mathfrak{U}\mathfrak{G}$ of order 2 then $|C_{\mathfrak{B}}(\mathfrak{Z})| \leq q^{n/2} = q^{16}$. We have shown that with the above notation

$$\mathfrak{B} = \bigcup_{\mathfrak{J}} C_{\mathfrak{B}}(\mathfrak{J}) \cup \bigcup_{\mathfrak{Z}} C_{\mathfrak{B}}(\mathfrak{Z}).$$

Now \mathfrak{U} is cyclic and central and by Lemma 2.6, $4 \nmid |\mathfrak{U}|$. Hence the number of choices for \mathfrak{Z} is at most $|\mathfrak{G}| = 2^{11}$ and the number of choices for \mathfrak{J} is at most $1/4 |\mathfrak{G}/\mathfrak{U}_2|$. Here \mathfrak{U}_2 is the normal 2-complement of \mathfrak{U} and the $1/4$ factor comes from the fact that \mathfrak{J} has four distinct generators. Since $|\mathfrak{G}/\mathfrak{U}_2| \leq |\mathfrak{G}|$ $|Sp(10, 2)| \leq 2^{66}$, the above union yields

$$q^{32} = |\mathfrak{B}| \leq 2^{66} q^{16}/4 + 2^{11} q^{16}.$$

Putting $q^{15} < q^{16}/2$ in the above we have

$$q^{32} < (2^{63} + 2^{11})q^{16} < 2^{64} q^{16}$$

so $q^{16} < 2^{64}$ and $q < 2^4 = 16$. On the other hand by Lemma 2.4 $2^5 | q^2 - 1$ so $16 | q \pm 1$. This yields $q \geq 17$, a contradiction.

The following partial result will be completed later under the additional assumption of solvability.

LEMMA 2.12. *In the case type II, $m = 4$ we have $q \geq 7$ and $|\mathfrak{G}/\mathfrak{U}\mathfrak{G}| > 10^4$.*

Proof. In the inequalities of the proof of Lemma 2.6 for type II, $m = 4$, we see that a strengthening by a factor of $p^2 = 4$ will yield a contradiction. Suppose $x \in \mathfrak{B}^*$ with $\mathbb{G}_x \cap \mathfrak{A}\mathfrak{C} = \langle 1 \rangle$ and let \mathfrak{P} be a Sylow 2-subgroup of \mathbb{G} extending one of \mathbb{G}_x . Using the same argument as in the preceding lemma we conclude that $\mathfrak{P}_x\mathfrak{C}$ has period ≥ 16 and hence $\mathfrak{P}_x\mathfrak{C}/Z(\mathfrak{C})$ has period ≥ 8 .

Suppose first that \mathfrak{P}_x has a cyclic subgroup $\mathfrak{J} = \langle J \rangle$ of order 8. Then in its action on $\mathfrak{B} = \mathfrak{C}/Z(\mathfrak{C})$, J has a Jordan block of rank at least 5 so \mathfrak{J} has at most

$$\frac{2^8 - 2^7}{8} + \frac{2^7 - 2^5}{4} + \frac{2^5 - 2^4}{2} + 2^4 = 64$$

orbits on \mathfrak{B} . By Lemma 1.6 if $a_0 = \dim C_{\mathfrak{B}}(J)$ then

$$a_0^2 + a_1^2 + \cdots + a_7^2 \leq 64$$

and $a_0 \leq 8$.

Now suppose \mathfrak{P}_x has period 4. Then since $\mathfrak{P}_x\mathfrak{C}/Z(\mathfrak{C})$ has period 8, \mathfrak{P}_x must contain an element J of order 4 with a Jordan block of rank 4. If $\mathfrak{J} = \langle J \rangle$, then \mathfrak{J} has at most

$$\frac{2^8 - 2^6}{4} + \frac{2^6 - 2^5}{2} + 2^5 = 96$$

orbits on \mathfrak{B} . By Lemma 1.6 if $a_0 = \dim C_{\mathfrak{B}}(J)$ then

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= p^m = 16 \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 &\leq 96. \end{aligned}$$

It is easy to see that the possibility $a_0 = 9$ is excluded and hence in both cases $a_0 \leq 8$.

We have

$$\mathfrak{B} = \bigcup C_{\mathfrak{B}}(\mathfrak{J}) \cup \bigcup C_{\mathfrak{B}}(\mathfrak{I})$$

where the subgroups \mathfrak{J} are as above and the subgroups \mathfrak{I} have order 2 and are contained in \mathfrak{C} . This follows since $4 \nmid |\mathfrak{A}|$ by Lemma 2.6. The number of choices for \mathfrak{J} or \mathfrak{I} is clearly at most $|\mathbb{G}/\mathfrak{A}_{2'}|$ where $\mathfrak{A}_{2'}$ is the normal 2-complement of \mathfrak{A} . Since $4 \nmid |\mathfrak{A}|$ we have $|\mathbb{G}/\mathfrak{A}_{2'}| = 2^9 |\mathbb{G}/\mathfrak{A}\mathfrak{C}|$. Therefore the above union yields

$$q^{16} = |\mathfrak{B}| \leq q^8 2^9 |\mathbb{G}/\mathfrak{A}\mathfrak{C}|$$

since $|C_{\mathfrak{B}}(\mathfrak{J})|$ and $|C_{\mathfrak{B}}(\mathfrak{I})|$ are both at most q^8 . Thus $|\mathbb{G}/\mathfrak{A}\mathfrak{C}| \geq q^8/2^9$. By Lemma 2.4, $2^4 |q^2 - 1|$ so $q \geq 7$. This yields

$$|\mathbb{G}/\mathfrak{A}\mathfrak{C}| \geq 7^8/2^9 = (2401)^2/2^9 > 10^4$$

and the result follows.

We now temporarily drop the assumptions stated at the beginning of this section and prove the first of our three theorems.

Proof of Theorem A. Let \mathfrak{G} be a linear group acting on vector space \mathfrak{V} of order q^n and suppose that \mathfrak{G} acts half-transitively but not semiregularly on \mathfrak{V}^* . Let $\mathfrak{P} = O_p(\mathfrak{G})$ be the maximal normal p -subgroup of \mathfrak{G} . By assumption \mathfrak{G} is primitive so \mathfrak{P} is of symplectic type. Suppose first that $p > 2$. If \mathfrak{P} is not cyclic, then \mathfrak{P} contains a characteristic subgroup \mathfrak{C} of type $E(p, m)$. By the Reduction Lemma (Lemma 1.8) and Lemmas 2.6, 2.7 and 2.8 we have a contradiction.

Now let $p = 2$ so that $\Phi(\mathfrak{P})$ is cyclic. Suppose $[\mathfrak{P} : \Phi(\mathfrak{P})] > 2^8$. Then \mathfrak{P} has a characteristic subgroup \mathfrak{C} of type $E(2, m)$ with $m > 3$. Thus by the Reduction Lemma and Lemmas 2.6 through 2.11 we see that $m = 4$ and $|Z(\mathfrak{C})| = 2$. But then $|\Phi(\mathfrak{P})| = 2$ so $\mathfrak{P} = \mathfrak{C}$ and $[\mathfrak{P} : \Phi(\mathfrak{P})] \leq 2^8$ here also. This completes the proof.

3. Solvable cases, $m = 1$. We have seen in the preceding section that if \mathfrak{C} is a group of type $E(p, m)$ normal in a half-transitive linear group \mathfrak{G} , then $p = 2$ and $m \leq 4$. We will consider these cases in the next few sections under the additional assumption that \mathfrak{G} is solvable.

For convenience we restate Lemmas 1.3 and 1.4 of [5].

LEMMA 3.1. *Suppose \mathfrak{G} is an irreducible linear group of degree n over $GF(q)$ and $\mathfrak{A} = \langle \mathfrak{A} \rangle$ is a cyclic normal subgroup all of whose irreducible constituents are similar. Let ζ be an eigenvalue of A with $GF(q)(\zeta) = GF(q^r)$ and $n/r = k$. Let p be a prime and suppose that for all vectors x , $p \mid |\mathfrak{G}_x|$. Consider those subgroups $\mathfrak{P}/\mathfrak{A}$ of $\mathfrak{G}/\mathfrak{A}$ of order p for which there exists an $x \neq 0$ with $\mathfrak{P} \cap \mathfrak{G}_x \neq \langle 1 \rangle$. If λ_1 of the \mathfrak{P} are contained in $C_{\mathfrak{G}}(\mathfrak{A})$ and λ_2 are not, then*

- (i) $\frac{q^{kr} - 1}{q^r - 1} \leq \lambda_1 \left\{ 1 + \frac{q^{r(k-1)} - 1}{q^r - 1} \right\} + \lambda_2 \left\{ \frac{q^{rk/p} - 1}{q^{r/p} - 1} \right\}$
- (ii) $q^r + 1 \leq 2\lambda_1 + \lambda_2(q^{r/p} + 1)$ for $k = 2$
- (iii) $q^r < 2(\lambda_1 + \lambda_2)$ for $k > 2$.

This is a very coarse statement which we will have to strengthen at times. The following assumptions hold throughout the remainder of this section.

ASSUMPTIONS. Group \mathfrak{G} acts faithfully on vector space \mathfrak{V} of order q^n and half-transitively but not semiregularly on \mathfrak{V}^* . \mathfrak{C} is a group of type $E(2, 1)$ which is normal in \mathfrak{G} and acts irreducibly on \mathfrak{V} .

Note that we do not assume that \mathfrak{G} is primitive here. The

reason for this, is that part (v) of the Reduction Lemma does not guarantee primitivity in this case.

LEMMA 3.2. *Let $\mathfrak{G} \cong \mathfrak{Q}$ (that is, $\mathfrak{G} = \text{iso } I$). Then $q^n = 3^2, 5^2, 7^2, 11^2$ or 17^2 .*

Proof. Clearly $q^n = q^2$ and hence $C_{\mathfrak{G}}(\mathfrak{G})$ consists of scalar matrices so $C_{\mathfrak{G}}(\mathfrak{G}) = Z(\mathfrak{G})$. Note that $\text{Aut } \mathfrak{G} \cong \text{Sym}_4$, the symmetric group of degree 4.

Suppose first that $3 \nmid |\mathfrak{G}/Z(\mathfrak{G})|$. Then $|\mathfrak{G}/Z(\mathfrak{G})| = 4$ or 8 and hence \mathfrak{G} is nilpotent. Thus $\mathfrak{G}_2 = O_2(\mathfrak{G})$ is half-transitive. Since $O_2(\mathfrak{G}) \subseteq Z(\mathfrak{G})$ acts semiregularly, we conclude that \mathfrak{G}_2 is not semi-regular. Hence $\mathfrak{G}_2 > \mathfrak{G}$ and since $[\mathfrak{G}_2 : Z(\mathfrak{G}_2)] = 4$ or 8 we have by Theorem II of [4], $q^n = 3^2, 5^2$ or 7^2 .

We assume now that $3 \mid |\mathfrak{G}/Z(\mathfrak{G})|$. We consider the possibility $3 \mid |\mathfrak{G}_x|$ first. If \mathfrak{Q} is a subgroup of \mathfrak{G} of order 3 fixing a vector x , then $|C_{\mathfrak{G}}(\mathfrak{Q})| = q$ clearly. Also either $q = 3$ or by complete reducibility $3 \mid q - 1$. Now $\mathfrak{G}/Z(\mathfrak{G})$ has at most 4 subgroups of order 3 and since $Z(\mathfrak{G})$ is cyclic, we see that \mathfrak{G} contains at most $4 \cdot 3 = 12$ subgroups of order 3 not contained in $Z(\mathfrak{G})$. From $\mathfrak{B} = \bigcup C_{\mathfrak{G}}(\mathfrak{Q})$ we obtain easily

$$q^2 - 1 = |\mathfrak{B}^{\#}| \leq 12(q - 1)$$

so $q + 1 \leq 12$. Since either $q = 3$ or $3 \mid q - 1$ we have $q = 3$ or 7 here.

We now assume that $3 \nmid |\mathfrak{G}_x|$. If $\mathfrak{G}_x \cap Z(\mathfrak{G})\mathfrak{G} \neq \langle 1 \rangle$ for all $x \in \mathfrak{B}^{\#}$, then by Lemma 1.5 $q^n = 3^2$ or 5^2 . Thus we can suppose that some $x \in \mathfrak{B}^{\#}$, $\mathfrak{G}_x \cap Z(\mathfrak{G})\mathfrak{G} = \langle 1 \rangle$. This yields $|\mathfrak{G}_x| = 2$ and by Lemma 1.9, $I(\mathfrak{G}) = q + 1$. We have actually shown above that $|\mathfrak{G}/Z(\mathfrak{G})|$ is divisible by $3 \cdot 8$ so $\mathfrak{G}/Z(\mathfrak{G}) \cong \text{Sym}_4$ and this group has two conjugacy classes of involutions, \mathfrak{C}_1 of size 3 and \mathfrak{C}_2 of size 6. If $\bar{T} \in \mathfrak{G}/Z(\mathfrak{G})$ is an involution then since $Z(\mathfrak{G})$ is cyclic of even order and central, the coset corresponding to \bar{T} will contain either 0 or 2 noncentral involutions of \mathfrak{G} and this number is the same for all conjugates of \bar{T} . Thus we have

$$q + 1 = I(\mathfrak{G}) = \delta_1 \cdot 2 \cdot 3 + \delta_2 \cdot 2 \cdot 6$$

where $\delta_1, \delta_2 = 0$ or 1. Moreover since for some $x \in \mathfrak{B}^{\#}$, $\mathfrak{G}_x \cap Z(\mathfrak{G})\mathfrak{G} = \langle 1 \rangle$ we have $\delta_2 = 1$. Thus $q + 1 = 6\delta_1 + 12$ and $q = 11$ or 17. This completes the proof.

LEMMA 3.3. *Let $\mathfrak{G} \cong \mathfrak{D}$ (that is, $\mathfrak{G} = \text{iso } II$). Then $q^n = 3^2, 5^2$ or 7^2 .*

Proof. Clearly $q^n = q^2$ so $C_{\mathfrak{G}}(\mathfrak{G}) = Z(\mathfrak{G})$ consists of scalar matrices.

Now $|\text{Aut } \mathfrak{G}| = 8$ so $[\mathfrak{G} : Z(\mathfrak{G})] = 4$ or 8 and hence \mathfrak{G} is nilpotent. Then $\mathfrak{G}_2 = O_2(\mathfrak{G})$ is half-transitive but not semiregular and $[\mathfrak{G}_2 : Z(\mathfrak{G}_2)] = 4$ or 8. By Theorem II of [4], $q^n = q^2 = 3^2, 5^2$ or 7^2 .

LEMMA 3.4. *Let $\mathfrak{G} \cong 3\mathfrak{Q}$ (that is, $\mathfrak{G} = \text{iso III}$). Then $q^n = 5^2, 17^2$ or \mathfrak{G} is imprimitive and $q^n = 3^4$.*

Proof. Here $q^n = q^2$ if $q \equiv 1 \pmod{4}$ and $q^n = q^4$ if $q \equiv -1 \pmod{4}$. Say $q^n = q^{2r}$.

Suppose first that \mathfrak{G} is imprimitive. Here we can apply Theorem 1.1. Note that if $q^r - 1$ is not a power of 2 then $O_2(\mathcal{S}_0(q^r))$ is abelian. Hence by Theorem 1.1 either $q^n = 3^4$ or $\mathfrak{G} = \mathcal{S}_0(q)$ for Fermat prime q . Here $q \equiv 1 \pmod{4}$ so $q \geq 5$. Let \mathfrak{B} be the diagonalized subgroup of \mathfrak{G} of index 2 so \mathfrak{B} is abelian. Then $\mathfrak{G} = \mathfrak{B}\mathfrak{C}$ and $\mathfrak{G}' = (\mathfrak{B}, \mathfrak{C}) \subseteq \mathfrak{C}$. Since \mathfrak{G}' is cyclic of order $(q-1)/2$ we have $(q-1)/2 \leq 4$ so $q \leq 9$ and hence since q is a Fermat prime $q = 5$ and $q^n = 5^2$.

Now we assume that \mathfrak{G} is primitive and we use the notation of Lemma 2.5. Then $[\mathfrak{G} : \mathfrak{H}] = 1$ or 2 where $\mathfrak{H} = C_{\mathfrak{G}}(Z(\mathfrak{G}))$ and $\mathfrak{H}/\mathfrak{H}\mathfrak{C} \subseteq Sp(2, 2) = SL(2, 2)$, a group of order 6. Now \mathfrak{G} has precisely 3 abelian subgroups of order 8 and these are not cyclic. Since \mathfrak{G} is primitive none of these groups is normal. Hence \mathfrak{G} permutes these transitively so $3 \mid |\mathfrak{H}/\mathfrak{H}\mathfrak{C}|$. Now $\mathfrak{G}/\mathfrak{H}\mathfrak{C}$ also acts on $\mathfrak{G}/\mathfrak{C}'$ and this action is clearly faithful on $\mathfrak{H}/\mathfrak{H}\mathfrak{C}$. If $\bar{\mathfrak{F}} = \mathfrak{F}/\mathfrak{H}\mathfrak{C}$ is the normal 3-subgroup of $\mathfrak{H}/\mathfrak{H}\mathfrak{C}$ then $\bar{\mathfrak{F}}$ centralizes $Z(\mathfrak{G})/\mathfrak{C}'$ and acts faithfully on the commutator $\mathfrak{G}_0/\mathfrak{C}'$, a 2-dimensional complement. Clearly $\mathfrak{G}_0 \cong \mathfrak{Q}$ and $\mathfrak{G}_0 \triangle \mathfrak{G}$. If $n = 2$, then by Lemma 3.2 and the fact that $q \equiv 1 \pmod{4}$ we have $q^n = 5^2$ or 17^2 .

Let $n = 4$ so $q \equiv -1 \pmod{4}$. $\mathfrak{G}/\mathfrak{H}$ acts on \mathfrak{G}_0 and the kernel acts faithfully on $Z(\mathfrak{G})$. Thus we see that either $\mathfrak{G}/\mathfrak{H} \subseteq \text{Aut } \mathfrak{G}_0 = \text{Sym}_4$ or $\mathfrak{G}/\mathfrak{H} \subseteq \mathfrak{H}/\mathfrak{H}\mathfrak{C} \times \bar{\mathfrak{F}}/\bar{\mathfrak{F}}\mathfrak{H}\mathfrak{C} \subseteq \text{Sym } 4 \times \bar{\mathfrak{F}}$ where $|\bar{\mathfrak{F}}| = |\bar{\mathfrak{F}}\mathfrak{H}\mathfrak{C}| = 2$. We apply Lemma 3.1 with $p = 2$. We have clearly $\lambda_1 \leq 9$, $\lambda_2 \leq 10$ and since $r = 2$, $k = 2$, $n = 4$ we obtain

$$q^2 + 1 \leq 18 + 10(q + 1)$$

or $q(q-10) \leq 27$ so $q < 13$. Since $q \equiv 3 \pmod{4}$ we have $q \equiv 3, 7$ or 11 . Suppose $3 \mid |\mathfrak{G}_x|$. Let T be a noncentral involution of \mathfrak{G} . By Lemma 1.5 there exists a point $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x = \langle T \rangle$. Let \mathfrak{L} be a subgroup of \mathfrak{G}_x of order 3. Then $\mathfrak{L} \cap \mathfrak{H}\mathfrak{C} = \langle 1 \rangle$, $\mathfrak{L} \subseteq \mathfrak{H}$ and \mathfrak{L} normalizes $\mathfrak{G}_x \cap \mathfrak{C} = \mathfrak{G}_x$, a contradiction since \mathfrak{L} acts irreducibly on $\mathfrak{G}/Z(\mathfrak{G})$. Hence $3 \nmid |\mathfrak{G}_x|$ and since $3 \mid |\mathfrak{G}|$ we conclude that $q \neq 3$.

Let $q = 7$ or 11 . By Lemma 1.5 there exists a point $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{H}\mathfrak{C} = \langle 1 \rangle$. Since $3 \nmid |\mathfrak{G}_x|$ we see that $|\mathfrak{G}_x| = 2$ or 4. Suppose $|\mathfrak{G}_x| = 4$. Then certainly $2 \mid |\mathfrak{H}_x|$ for all $x \in \mathfrak{B}^*$ and Lemma 3.1

applies to \mathfrak{G} . Here $\lambda_1 \leq 9$, $\lambda_2 = 0$, $r = 2$, $n = 4$, $k = 2$ so

$$q^2 + 1 \leq 2 \cdot 9 + 0,$$

a contradiction. Thus $|\mathfrak{G}_x| = 2$ and by Lemma 1.9, $I(\mathfrak{G}) = q^2 + 1$. Let \mathfrak{Z} be a Sylow 3-subgroup of \mathfrak{G} . Then \mathfrak{Z} permutes by conjugation the noncentral involutions of \mathfrak{G} . Since $3 \nmid (q^2 + 1)$, \mathfrak{Z} must centralize such an involution. Now subgroups of Sym_4 of order 3 are self-centralizing so this implies that $\mathfrak{G}/\mathfrak{A} \not\subseteq \text{Sym}_4$. Hence $\mathfrak{G}/\mathfrak{A} \cong \mathfrak{S}/\mathfrak{A} \times \mathfrak{J}/\mathfrak{A}$ where $\mathfrak{S}/\mathfrak{A} \cong \text{Sym}_4$ and $|\mathfrak{J}/\mathfrak{A}| = 2$. Clearly $\mathfrak{S}/\mathfrak{A} \cong \text{Alt}_4$ and if $\mathfrak{S}/\mathfrak{A} \cong \text{Alt}_4$ then in the notation of Lemma 3.1 with $p = 2$, $\lambda_1 \leq 3$, $\lambda_2 \leq 4$ and

$$(q^2 + 1) \leq 2\lambda_1 + (q + 1)\lambda_2 \leq 6 + 4(q + 1)$$

a contradiction for $q = 7, 11$. Hence $\mathfrak{S}/\mathfrak{A} \cong \text{Sym}_4$ and $\mathfrak{G}/\mathfrak{A}$ has five classes \mathfrak{C}_i of involutions. These satisfy $\mathfrak{C}_1, \mathfrak{C}_2 \subseteq \mathfrak{S}/\mathfrak{A}$ with $|\mathfrak{C}_1| = 3$, $|\mathfrak{C}_2| = 6$ and $\mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5 \not\subseteq \mathfrak{S}/\mathfrak{A}$ with $|\mathfrak{C}_3| = 1$, $|\mathfrak{C}_4| = 3$, $|\mathfrak{C}_5| = 6$.

Let \bar{T} be an involution of $\mathfrak{G}/\mathfrak{A}$. If the coset of \bar{T} contains α involutions, then the same is true for all conjugates of \bar{T} . If $\bar{T} \in \mathfrak{S}/\mathfrak{A}$ then certainly $\alpha = 0$ or 2 . If $\bar{T} \notin \mathfrak{S}/\mathfrak{A}$, then by Lemma 1.1 of [5] \bar{T} acts on \mathfrak{A} like a field automorphism of $GF(q^2)$ of order 2 (that is, the map $x \rightarrow x^q$). Suppose the coset contains an involution T . Then for $B \in \mathfrak{A}$, BT is an involution if and only if $B^q = B^T = B^{-1}$. Hence $\alpha = 0$ or the number N of elements of \mathfrak{A} of order dividing $q + 1$. Note that since $|Z(\mathfrak{G})| = 4$ we have $N = 4$ or 8 for $q = 7$ and $N = 4$ or 12 for $q = 11$. Now if $\delta_i = 1$ or 0 according to whether the coset of $\bar{T} \in \mathfrak{C}_i$ contains an involution of \mathfrak{G} then we obtain

$$q^2 + 1 = I(\mathfrak{G}) = 6\delta_1 + 12\delta_2 + N(\delta_3 + 3\delta_4 + 6\delta_5).$$

Considering this modulo 3 we have

$$2 \equiv q^2 + 1 \equiv N\delta_3 \pmod{3}.$$

This shows that $q \neq 11$. If $q = 7$ then $N = 8$ so $8 \mid |\mathfrak{A}|$ and $\delta_3 = 1$. Furthermore $\delta_5 = 0$ and then $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$.

Since $\delta_2 = 1$ we can find an involution $T \in \mathfrak{S}$ corresponding to a transposition in $\mathfrak{S}/\mathfrak{A} \cong \text{Sym}_4$. Now T normalized $\mathfrak{G}_0 \cong \Omega$ as mentioned before and T does not fix $\mathfrak{G}_0/\mathfrak{G}'_0$ since T does not fix $\mathfrak{G}/Z(\mathfrak{G})$. Thus $\langle \mathfrak{G}_0, T \rangle$ is a maximal class group of order 16 and hence this group has a cyclic subgroup \mathfrak{B} of order 8. The group $\mathfrak{A}_2\mathfrak{B}$ is abelian and has period $|\mathfrak{A}_2|$ since $\mathfrak{B} \subseteq \mathfrak{S}$ and $|\mathfrak{A}_2| \geq |\mathfrak{B}|$. Also $|\mathfrak{B} \cap \mathfrak{A}_2| = 2$ so $|\mathfrak{A}_2\mathfrak{B}| = 4|\mathfrak{A}_2|$. Let $\mathfrak{U} \subseteq \mathfrak{B}$ be an irreducible $\mathfrak{A}_2\mathfrak{B}$ -submodule and let $\mathfrak{K} \subseteq \mathfrak{A}_2\mathfrak{B}$ be the kernel. Then $\mathfrak{A}_2\mathfrak{B}/\mathfrak{K}$ is cyclic so $|\mathfrak{A}_2\mathfrak{B}/\mathfrak{K}| \leq |\mathfrak{A}_2|$ and

hence $|\mathfrak{R}| \geq 4$. If $x \in \mathfrak{U}^*$, then $\mathfrak{G}_x \cong \mathfrak{R}$ and $|\mathfrak{G}_x| \geq 4$, a contradiction. This completes the proof.

EXAMPLES. The examples with $q^n = 3^2, 5^2, 7^2$ and 11^2 can occur as transitive groups and these are given in [3]. We consider the case $q^n = 17^2$. Let $SL(2, 17)^*$ denote the subgroup of $GL(2, 17)$ consisting of those matrices with determinant ± 1 . Let $\mathfrak{G} = \mathfrak{Q}\mathfrak{W}$ where \mathfrak{Q} is the quaternion group of order 8, $\mathfrak{Q} \triangle \mathfrak{G}$ and $\mathfrak{W} \cong \text{Sym}_3$ acts faithfully on $\mathfrak{Q}/\mathfrak{Q}'$. Clearly $\mathfrak{G}' = \mathfrak{Q}\mathfrak{W}' \cong SL(2, 3)$. This group has a unique faithful irreducible rational character of degree 2. Hence \mathfrak{G} has a faithful character χ of degree 2 with $\chi|_{\mathfrak{G}'}$ rational. Now all elements of $\mathfrak{G} - \mathfrak{G}'$ are 2-elements and a Sylow 2-subgroup of \mathfrak{G} has period 8. Thus $Q(\chi) \subseteq Q(\varepsilon)$ where ε is a primitive 8th root of unity. Since $8 \nmid |GF(17)^*|$, this representation of \mathfrak{G} is realizable over $GF(17)$ and hence we can assume $\mathfrak{G} \subseteq GL(2, 17)$. All subgroups of \mathfrak{G} of order 3 are contained in $SL(2, 17)$ since $3 \nmid |GF(17)^*|$ so $\mathfrak{G}' \subseteq SL(2, 17)$ and $\mathfrak{G} \subseteq SL(2, 17)^*$. Let $i = \sqrt{-1} \in GF(17)$ and let $\mathfrak{Z} = \left\langle \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\rangle$. Then \mathfrak{Z} is cyclic of order 4, $\mathfrak{Z} \subseteq SL(2, 17)^*$ and \mathfrak{Z} is central in $GL(2, 17)$. Set $\mathfrak{G} = \mathfrak{Z}\mathfrak{G}$ so $\mathfrak{G} \subseteq SL(2, 17)^*$.

We show first that \mathfrak{G} has precisely $17 + 1 = 18$ noncentral involutions. Now $|\mathfrak{Z}| = 4$ and $\mathfrak{G}/\mathfrak{Z} \cong \text{Sym}_4$. This quotient group has two classes of involutions $\mathfrak{C}_1, \mathfrak{C}_2$ with $|\mathfrak{C}_1| = 3$, $|\mathfrak{C}_2| = 6$. If $\bar{T} \in \mathfrak{C}_i$ and the coset of \bar{T} contains an involution of \mathfrak{G} , then the same is true for all conjugates of \bar{T} . Moreover the coset would then clearly contain precisely two such involutions. Thus if $\delta_i = 0, 1$ has the obvious meaning, then

$$I(\mathfrak{G}) = 2\delta_1 |\mathfrak{C}_1| + 2\delta_2 |\mathfrak{C}_2| = 6\delta_1 + 12\delta_2.$$

Let $W \subseteq \mathfrak{W}$ have order 2. Then $\bar{W} \in \mathfrak{C}_2$ so $\delta_2 = 1$. Let $Q \in \mathfrak{Q}$ have order 4 and let $\mathfrak{Z} = \langle Z \rangle$. Then QZ has order 2 and $\overline{QZ} \in \mathfrak{C}_1$. Hence $\delta_1 = 1$ and $I(\mathfrak{G}) = 18$.

Let \mathfrak{B} be a 2-dimensional $GF(17)$ -vector space and let $x \in \mathfrak{B}^*$. Since $|\mathfrak{G}|$ is prime to 17 we can write $\mathfrak{G}_x \subseteq \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in GF(17)^* \right\}$ by taking a suitable basis. Now $\mathfrak{G} \subseteq SL(2, 17)^*$ and $\det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a$ so we see that $|\mathfrak{G}_x| = 1$ or 2. If T is a noncentral involution of \mathfrak{G} , then $\mathfrak{B} > C_{\mathfrak{B}}(T) > \{0\}$ and hence $|C_{\mathfrak{B}}(T)^*| = 17 - 1$. By the above the centralizer spaces for the involutions are disjoint. Hence

$$\begin{aligned} |\bigcup_x C_{\mathfrak{B}}(T)^*| &= I(\mathfrak{G})(17 - 1) = (17 + 1)(17 - 1) \\ &= 17^2 - 1 = |\mathfrak{B}^*|. \end{aligned}$$

Thus $\bigcup_x C_{\mathfrak{B}}(T) = \mathfrak{B}$ and so for all $x \in \mathfrak{B}^*$, $|\mathfrak{G}_x| \geq 2$. This yields $|\mathfrak{G}_x| = 2$ and \mathfrak{G} is half-transitive but not semiregular. Finally

$\mathfrak{G} \not\subseteq \mathcal{T}(17^2)$, the semilinear transformations, since \mathfrak{G} does not have a cyclic subgroup of index 2.

We close this section with some additional information about the degree 17^2 group.

LEMMA 3.5. *If $q^n = 17^2$, then $|\mathfrak{G}| = 96$.*

Proof. These groups occur in Lemmas 3.2 and 3.4. However the latter case was deduced from the former so we can assume \mathfrak{G} is as described in the proof of Lemma 3.2. We showed there that $|\mathfrak{G}_x| = 2$, $\mathfrak{G}/Z(\mathfrak{G}) \cong \text{Sym}_4$ and $\delta_1 = \delta_2 = 1$. The latter says that if \bar{T} is any involution of $\mathfrak{G}/Z(\mathfrak{G})$, then its coset contains an involution of \mathfrak{G} .

Now $\mathfrak{A} = Z(\mathfrak{G})$ has order dividing $|GF(17)^*| = 16$. If $|\mathfrak{A}| = 2$, then an involution T in the four groups of Sym_4 would not have an involution of \mathfrak{G} in its coset. We assume that $|\mathfrak{A}| \geq 8$ and derive a contradiction. Let T be an involution of \mathfrak{G} corresponding to a transposition of Sym_4 . Then $\langle \mathfrak{G}, T \rangle$ is a maximal class group of order 16 and this group has a cyclic subgroup \mathfrak{B} of order 8. We see that $|\mathfrak{A} \cap \mathfrak{B}| = 2$ so $|\mathfrak{A}\mathfrak{B}| = 4|\mathfrak{A}|$ and $\mathfrak{A}\mathfrak{B}$ has period $|\mathfrak{A}|$ since $|\mathfrak{A}| \geq |\mathfrak{B}| = 8$. As in the last paragraph of the proof of the preceding lemma, this implies that $|\mathfrak{G}_x| \geq 4$, a contradiction. Thus $|\mathfrak{A}| = 4$ and since $\mathfrak{G}/\mathfrak{A} \cong \text{Sym}_4$ we have $|\mathfrak{G}| = 4 \cdot 24 = 96$. This completes the proof of the lemma.

4. Solvable case, $m = 2$. In this and the next section the following assumptions hold.

ASSUMPTIONS. Group \mathfrak{G} acts faithfully on vector space \mathfrak{V} of order q^n and half-transitively but not semiregularly on \mathfrak{V}^* . \mathfrak{G} is a group of type $E(2, m)$ with $\mathfrak{G} \triangle \mathfrak{G}$. In addition \mathfrak{G} acts irreducibly on \mathfrak{V} , \mathfrak{G} is primitive as a linear group and \mathfrak{G} is solvable.

We will use the notation of Lemma 2.5. Moreover set $\tilde{\mathfrak{G}} = \mathfrak{G}/\mathfrak{A}\mathfrak{G}$ so that $\tilde{\mathfrak{G}}$ is a solvable subgroup of $Sp(2m, 2)$. We let $\tilde{\mathfrak{F}} = F(\tilde{\mathfrak{G}})$, the Fitting subgroup of $\tilde{\mathfrak{G}}$, and for each prime p we let $\tilde{\mathfrak{F}}_p$ be the normal Sylow p -subgroup of $\tilde{\mathfrak{F}}$. By Fitting's theorem, $C_{\tilde{\mathfrak{G}}}(\tilde{\mathfrak{F}}) \subseteq \tilde{\mathfrak{F}}$. Recall the possible isomorphism classes for \mathfrak{G} namely: iso I if $\mathfrak{G} \cong \Omega\Omega \cdots \Omega$, iso II if $\mathfrak{G} \cong \mathfrak{D}\Omega\Omega \cdots \Omega$ and iso III if $\mathfrak{G} \cong \mathfrak{B}\Omega\Omega\Omega \cdots \Omega$.

LEMMA 4.1. *Suppose $\tilde{\mathfrak{F}}_2 \neq \langle 1 \rangle$. Then $|\tilde{\mathfrak{F}}_2| = 2$, $\mathfrak{G} = \text{iso I or II}$ and \mathfrak{G} has a normal subgroup \mathfrak{G}_0 of type $E(2, m - 1)$ with $\mathfrak{G}_0 = \text{iso III}$.*

Proof. Let \mathfrak{C} be the complete inverse image of $\tilde{\mathfrak{F}}_2$ in \mathfrak{G} so $\mathfrak{C}/\mathfrak{A}\mathfrak{G} = \tilde{\mathfrak{F}}_2$. Then $\mathfrak{C}/\mathfrak{A}$ is a 2-group and since \mathfrak{A} is central in \mathfrak{G} , \mathfrak{C}

is nilpotent. If \mathfrak{S}_2 is the normal Sylow 2-subgroup of \mathfrak{S} , then $\mathfrak{S}_2 \cong \mathfrak{G}$ and $\mathfrak{S}_2 \triangleleft \mathfrak{G}$. Since \mathfrak{G} is primitive, \mathfrak{S}_2 is of symplectic type. Suppose $4 \mid |\mathfrak{U}_2|$. Then since \mathfrak{U}_2 is central in \mathfrak{S}_2 , \mathfrak{S}_2 has a center of order at least 4 and hence \mathfrak{S}_2 is the central product of $Z(\mathfrak{S}_2)$ with a number of nonabelian groups of order 8. Note that since $\mathfrak{G} \subseteq \mathfrak{S}_2$, $Z(\mathfrak{S}_2) \subseteq C_{\mathfrak{G}}(\mathfrak{G}) = \mathfrak{U}$ so that $Z(\mathfrak{S}_2) = \mathfrak{U}_2$. Since $|\tilde{\mathfrak{G}}_2| > 1$, $\mathfrak{S}_2 \neq \mathfrak{U}_2\mathfrak{G}$ and thus $\mathfrak{S}_2 \supseteq \mathfrak{U}_2\mathfrak{B}$ where $|\mathfrak{B}| = 8$, $\mathfrak{B} \not\subseteq \mathfrak{U}_2$ and $\mathfrak{B} \subseteq C_{\mathfrak{G}}(\mathfrak{G})$, a contradiction. Thus $|\mathfrak{U}_2| = 2$ and hence $|Z(\mathfrak{G})| = 2$. This implies that $\dim \mathfrak{B} = 2^m$ and since \mathfrak{S}_2 acts faithfully on \mathfrak{B} , \mathfrak{S}_2 has at most m nonabelian factors. Since $|Z(\mathfrak{S}_2)| = |\mathfrak{U}_2| = 2$ we see that $\mathfrak{S}_2 = \mathfrak{B}_0\mathfrak{B}_1 \cdots \mathfrak{B}_{m-1}$, a central product of nonabelian groups with $|\mathfrak{B}_i| = 8$ if $i > 0$ and \mathfrak{B}_0 a maximal class group. Now $\mathfrak{B}_0 \cap \mathfrak{G}$ is a 2-generator subgroup of \mathfrak{G} so $|\mathfrak{B}_0 \cap \mathfrak{G}| \leq 8$. Thus $|\mathfrak{B}_0\mathfrak{G}| \geq |\mathfrak{B}_0| \cdot |\mathfrak{G}|/8 = |\mathfrak{B}_0| \cdot 2^{2(m-1)} = |\mathfrak{S}_2|$. Hence we have equality throughout and $|\mathfrak{B}_0 \cap \mathfrak{G}| = 8$. Now $\mathfrak{B}_0 \cap \mathfrak{G} \triangleleft \mathfrak{B}_0$ and $\mathfrak{B}_0 \cap \mathfrak{G}$ is noncyclic. As is well known this implies that $|\mathfrak{B}_0 : \mathfrak{B}_0 \cap \mathfrak{G}| \leq 2$ so $|\mathfrak{B}_0| \leq 16$ and $|\mathfrak{S}_2 : \mathfrak{G}| \leq 2$. If $|\tilde{\mathfrak{G}}_2| \geq 1$, then $|\tilde{\mathfrak{G}}_2| = 2$. Finally $\phi(\mathfrak{S}_2)$ is cyclic of order 4 and from $\mathfrak{S}_2 = \mathfrak{B}_0\mathfrak{G}$ we see that $\mathfrak{G}_0 = C_{\mathfrak{G}}(\phi(\mathfrak{S}_2))$ has the appropriate properties. Thus the result follows.

We assume throughout the remainder of this section that $m = 2$. Since $\tilde{\mathfrak{G}} \subseteq Sp(4, 2)$ here, we make some comments about this latter group. Suppose $Sp(4, 2)$ acts on symplectic space \mathfrak{W} . If \mathfrak{U} is an isotropic subspace of \mathfrak{W} of dimension 2, then the symplectic form restricted to \mathfrak{U} is trivial. We see easily that \mathfrak{W} contains 15 such subspaces. Note that $|Sp(4, 2)| = 2^4 \cdot 3^2 \cdot 5$.

Let $\tilde{\mathfrak{K}}$ be a Sylow 3-subgroup of $Sp(4, 2)$. Then $\tilde{\mathfrak{K}}$ is abelian of type $(3, 3)$ and contains the four subgroups $\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2, \tilde{\mathfrak{X}}_3, \tilde{\mathfrak{X}}_4$ of order 3. We can take (see [10]) the following concrete realization for $\tilde{\mathfrak{K}}$. Write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$, a direct sum of two nonisotropic 2-dimensional subspaces and then let $\tilde{\mathfrak{X}}_1$ centralize \mathfrak{W}_2 and act irreducibly on \mathfrak{W}_1 and $\tilde{\mathfrak{X}}_2$ centralize \mathfrak{W}_1 and act irreducibly on \mathfrak{W}_2 .

Let $\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}_1$ or $\tilde{\mathfrak{X}}_2$. Then $\mathfrak{W} = C_{\mathfrak{W}}(\tilde{\mathfrak{X}}) \oplus (\mathfrak{W}, \tilde{\mathfrak{X}})$ a direct sum of 2-dimensional subspaces. Let \mathfrak{U} be a 2-dimensional $\tilde{\mathfrak{X}}$ -subspace of \mathfrak{W} . If $\mathfrak{U} \cap (\mathfrak{W}, \tilde{\mathfrak{X}}) = \{0\}$, then certainly $\mathfrak{U} \subseteq C_{\mathfrak{W}}(\tilde{\mathfrak{X}})$ so $\mathfrak{U} = C_{\mathfrak{W}}(\tilde{\mathfrak{X}})$. If $\mathfrak{U} \cap (\mathfrak{W}, \tilde{\mathfrak{X}}) \neq \{0\}$, then since $\tilde{\mathfrak{X}}$ acts irreducibly on $(\mathfrak{W}, \tilde{\mathfrak{X}})$ we have $\mathfrak{U} \supseteq (\mathfrak{W}, \tilde{\mathfrak{X}})$ so $\mathfrak{U} = (\mathfrak{W}, \tilde{\mathfrak{X}})$. Thus $\mathfrak{U} = \mathfrak{W}_1$ or \mathfrak{W}_2 . In particular $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ do not normalize a 2-dimensional isotropic subspace of \mathfrak{W} . If \mathfrak{U} is a 1-dimensional $\tilde{\mathfrak{X}}$ -subspace, then certainly $\mathfrak{U} \subseteq C_{\mathfrak{W}}(\tilde{\mathfrak{X}})$ so $\mathfrak{U} \subseteq \mathfrak{W}_1$ or \mathfrak{W}_2 .

Now let $\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}_3$ or $\tilde{\mathfrak{X}}_4$. Then $\tilde{\mathfrak{X}}$ acts irreducibly on both \mathfrak{W}_1 and \mathfrak{W}_2 so $\tilde{\mathfrak{X}}$ has no 1-dimensional invariant subspace. Let \mathfrak{U} be a 2-dimensional $\tilde{\mathfrak{X}}$ -invariant subspace. If $\mathfrak{U} = \mathfrak{W}_1$ or \mathfrak{W}_2 , then \mathfrak{U} is nonisotropic.

Suppose $\mathfrak{U} \neq \mathfrak{W}_1$ or \mathfrak{W}_2 and $w_1 + w_2 \in \mathfrak{U}$ with $w_i \in \mathfrak{W}_i$. Clearly $w_1, w_2 \neq 0$. It is now easy to see that we get precisely three subspaces \mathfrak{U} and since \mathfrak{W}_1 and \mathfrak{W}_2 are orthogonal each such \mathfrak{U} is isotropic. Thus $\bar{\mathfrak{G}}$ normalizes two nonisotropic 2-dimensional subspaces and three isotropic ones.

If $\bar{\mathfrak{F}}$ is a subgroup of $Sp(4, 2)$ of order 5, then $\bar{\mathfrak{F}}$ acts irreducibly on \mathfrak{W} . Then $|C(\bar{\mathfrak{F}})| \mid 2^4 - 1$ so $|C(\bar{\mathfrak{F}})| = 5$ or 15. In the latter case let $\bar{\mathfrak{E}}$ be a subgroup of order 3 centralizing $\bar{\mathfrak{F}}$. Then $\bar{\mathfrak{F}}$ permutes the two 2-dimensional nonisotropic subspaces normalized by $\bar{\mathfrak{E}}$ and hence $\bar{\mathfrak{F}}$ normalizes each, a contradiction. Thus $Sp(4, 2)$ has no elements of order 10 or 15.

LEMMA 4.2. $\bar{\mathfrak{F}}$ is a normal 2-complement of $\bar{\mathfrak{G}}$ and $|\bar{\mathfrak{F}}| = 3, 5$ or $\bar{\mathfrak{F}}$ is abelian of type $(3, 3)$.

Proof. Suppose first that $\bar{\mathfrak{G}}_2 \neq \langle 1 \rangle$. By Lemma 4.1, \mathfrak{G} has a normal subgroup $\mathfrak{G}_0 \cong 3\Omega$ and moreover $4 \nmid |Z(\mathfrak{G})|$. By the Reduction Lemma and Lemma 3.4 we have $q = 3, 5$ or 17. Suppose $q = 3$. Since $|Z(\mathfrak{G})| = 2$ and \mathfrak{G} acts irreducibly, $q^n = 3^4$ and thus \mathfrak{G}_0 also acts irreducibly. By Lemma 4.1 \mathfrak{G} is imprimitive, a contradiction. Let $q = 5$ or 17. Then $4 \mid q - 1$ and since \mathfrak{G} is primitive and $Z(\mathfrak{G}_0) \trianglelefteq \mathfrak{G}$ with $|Z(\mathfrak{G}_0)| = 4$ we conclude that $Z(\mathfrak{G}_0)$ consists of scalar matrices and $4 \mid |Z(\mathfrak{G})|$, a contradiction.

Now suppose $\bar{\mathfrak{F}} = \langle 1 \rangle$. Then $\bar{\mathfrak{G}} = \langle 1 \rangle$. If $Z(\mathfrak{G})$ is central then $\mathfrak{G} = \mathfrak{U}\mathfrak{G}$ is nilpotent so $\mathfrak{G}_2 \supseteq \mathfrak{G}$ is half-transitive. By Theorem II of [4], $\mathfrak{G}_2 \cong \mathfrak{D}\Omega$ and $q^n = 3^4$. Then $|\mathfrak{U}| \mid q - 1$ so $\mathfrak{G} = \mathfrak{G}_2 \cong \mathfrak{D}\Omega$ and this group is imprimitive, a contradiction. Thus $Z(\mathfrak{G})$ is not central and in particular $|Z(\mathfrak{G})| = 4$. Since $|\mathfrak{G}/\bar{\mathfrak{G}}| = 2$ we see that \mathfrak{G} normalizes a hyperplane in $\mathfrak{W} = \mathfrak{G}/Z(\mathfrak{G})$, say $\mathfrak{W}_0 = \mathfrak{G}_0/Z(\mathfrak{G})$. Then $\mathfrak{G}_0 \trianglelefteq \mathfrak{G}$ and \mathfrak{G}_0 has period 4. Since \mathfrak{G} is primitive $Z(\mathfrak{G}_0)$ is cyclic so $Z(\mathfrak{G}_0) = Z(\mathfrak{G})$ and then $\mathfrak{G}_0/Z(\mathfrak{G}_0)$ has odd dimension, a contradiction.

Using the fact that $Sp(4, 2)$ has no elements of order 15 we conclude that $\bar{\mathfrak{F}}$ is one of the three possibilities mentioned in the statement of the lemma. Since $\bar{\mathfrak{F}}$ is abelian, $\bar{\mathfrak{G}}/\bar{\mathfrak{F}} \subseteq \text{Aut } \bar{\mathfrak{F}}$ and from this we see easily that $\bar{\mathfrak{F}}$ is a normal 2-complement.

LEMMA 4.3. $\mathfrak{G} = \text{iso I}$ does not occur.

Proof. Suppose $\mathfrak{G} \cong \Omega\Omega \cong \mathfrak{D}\mathfrak{D}$. Then $\bar{\mathfrak{G}} = \mathfrak{G}$ and $\bar{\mathfrak{G}}$ permutes the involution vectors of $\mathfrak{W} = \mathfrak{G}/Z(\mathfrak{G})$. By Lemma 1.3, $i(\mathfrak{W}) = 9$ and this clearly implies that $|\bar{\mathfrak{F}}| \neq 5$. Thus $\bar{\mathfrak{F}}$ is abelian of type (3) or $(3, 3)$. Since $\mathfrak{G} \cong \mathfrak{D}\mathfrak{D}$ we see easily that \mathfrak{G} contains an abelian subgroup \mathfrak{B} of type $(2, 2, 2)$. If \mathfrak{W}_1 is an irreducible \mathfrak{B} -submodule of \mathfrak{W} then by Schur's lemma, $[\mathfrak{B} : C_{\mathfrak{B}}(\mathfrak{W}_1)] \leq 2$ so for $x \in \mathfrak{W}_1^*$, $4 \mid |\mathfrak{B}_x|$ and hence $4 \mid |\mathfrak{G}_x|$. Moreover since \mathfrak{G}_x is abelian and $\mathfrak{G}_x \cap Z(\mathfrak{G}) = \langle 1 \rangle$ we

see easily that $\mathbb{G}_x = \mathbb{B}_x$. Suppose $|\tilde{\mathfrak{F}}| = 3$. By Lemma 1.5 there exists $y \in \mathbb{B}^*$ with $\mathbb{G}_y \cap \mathfrak{A}\mathbb{G} = \langle 1 \rangle$. Since $\mathbb{G} = \mathfrak{F}$ and $|\tilde{\mathfrak{F}}| = 3$ or 6 by Fitting's theorem, we have $|\mathbb{G}_y| \nmid 6$, a contradiction. Thus $\tilde{\mathfrak{F}}$ is abelian of type (3, 3).

First suppose $q = 3$. Then a Sylow 3-subgroup of \mathbb{G} has a fixed point in \mathbb{B}^* and thus by half-transitivity $\mathbb{G}_x \cong \mathfrak{R}$ where x is the above mentioned point and \mathfrak{R} is a Sylow 3-subgroup of \mathbb{G} . Note that if $\bar{\mathfrak{R}}$ is the image of \mathfrak{R} in $\tilde{\mathfrak{F}}$ then $\bar{\mathfrak{R}} = \tilde{\mathfrak{F}}$. Since $\mathbb{G}_x = \mathbb{G} \cap \mathbb{G}_x \triangle \mathbb{G}_x$ we see that $\bar{\mathfrak{R}}$ normalizes $Z(\mathbb{G})\mathbb{G}_x/Z(\mathbb{G}) = \mathfrak{B}/Z(\mathbb{G})$ a 2-dimensional isotropic subspace of symplectic space \mathbb{B} . This contradicts our preceding remarks about $Sp(4, 2)$ since the subgroup $\bar{\mathfrak{L}}$ of $\bar{\mathfrak{R}}$ normalizes no such subspaces. Thus $q \neq 3$.

Now $\tilde{\mathfrak{F}}$ acts on $\mathbb{B} = \mathbb{G}/Z(\mathbb{G})$ and let $\mathbb{B} = \mathbb{B}_1 \oplus \mathbb{B}_2$ be the decomposition of \mathbb{B} given in our earlier discussion of $Sp(4, 2)$. If $Z(\mathbb{G}) \subseteq \mathbb{G}_i \subseteq \mathbb{G}$ with $\mathbb{G}_i/Z(\mathbb{G}) = \mathbb{B}_i$, then \mathbb{G}_i is nonabelian since \mathbb{B}_i is nonisotropic, and since \mathbb{G}_i admits an automorphism of order 3 we have $\mathbb{G}_i \cong \mathfrak{Q}$. Hence we can find a noncentral involution $T \in \mathbb{G} - (\mathbb{G}_1 \cup \mathbb{G}_2)$. By Lemma 1.5 there exists $x \in \mathbb{B}^*$ with $\mathbb{G}_x = \langle T \rangle$. Now a Sylow 3-subgroup of \mathbb{G} is not cyclic, since $\tilde{\mathfrak{F}}$ is not cyclic and hence it cannot act semiregularly. By half-transitivity \mathbb{G}_x contains a subgroup \mathfrak{L} of order 3. Then $\mathfrak{L} \cap \mathfrak{A}\mathbb{G} = \langle 1 \rangle$ so if $\bar{\mathfrak{L}}$ denotes the image of \mathfrak{L} in $\tilde{\mathfrak{F}}$, then $|\bar{\mathfrak{L}}| = 3$. Since $\langle T \rangle = \mathbb{G}_x = \mathbb{G} \cap \mathbb{G}_x \triangle \mathbb{G}_x$ we see that $\bar{\mathfrak{L}}$ normalizes the 1-dimensional subspace $\mathbb{G}_x Z(\mathbb{G})/Z(\mathbb{G}) = \mathfrak{U}$. Now T was chosen in such a way that $\mathfrak{U} \not\subseteq \mathbb{B}_1$ or \mathbb{B}_2 . Hence in the notation of our discussion of $Sp(4, 2)$ we see that $\bar{\mathfrak{L}} \neq \bar{\mathfrak{L}}_1$ or $\bar{\mathfrak{L}}_2$. On the other hand $\bar{\mathfrak{L}}_3$ and $\bar{\mathfrak{L}}_4$ do not normalize 1-dimensional subspaces. Hence $\bar{\mathfrak{L}} \neq \bar{\mathfrak{L}}_1, \bar{\mathfrak{L}}_2, \bar{\mathfrak{L}}_3$ or $\bar{\mathfrak{L}}_4$, a contradiction.

LEMMA 4.4. *If $\mathbb{G} = \text{iso II}$, then $q^n = 3^4$.*

Proof. Let us assume that $q^n \neq 3^4$. Since \mathbb{G} acts irreducibly on \mathbb{B} we have $|\mathbb{B}| = q^n = q^4$ so $q \geq 5$. We consider the possibilities for $\tilde{\mathfrak{F}}$. Suppose $\tilde{\mathfrak{F}}$ is abelian of type (3, 3). Then $\tilde{\mathfrak{F}}$ is a Sylow 3-subgroup of $Sp(4, 2)$ and we can write $\mathbb{B} = \mathbb{B}_1 \oplus \mathbb{B}_2$, the corresponding decomposition of $\mathbb{G}/Z(\mathbb{G}) = \mathbb{B}$. If $\mathbb{G}_i/Z(\mathbb{G}) = \mathbb{B}_i$, then since \mathbb{B}_i is nonisotropic, \mathbb{G}_i is nonabelian of order 8. Now \mathbb{G}_i admits an automorphism of order 3 so $\mathbb{G}_i \cong \mathfrak{Q}$ and $\mathbb{G} \cong \mathfrak{Q}\mathfrak{Q}$, a contradiction. Thus $|\tilde{\mathfrak{F}}| = p$ for $p = 3$ or 5.

Note that $\mathfrak{F} = \mathbb{G}$ and $|\tilde{\mathfrak{F}}/\tilde{\mathfrak{F}}| \mid (p-1)$. Thus $\tilde{\mathfrak{F}}/\tilde{\mathfrak{F}}$ is a cyclic 2-group. Suppose $p \mid |\mathbb{G}_x|$ for all $x \in \mathbb{B}^*$. Let T be a noncentral involution of \mathbb{G} . Since $q \neq 3$ there exists by Lemma 1.5 an $x \in \mathbb{B}^*$ with $\mathbb{G}_x = \langle T \rangle$. Let \mathfrak{L} be a subgroup of \mathbb{G}_x of order p . Since

$\mathfrak{Z} \cap \mathfrak{U}\mathfrak{G} = \langle 1 \rangle$, $\bar{\mathfrak{Z}}$, the image of \mathfrak{Z} in $\bar{\mathfrak{G}}$, has order p so $\bar{\mathfrak{Z}} = \bar{\mathfrak{Z}}$. Since $\langle T \rangle = \mathfrak{G}_x = \mathfrak{G}_x \cap \mathfrak{G}$ we see that $\bar{\mathfrak{Z}}$ centralizes the involution vector in \mathfrak{B} corresponding to T . By Lemma 1.2, $\bar{\mathfrak{Z}}$ centralizes \mathfrak{B} , a contradiction. Thus $p \nmid |\mathfrak{G}_x|$ and in particular $p \neq q$.

Suppose $p = 3$. By Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{U}\mathfrak{G} = \langle 1 \rangle$. Hence $|\mathfrak{G}_x| \mid |\bar{\mathfrak{G}}|$. Since $|\bar{\mathfrak{G}}| = 6$ we conclude that $|\mathfrak{G}_x| = 2$. We note now that $4 \nmid |\mathfrak{U}|$. Otherwise $\mathfrak{U}\mathfrak{G}$ contains $\mathfrak{G}^* \cong 3\mathfrak{D}\mathfrak{D}$ and this group contains an abelian subgroup of type $(2, 2, 2)$. This easily implies that $4 \mid |\mathfrak{G}_x|$, a contradiction. Let \mathfrak{Z} be a Sylow 3-subgroup of \mathfrak{G} . Since $\bar{\mathfrak{G}}/\bar{\mathfrak{Z}}$ acts faithfully on $\bar{\mathfrak{Z}}$ we see by the above that if T is a noncentral involution of \mathfrak{G} and $T \subseteq C_{\bar{\mathfrak{G}}}(\mathfrak{Z})$ then $T \in \mathfrak{G}$. Now $\bar{\mathfrak{Z}} = \bar{\mathfrak{Z}}$ permutes faithfully the $i(\mathfrak{B}) = 5$ involution vectors of \mathfrak{B} . Thus $\bar{\mathfrak{Z}}$ moves 3 such and fixes 2 such. Since each involution vector corresponds to two noncentral involutions of \mathfrak{G} we see that \mathfrak{Z} centralizes precisely four noncentral involutions of \mathfrak{G} . Thus clearly $I(\mathfrak{G}) \equiv 4 \pmod{3}$. On the other hand by Lemma 1.9 we have $I(\mathfrak{G}) = 1 + q^2$. Thus $q^2 \equiv 0 \pmod{3}$, a contradiction since $q \neq 3$.

We consider $p = 5$ so $q \geq 7$. Let \bar{I} denote the number of involutions of $\mathfrak{G}/\mathfrak{U}$. Since \mathfrak{U} is cyclic and central in \mathfrak{G} , each involution of $\mathfrak{G}/\mathfrak{U}$ corresponds to at most two noncentral involutions of \mathfrak{G} so $I(\mathfrak{G}) \leq 2\bar{I}$. Now $\mathfrak{B}\bar{\mathfrak{Z}} \trianglelefteq \mathfrak{G}/\mathfrak{U}$ where \mathfrak{B} is elementary abelian of order 2^4 , $|\bar{\mathfrak{Z}}| = 5$ and $\bar{\mathfrak{Z}}$ acts irreducibly on \mathfrak{B} . Furthermore $(\mathfrak{G}/\mathfrak{U})/(\mathfrak{B}\bar{\mathfrak{Z}})$ is a cyclic 2-group which acts faithfully on $(\mathfrak{B}\bar{\mathfrak{Z}})/\mathfrak{B}$. Hence we see easily that $\bar{I} \leq 15 + 5 \cdot 4 = 35$ and $I(\mathfrak{G}) \leq 70$.

Let T be a noncentral involution of \mathfrak{G} . If $T \in \mathfrak{U}\mathfrak{G}$ then certainly $|C_{\mathfrak{B}}(T)| = q^2$. Suppose $T \notin \mathfrak{U}\mathfrak{G}$. From the structure of $\bar{\mathfrak{G}}$ we see that for some $F \in \mathfrak{G}$, $\langle \bar{T}, \bar{T}^F \rangle \cong \bar{\mathfrak{Z}}$. Since $5 \nmid |\mathfrak{G}_x|$ we see that $C_{\mathfrak{B}}(T) \cap C_{\mathfrak{B}}(T^F) = \{0\}$. Hence $|C_{\mathfrak{B}}(T)| \leq q^2$ here also. Now every element of \mathfrak{B}^* is fixed by some noncentral involution of \mathfrak{G} so $\mathfrak{B}^* = \bigcup_r C_{\mathfrak{B}}(T)^*$ and hence

$$q^4 - 1 = |\mathfrak{B}| \leq I(\mathfrak{G})(q^2 - 1)$$

or $q^2 + 1 \leq I(\mathfrak{G}) \leq 70$. Since $q > 5$, we have $q = 7$.

For $q = 7$ the argument is somewhat involved. Since $|\mathfrak{U}| \mid q - 1$ we have $|\mathfrak{U}| = 2$ or 6 . Now $O_3(\mathfrak{U})$ is central in \mathfrak{G} and is a Sylow 3-subgroup of \mathfrak{G} . Thus \mathfrak{G} has a normal 3-complement. Since this group is also half-transitive we see that it suffices to assume that $O_3(\mathfrak{U}) = \langle 1 \rangle$ and hence $|\mathfrak{U}| = 2$, $\mathfrak{U}\mathfrak{G} = \mathfrak{G}$.

We can now get a tighter count on $I(\mathfrak{G})$. Let $\bar{I} = \bar{I}_1 + \bar{I}_2$ where \bar{I}_1 counts the number of involutions of $\mathfrak{G}/\mathfrak{U}$ and \bar{I}_2 counts those of $\mathfrak{G}/\mathfrak{U}$ not in $\mathfrak{G}/\mathfrak{U}$. We have as before $\bar{I}_1 = 15$, $\bar{I}_2 \leq 20$. If $I(\mathfrak{G}) = I_1 + I_2$ is the corresponding break up of $I(\mathfrak{G})$, then $I_2 \leq 2\bar{I}_2 \leq 40$ and

$I_1 = I(\mathfrak{G}) = 10$. Hence $I(\mathfrak{G}) \leq 50$ here. As above $\mathfrak{B}^* = \bigcup_{\mathfrak{T}} C_{\mathfrak{B}}(T)$ yields $50 = q^2 + 1 \leq I(\mathfrak{G}) \leq 50$. Thus we must have equality throughout and hence $\bigcup_{\mathfrak{T}} C_{\mathfrak{B}}(T)$ is a disjoint union. This implies that every element $x \in \mathfrak{B}^*$ is centralized by precisely one involution so \mathfrak{G}_x has a unique involution.

Let \mathfrak{R} be the subgroup of \mathfrak{G} with $\mathfrak{R} \supseteq \mathfrak{G}$ and $[\bar{\mathfrak{R}} : \bar{\mathfrak{G}}] = 2$. Since $\mathfrak{G}/\bar{\mathfrak{G}}$ is cyclic, \mathfrak{R} contains all the involutions of \mathfrak{G} . We study the group \mathfrak{R} . Note that $\bar{\mathfrak{R}}$ is dihedral of order 10 and $\bar{\mathfrak{G}}$ acts irreducibly on $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$. Let \mathfrak{S} be a Sylow 5-subgroup of \mathfrak{R} so that $|\mathfrak{S}| = 5$ and let $\mathfrak{R} = N_{\mathfrak{R}}(\mathfrak{S})$. From the above we see that $\mathfrak{R}/Z(\mathfrak{G})$ is dihedral of order 10. Let $\mathfrak{S} = \langle L \rangle$ and let $N \in \mathfrak{R} - Z(\mathfrak{G})$ be a 2-element. Then $L^N = L^{-1}$.

Now \mathfrak{R} permutes the 10 noncentral involutions of \mathfrak{G} and the corresponding five involution vectors of \mathfrak{B} . Using $(())$ to denote cyclic permutations, it is clear that we can label the involutions by $X_i, Y_i, i = 1, 2, \dots, 5$ such that $Y_i = -X_i$ and as a permutation

$$L = ((X_1, X_2, X_3, X_4, X_5))((Y_1, Y_2, Y_3, Y_4, Y_5)).$$

Here for convenience we denoted the central involution of \mathfrak{G} by -1 . We consider N . As a permutation, it has order 2. Since N acts on the five involution vectors of \mathfrak{B} , N must fix at least one such, say the one corresponding to $\{X_1, Y_1\}$. Then either N fixes both X_1 and Y_1 or N interchanges the two. Since $L^N = L^{-1}$ this completely determines the cycle structure of N and we have either

- (a) $N = ((X_1))((X_2, X_5))((X_3, X_4))((Y_1))((Y_2, Y_5))((Y_3, Y_4))$ or
- (b) $N = ((X_1, Y_1))((X_2, Y_5))((X_3, Y_4))((X_4, Y_3))((X_5, Y_2))$.

Note that it is easy to see that for $i \neq j$, $(X_i, X_j) = (Y_i, Y_j) = -1$. Now the sum of the five involution vectors of \mathfrak{B} is L invariant and hence must be 0. Thus $Z = X_1 X_2 X_3 X_4 X_5 \in Z(\mathfrak{G})$. If N acts like (b) above, then

$$\begin{aligned} Z &= Z^N = (X_1 X_2 X_3 X_4 X_5)^N = Y_1 Y_5 Y_4 Y_3 Y_2 \\ &= -X_1 (X_5 X_4 X_3 X_2) = -Z^{-1}. \end{aligned}$$

Thus $Z^2 = -1$, a contradiction and hence N must act like (a) above.

Suppose N has order 2. Then $\langle N, X_1, Y_1 \rangle$ is elementary abelian of order 8. This yields as usual an element $x \in \mathfrak{B}^*$ such that \mathfrak{G}_x contains a subgroup of type $(2, 2)$ and this contradicts our preceding remarks. Hence $N^2 = -1$.

Now $\mathfrak{S} = \langle \mathfrak{G}, N \rangle$ is a Sylow 2-subgroup of \mathfrak{R} . We show that every involution of \mathfrak{S} is contained in \mathfrak{G} . This will imply that \mathfrak{G} contains only 10 noncentral involutions and this will yield the required contradiction. Suppose $T \in \mathfrak{S} - \mathfrak{G}$ is an involution. Then $T = NE$ for some $E \in \mathfrak{G}$. Since $N^2 = -1$ we have

$$1 = T^2 = NENE = -E^N E$$

so $E^N = -E^{-1}$. In particular the image of E in $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$ is centralized by N . Now $C_{\mathfrak{B}}(N)$ is a 2-dimensional subspace which is clearly spanned by the images in \mathfrak{B} of X_1 and X_2X_5 . Note that X_1 and X_2X_5 commute and X_2X_5 has order 4. Hence $E \in \langle X_1, X_2X_5 \rangle = \mathfrak{B}$. We have $X_1^N = X_1 = X_1^{-1}$ and $(X_2X_5)^N = X_5X_2 = (X_2X_5)^{-1}$ so since \mathfrak{B} is abelian, N acts in a dihedral manner on \mathfrak{B} . Thus $E^N = E^{-1}$ which contradicts the previous relation $E^N = -E^{-1}$. This implies that T does not exist and the proof is complete.

If $q^n = 3^4$ above then $\mathfrak{G} = F(\mathfrak{G})$ is half-transitive. Thus these groups are given in [5] where uniqueness was proved. Since \mathfrak{G} is primitive, we see that \mathfrak{G} is transitive and hence it is one of the groups given in [3].

LEMMA 4.5. $\mathfrak{G} = \text{iso III}$ does not occur.

Proof. Suppose $\mathfrak{G} \cong 3\Omega\Omega$. Since $|Z(\mathfrak{G})| = 4$ and \mathfrak{G} acts irreducibly we see that $|\mathfrak{B}| = q^4$ if $q \equiv 1 \pmod{4}$ and $|\mathfrak{B}| = q^8$ if $q \equiv -1 \pmod{4}$. If $\mathfrak{H} = C_{\mathfrak{G}}(Z(\mathfrak{G}))$, then $[\mathfrak{G} : \mathfrak{H}] = 1$ or 2 . Moreover if $[\mathfrak{G} : \mathfrak{H}] = 2$ then $q \equiv -1$.

We consider $\tilde{\mathfrak{G}}$. Suppose $|\tilde{\mathfrak{G}}| = 5$ or 9 so that $C_{\mathfrak{B}}(\tilde{\mathfrak{G}}) = \langle 1 \rangle$. Clearly $\tilde{\mathfrak{G}}$ acts faithfully on $\mathfrak{G}/\mathfrak{G}'$ and centralizes $Z(\mathfrak{G})/\mathfrak{G}'$. Let \mathfrak{G}_0 be the commutator subgroup of $\mathfrak{G}\tilde{\mathfrak{G}}$. Then clearly $|\mathfrak{G}_0/\mathfrak{G}'| = 2^4$, $\mathfrak{G}_0 \triangle \mathfrak{G}$ and $\mathfrak{G}_0 = \text{iso I or II}$. By the Reduction Lemma and the previous two lemmas, $\mathfrak{G}_0 = \text{iso II}$ and $q = 3$. Since as we have seen, this group does not admit an automorphism group of type $(3, 3)$ we must have $|\tilde{\mathfrak{G}}| = 5$. Since $q = 3$, $q^n = 3^8$.

Now \mathfrak{G} has an abelian subgroup of type $(2, 2, 2)$ so it follows that $4 \mid |\mathfrak{G}_x|$ and hence $2 \mid |\mathfrak{H}_x|$ for all $x \in \mathfrak{B}^*$. As in the proof of the previous lemma we see that $5 \nmid |\mathfrak{G}_x|$ and hence if T is a noncentral involution of \mathfrak{H} , then $|C_{\mathfrak{B}}(T)| \leq 3^4$. Now $\mathfrak{H}/\mathfrak{A}$ contains at most $15 + 5 \cdot 4 = 35$ involutions and hence since \mathfrak{A} is central and cyclic we have $I(\mathfrak{H}) \leq 2 \cdot 35 = 70$. Since $\mathfrak{B} = \bigcup_T C_{\mathfrak{B}}(T)$ we have

$$3^8 = |\mathfrak{B}| \leq 3^4 I(\mathfrak{H}) \leq 3^4 \cdot 70$$

or $3^4 \leq 70$, a contradiction.

Finally let $|\tilde{\mathfrak{G}}| = 3$. As above we see that $4 \mid |\mathfrak{G}_x|$. Since by Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{G} = \langle 1 \rangle$, we conclude that $4 \mid |\mathfrak{G}/\mathfrak{A}\mathfrak{G}|$. Hence $|\tilde{\mathfrak{G}}| = 6$ and $[\mathfrak{G} : \mathfrak{H}] = 2$ so $q \equiv -1 \pmod{4}$, $q^n = q^8$ and $q \neq 5$. By Lemma 1.5, if T is a noncentral involution of \mathfrak{G} then for some $x \in \mathfrak{B}^*$, $\mathfrak{G}_x = \langle T \rangle$. Hence if $3 \mid |\mathfrak{G}_x|$, then $\tilde{\mathfrak{G}}$ fixes all involution vectors of \mathfrak{B} and $\tilde{\mathfrak{G}}$ centralizes \mathfrak{B} , a contradiction. Thus

$3 \nmid |\mathbb{G}_x|$ and this implies easily that if T is an involution of \mathfrak{F} , then $|\mathcal{C}_{\mathfrak{B}}(T)| \leq q^4$. Also $q \neq 3$ so $q \geq 7$. We have clearly $I(\mathfrak{F}) \leq 2 \cdot 2 \cdot 16 \cdot 3 = 192$ and since $\mathfrak{B} = \bigcup_T \mathcal{C}_{\mathfrak{B}}(T)$ we have

$$q^8 = |\mathfrak{B}| \leq q^4 I(\mathfrak{F}) \leq 192q^4.$$

Thus $7^4 \leq q^4 \leq 192$, a contradiction. This completes the proof of the lemma.

5. Solvable case, $m = 3$ and 4. We continue with the assumptions of the preceding section except that $m = 3$ or 4 here. First let $m = 3$. Now $|Sp(2m, 2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$. We consider the possibilities for \mathfrak{F} .

LEMMA 5.1. \mathfrak{F} is a 3-group.

Proof. If p is a prime, we let \mathfrak{F}_p denote the normal Sylow p -subgroup of \mathfrak{F} . We show here that $\mathfrak{F}_2 = \mathfrak{F}_5 = \mathfrak{F}_7 = \langle 1 \rangle$.

Suppose $\mathfrak{F}_2 \neq \langle 1 \rangle$. By Lemma 4.1 \mathbb{G} has a normal subgroup $\mathbb{G}_0 \cong 3\Omega\Omega$. By the Reduction Lemma and Lemma 4.5 this does not occur.

Suppose $\mathfrak{F}_7 \neq \langle 1 \rangle$. Then $|\mathfrak{F}_7| = 7$ and \mathfrak{F}_7 acts irreducibly on \mathfrak{B} . By Schur's lemma, $\mathcal{C}_{\mathfrak{B}}(\mathfrak{F}_7)$ is a cyclic group of odd order and $[\mathfrak{F} : \mathcal{C}_{\mathfrak{B}}(\mathfrak{F}_7)] \leq 6$. Hence if $\mathbb{G} = \text{iso I or II}$ then $4 \nmid [\mathbb{G} : \mathcal{U}(\mathbb{G})]$ while if $\mathbb{G} = \text{iso III}$, then $8 \nmid [\mathbb{G} : \mathcal{U}(\mathbb{G})]$. Now if $\mathbb{G} = \text{iso I or II}$ then \mathbb{G} has an abelian subgroup of type $(2, 2, 2)$ so for some $y \in \mathfrak{B}^*$, $4 \mid |\mathbb{G}_y|$. If $\mathbb{G} = \text{iso III}$, then \mathbb{G} has an abelian subgroup of type $(2, 2, 2, 2)$ so $8 \mid |\mathbb{G}_y|$. Finally by Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathbb{G}_x \cap \mathcal{U}(\mathbb{G}) = \langle 1 \rangle$ so $|\mathbb{G}_x| \mid [\mathbb{G} : \mathcal{U}(\mathbb{G})]$. Since $|\mathbb{G}_x| = |\mathbb{G}_y|$ we have a contradiction.

Suppose $\mathfrak{F}_5 \neq \langle 1 \rangle$. Then $|\mathfrak{F}_5| = 5$ and we can write $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ where $|\mathfrak{B}_1| = 2^2$, $|\mathfrak{B}_2| = 2^4$, both these spaces are \mathfrak{F}_5 invariant and $\mathfrak{B}_1 = \mathcal{C}_{\mathfrak{B}}(\mathfrak{F}_5)$. Let $\mathbb{G} \supseteq \mathbb{G}_i \supseteq \mathcal{Z}(\mathbb{G})$ with $\mathbb{G}_i/\mathcal{Z}(\mathbb{G}) = \mathfrak{B}_i$. Clearly $\mathbb{G}_i \triangleleft \mathbb{G}$ and since \mathbb{G} is primitive each \mathbb{G}_i is of symplectic type. By the Reduction Lemma applied to \mathbb{G}_2 and Lemmas 4.3, 4.4 and 4.5 we have $q = 3$ and $\mathbb{G}_2 \cong \mathcal{D}\Omega$. Hence $|\mathcal{Z}(\mathbb{G})| = 2$ so $\mathbb{G} \neq \text{iso III}$.

Now \mathfrak{B}_1 and \mathfrak{B}_2 are nonisotropic and we know that \mathfrak{F}_5 is self-centralizing in its action on \mathfrak{B}_2 . Write $\mathcal{C}_{\mathfrak{B}}(\mathfrak{F}_5) = \bar{\mathfrak{B}} \times \mathfrak{F}_5$ where $\bar{\mathfrak{B}} \triangleleft \mathfrak{F}_5$. Then $\bar{\mathfrak{B}}$ acts faithfully on \mathfrak{B}_1 so since $\mathfrak{F}_2 = \langle 1 \rangle$, either $\bar{\mathfrak{B}} = \langle 1 \rangle$ or $\bar{\mathfrak{B}}$ has a normal 3-subgroup of order 3 which is clearly \mathfrak{F}_3 . Suppose $\bar{\mathfrak{B}} = \langle 1 \rangle$. Then $\mathfrak{F}_5/\mathfrak{F}_3$ is a 2-group which acts on \mathfrak{B}_1 and hence there is a 1-dimensional \mathfrak{F}_5 -invariant subspace \mathfrak{B}_0 of \mathfrak{B}_1 . Note that $\bar{\mathfrak{G}} = \mathbb{G}$ since $\mathbb{G} \neq \text{iso III}$ and thus if $\mathbb{G} \supseteq \mathbb{G}_3 \supseteq \mathcal{Z}(\mathbb{G})$ with $\mathbb{G}_3/\mathcal{Z}(\mathbb{G}) = \mathfrak{B}_0 \oplus \mathfrak{B}_2$ then $\mathbb{G}_3 \triangleleft \mathbb{G}$. By the Reduction Lemma and Lemma 4.5 we have a contradiction since clearly $\mathbb{G}_3 \cong 3\Omega\Omega$.

Thus $\bar{\mathfrak{B}} \cong \bar{\mathfrak{F}}_3$ and $|\bar{\mathfrak{F}}_3| = 3$. Since $q = 3$ we see that the Sylow 3-subgroups of \mathfrak{G} have order 3. Now $\bar{\mathfrak{F}}_3$ centralizes \mathfrak{B}_2 so clearly \mathfrak{G} contains precisely four Sylow 3-subgroups say \mathfrak{L}_i for $i = 1, 2, 3, 4$. Since $q = 3$ each \mathfrak{L}_i has a fixed point on \mathfrak{B}^* so by half-transitivity $\mathfrak{B} = \bigcup_i C_{\mathfrak{B}}(\mathfrak{L}_i)$. Hence since the \mathfrak{L}_i are all conjugate in \mathfrak{G} we see that each $C_{\mathfrak{B}}(\mathfrak{L}_i)$ has codimension 1 in \mathfrak{B} . But $|\mathfrak{B}| = 3^8$ so $\mathfrak{B}_0 = \bigcap C_{\mathfrak{B}}(\mathfrak{L}_i) \neq \{0\}$. Since \mathfrak{B}_0 is clearly a proper \mathfrak{G} -invariant subspace of \mathfrak{B} we have a contradiction.

LEMMA 5.2. $\bar{\mathfrak{F}}$ is not cyclic and $q \neq 3$.

Proof. We have shown that $\bar{\mathfrak{F}} = \bar{\mathfrak{F}}_3$. If $\bar{\mathfrak{F}}$ is cyclic (including the possibility that $\bar{\mathfrak{F}} = \langle 1 \rangle$) then clearly $4 \nmid |\bar{\mathfrak{F}}|$. If $\mathfrak{G} = \text{iso I or II}$ then $\mathfrak{G} = \mathfrak{S}$ so $4 \nmid |\mathfrak{G}/\mathfrak{M}\mathfrak{G}|$. If $\mathfrak{G} = \text{iso III}$ then $8 \nmid |\mathfrak{G}/\mathfrak{M}\mathfrak{G}|$. If $\mathfrak{G} = \text{iso I or II}$, then \mathfrak{G} has an abelian subgroup of type $(2, 2, 2)$ so we see that $4 \mid |\mathfrak{G}_x|$. If $\mathfrak{G} = \text{iso III}$, then \mathfrak{G} has an abelian subgroup of type $(2, 2, 2, 2)$ so $8 \mid |\mathfrak{G}_x|$. Now by Lemma 1.5 there exists $y \in \mathfrak{B}^*$ with $\mathfrak{G}_y \cap \mathfrak{M}\mathfrak{G} = \langle 1 \rangle$. Hence $|\mathfrak{G}_y| \mid |\mathfrak{G} : \mathfrak{M}\mathfrak{G}|$, a contradiction.

Let $q = 3$ so that for all $x \in \mathfrak{B}^*$, \mathfrak{G}_x contains a Sylow 3-subgroup of \mathfrak{G} . Let $\bar{\mathfrak{F}}$ be the complete inverse image of $\bar{\mathfrak{F}}$ in \mathfrak{S} . For any $x \in \mathfrak{B}^*$, let \mathfrak{L} be a Sylow 3-subgroup of $\bar{\mathfrak{F}}_x$. Then clearly $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{M}\mathfrak{G}/\mathfrak{M}\mathfrak{G} = \bar{\mathfrak{F}}$ and since $\mathfrak{G}_x = \mathfrak{G} \cap \mathfrak{G}_x \triangle \mathfrak{G}_x$ we see that $\bar{\mathfrak{F}}$ normalizes $\mathfrak{G}_x\mathfrak{Z}(\mathfrak{G})/\mathfrak{Z}(\mathfrak{G})$. If $\mathfrak{G} = \text{iso II or III}$, then by Lemma 1.5 if T is any noncentral involution of \mathfrak{G} then for some $x \in \mathfrak{B}^*$, $\mathfrak{G}_x = \langle T \rangle$. This implies that $\bar{\mathfrak{F}}$ fixes all involution vectors and $\bar{\mathfrak{F}} = \langle 1 \rangle$, a contradiction. If $\mathfrak{G} = \text{iso I}$ then by Lemma 1.5, $|\mathfrak{G}_x| = 1$ or 4. However here it is easy to see that for each such T we can find two points $x_1, x_2 \in \mathfrak{B}^*$ with $\langle T \rangle = \mathfrak{G}_{x_1} \cap \mathfrak{G}_{x_2}$. This again implies that $\bar{\mathfrak{F}}$ fixes all involution vectors and the result follows.

LEMMA 5.3. $\mathfrak{G} = \text{iso I}$ does not occur.

Proof. Here $\mathfrak{G} \cong \Omega\Omega\Omega$ and we see easily that $\text{Aut } \mathfrak{G}$ contains $\bar{\mathfrak{F}} \sim \bar{\mathfrak{F}}$ where $|\bar{\mathfrak{F}}| = 3$ and this is a full Sylow 3-subgroup of $Sp(6, 2)$. Then any 3-group acting on \mathfrak{G} can be embedded in this Sylow 3-subgroup. Let $\bar{\mathfrak{K}}$ be a Sylow 3-subgroup of $\text{Aut } \mathfrak{G}$. Then $\bar{\mathfrak{K}}$ acts faithfully on $\mathfrak{B} = \mathfrak{G}/\mathfrak{Z}(\mathfrak{G})$. As a Sylow 3-subgroup of $Sp(6, 2)$ we know that it has the following structure. We can write $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3$, a direct sum of orthogonal 2-dimensional nonisotropic subspaces. $\bar{\mathfrak{K}}$ has a subgroup $\bar{\mathfrak{K}}$ of index 3 with $\bar{\mathfrak{K}} = \bar{\mathfrak{L}}_1 \times \bar{\mathfrak{L}}_2 \times \bar{\mathfrak{L}}_3$. Here $|\bar{\mathfrak{L}}_i| = 3$ and $\bar{\mathfrak{L}}_i$ acts irreducibly on \mathfrak{B}_i and centralizes the remaining \mathfrak{B}_j . Further, any element of $\bar{\mathfrak{K}} - \bar{\mathfrak{K}}$ permutes these three subspaces. Now let \mathfrak{G}_i

be the subgroup of \mathfrak{G} with $\mathfrak{G}_i/Z(\mathfrak{G}) = \mathfrak{B}_i$. Then \mathfrak{G}_i is nonabelian of order 8 and admits an automorphism of order 3. Thus $\mathfrak{G}_i \cong \mathfrak{Q}$. Suppose $T = T_1 T_2 T_3$ is a noncentral involution of \mathfrak{G} with $T_i \in \mathfrak{G}_i$. Since $\mathfrak{G}_i \cong \mathfrak{Q}$ we see that precisely one of the T_i is contained in $Z(\mathfrak{G})$, say for example T_1 . Then we can write $T = T_2 T_3$. If some subgroup $\bar{\mathfrak{L}}$ of $\bar{\mathfrak{K}}$ centralizes the involution vector corresponding to T then clearly $\bar{\mathfrak{L}}$ normalizes \mathfrak{B}_1 . Thus $\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{N}}$ so $\bar{\mathfrak{L}}$ normalizes \mathfrak{B}_2 and \mathfrak{B}_3 . This clearly implies that $\bar{\mathfrak{L}}$ centralizes \mathfrak{B}_2 and \mathfrak{B}_3 and thus $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_1$. Hence the only subgroups of $\bar{\mathfrak{K}}$ which centralize involution vectors are $\bar{\mathfrak{L}}_1, \bar{\mathfrak{L}}_2$ and $\bar{\mathfrak{L}}_3$.

Now $\bar{\mathfrak{F}}$ is not cyclic and hence a Sylow 3-subgroup of \mathfrak{G} is not cyclic. Thus $3 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{B}^*$. By the preceding lemma again $q \neq 3$. Hence if $T \in \mathfrak{G}$ is an involution, then by Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x = \langle T \rangle$. Let \mathfrak{L} be a Sylow 3-subgroup of \mathfrak{G}_x so $|\mathfrak{L}| \geq 3$ and $\mathfrak{L} \cap \mathfrak{A}\mathfrak{G} = \langle 1 \rangle$. Then $\bar{\mathfrak{L}}$ acts faithfully on \mathfrak{G} so we can extend \mathfrak{L} to $\bar{\mathfrak{K}}$ as above. Since $\bar{\mathfrak{L}}$ normalizes the involution vector corresponding to T we see that $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_i$ for some i . Thus $|\bar{\mathfrak{L}}| = 3$ and $9 \nmid |\mathfrak{G}_x|$.

Suppose $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{A}\mathfrak{G}$ contains a copy of $\bar{\mathfrak{N}} \subseteq \bar{\mathfrak{K}}$. Then let \mathfrak{S} be a 3-subgroup of \mathfrak{G} with $\mathfrak{S}\mathfrak{A}\mathfrak{G}/\mathfrak{A}\mathfrak{G} = \bar{\mathfrak{N}}$. Certainly $\mathfrak{S}' \subseteq \mathfrak{A}$. Now \mathfrak{S} acts on \mathfrak{B} , a vector space of dimension $n = 2^4$. Since \mathfrak{S} is a 3-group we conclude that \mathfrak{S}' is in the kernel of some irreducible constituent and hence \mathfrak{S}' has a fixed point in \mathfrak{B}^* . Since $\mathfrak{S}' \subseteq \mathfrak{A}$ we see that $\mathfrak{S}' = \langle 1 \rangle$ and \mathfrak{S} is abelian. Now $\mathfrak{S}/\mathfrak{S}' \cap \mathfrak{A}$ is abelian of type $(3, 3, 3)$ and hence \mathfrak{S} contains a subgroup of type $(3, 3, 3)$. But this implies that $9 \mid |\mathfrak{G}_x|$, a contradiction. In particular we see that a Sylow 3-subgroup of $\bar{\mathfrak{G}}$ has order $\leq 3^3$.

Let T and \mathfrak{L} be as above and set $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{A}\mathfrak{G}/\mathfrak{A}\mathfrak{G}$. This time embed 3-group $\bar{\mathfrak{F}}\bar{\mathfrak{L}}$ in $\bar{\mathfrak{K}}$. Again $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_i$ for some i . Now $\bar{\mathfrak{K}}$ is generated by $\bar{\mathfrak{L}}_i$ and any element outside $\bar{\mathfrak{N}}$. Since $\bar{\mathfrak{F}}\bar{\mathfrak{L}} < \bar{\mathfrak{K}}$ we must have $\bar{\mathfrak{F}} \subseteq \bar{\mathfrak{N}}$ and hence $\bar{\mathfrak{F}}\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{N}}$. Since $\bar{\mathfrak{L}}$ centralizes $\bar{\mathfrak{F}}$ we have $\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{F}}$.

Now embed $\bar{\mathfrak{F}}$ alone in $\bar{\mathfrak{K}}$. We have shown that for each involution vector of \mathfrak{B} , $\bar{\mathfrak{F}}$ contains a subgroup of order 3 centralizing it. Thus $\bar{\mathfrak{F}} \supseteq \bar{\mathfrak{L}}_1, \bar{\mathfrak{L}}_2, \bar{\mathfrak{L}}_3$ and $\bar{\mathfrak{F}} \supseteq \bar{\mathfrak{N}}$, a contradiction since $\bar{\mathfrak{G}} \not\supseteq \bar{\mathfrak{N}}$. This completes the proof of this result.

LEMMA 5.4. $\mathfrak{G} = \text{iso II and III do not occur.}$

Proof. Suppose $C_{\mathfrak{B}}(\bar{\mathfrak{F}}) = \mathfrak{B}_1 \neq \langle 1 \rangle$. Then $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ where $\mathfrak{B}_2 = (\mathfrak{B}, \bar{\mathfrak{F}})$. Since \mathfrak{B}_2 has even dimension (the nonprincipal irreducible representations of a 3-group over $GF(2)$ have even dimension) so does \mathfrak{B}_1 . One of these two subspaces, say \mathfrak{B}_i has dimension equal to 4.

Let \mathfrak{G}_i be the subgroup of \mathfrak{G} with $\mathfrak{G}_i/Z(\mathfrak{G}) = \mathfrak{W}_i$. Then $\mathfrak{G}_i \trianglelefteq \mathfrak{G}$ and \mathfrak{G} is primitive so \mathfrak{G}_i is of symplectic type. By the Reduction Lemma and Lemmas 4.3, 4.4 and 4.5 we have $q = 3$, a contradiction by Lemma 5.2.

Now let $\mathfrak{G} = \text{iso II}$. By Lemma 1.3, $\bar{\mathfrak{F}}$ permutes the $i(\mathfrak{W}) = 35$ involution vectors. Hence $\bar{\mathfrak{F}}$ must fix one of these and $C_{\mathfrak{W}}(\bar{\mathfrak{F}}) \neq \langle 1 \rangle$, a contradiction.

Having already eliminated $\mathfrak{G} = \text{iso I}$ and II we now eliminate iso III. $\bar{\mathfrak{F}}$ acts on $\mathfrak{G}/\mathfrak{G}' = \mathfrak{U}$ and centralizes $Z(\mathfrak{G})/\mathfrak{G}'$. Since $C_{\mathfrak{W}}(\bar{\mathfrak{F}}) = \langle 1 \rangle$ we see that $\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_2$ where $\mathfrak{U}_1 = C_{\mathfrak{W}}(\bar{\mathfrak{F}})$, $\mathfrak{U}_2 = (\mathfrak{U}, \bar{\mathfrak{F}})$, $|\mathfrak{U}_1| = 2$, $|\mathfrak{U}_2| = 2^4$. Let \mathfrak{G}_2 be a subgroup of \mathfrak{G} with $\mathfrak{G}_2/\mathfrak{G}' = \mathfrak{U}_2$. Then $\mathfrak{G}_2 \trianglelefteq \mathfrak{G}$ and \mathfrak{G}_2 is type $E(2, 3)$ and iso I or II. By the Reduction Lemma and the above we have a contradiction.

We now consider $m = 4$. Here we have partial results in Lemmas 2.6, 2.10 and 2.12. Thus $\mathfrak{G} \neq \text{iso III}$, $q \geq 7$ and $|\mathfrak{G}/\mathfrak{W}\mathfrak{G}| > 10^4$. We consider $\bar{\mathfrak{F}}$.

LEMMA 5.5. *All irreducible constituents of $\bar{\mathfrak{F}}_p$ on \mathfrak{W} have the same degree. Thus $\bar{\mathfrak{F}}_2 = \langle 1 \rangle$, $\bar{\mathfrak{F}}_p = \langle 1 \rangle$ if $p \nmid i(\mathfrak{W})$ and $\bar{\mathfrak{F}}_3$ is elementary abelian.*

Proof. Suppose $\bar{\mathfrak{F}}_2 \neq \langle 1 \rangle$. Then by Lemma 4.1, \mathfrak{G} has a normal subgroup \mathfrak{G}_0 of type $E(2, 3)$ and iso III. By the Reduction Lemma and Lemma 5.4 this is a contradiction.

If $p \neq 2$ then $\bar{\mathfrak{F}}_p$ acts in a completely reducible manner on \mathfrak{W} . If all its irreducible constituents do not have the same degree, then certainly we can write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$ where $\mathfrak{W}_i \neq \langle 1 \rangle$ and \mathfrak{W}_i is \mathfrak{G} invariant. One of these two, say \mathfrak{W}_1 , has dimension at least 4. If $\mathfrak{G}_1/Z(\mathfrak{G}) = \mathfrak{W}_1$ then $\mathfrak{G}_1 \trianglelefteq \mathfrak{G}$ and since \mathfrak{G} is primitive, \mathfrak{G}_1 is type $E(2, m')$ with $m' = 2$ or 3. Since $q \geq 7$ the Reduction Lemma and the $m = 2$ and 3 results yield a contradiction. Now if $p \nmid i(\mathfrak{W})$, then certainly $\bar{\mathfrak{F}}_p$ has a 1-dimensional constituent so they are all 1-dimensional and over $GF(2)$ this implies that $\bar{\mathfrak{F}}_p$ centralizes \mathfrak{W} so $\bar{\mathfrak{F}}_p = \langle 1 \rangle$.

Finally we consider $\bar{\mathfrak{F}}_3$. If $\bar{\mathfrak{F}}_3$ is nonabelian then the degree of an irreducible representation of $\bar{\mathfrak{F}}_3$, with $\bar{\mathfrak{F}}'_3$ not in the kernel is divisible by 3. Since $3 \nmid \dim \mathfrak{W}$, $\bar{\mathfrak{F}}'_3$ is in the kernel of all constituents so $\bar{\mathfrak{F}}'_3 = \langle 1 \rangle$ and $\bar{\mathfrak{F}}_3$ is abelian. Let \mathfrak{W}_0 be an irreducible $\bar{\mathfrak{F}}_3$ -constituent of \mathfrak{W} with dimension j . Then $j \mid \dim \mathfrak{W}$ so $j = 1, 2, 4$ or 8. In all these cases $9 \nmid 2^j - 1$ and hence clearly $\bar{\mathfrak{F}}_3$ is elementary abelian.

LEMMA 5.6. *$\mathfrak{G} = \text{iso I}$ does not occur.*

Proof. Here by Lemma 1.3, $i(\mathfrak{W}) = 3^3 \cdot 5$ so only $\bar{\mathfrak{F}}_5$ and $\bar{\mathfrak{F}}_3$ can

be nontrivial. We show first that $\bar{\mathfrak{F}}_5 = \langle 1 \rangle$. Note that a Sylow 5-subgroup of $Sp(8, 2)$ is abelian of type (5, 5).

Suppose first that $|\bar{\mathfrak{F}}_5| = 5^2$. Then $\bar{\mathfrak{F}}_5$ is elementary abelian and a Sylow 5-subgroup of $\bar{\mathfrak{G}}$. We can write $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$, $\bar{\mathfrak{F}}_5 = \bar{\mathfrak{L}}_1 \bar{\mathfrak{L}}_2$ where $\dim \mathfrak{B}_i = 4$, $|\bar{\mathfrak{L}}_i| = 5$ and $\bar{\mathfrak{L}}_i$ acts irreducibly on \mathfrak{B}_i and centralizes the other \mathfrak{B}_j . Now a Sylow 5-subgroup of \mathfrak{G} is not cyclic so $5 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{B}^*$. We have $i(\mathfrak{B}) = 135$ and $|\mathfrak{B}_1 \cup \mathfrak{B}_2| = 31$. Hence we can find a noncentral involution $T \in \mathfrak{G}$ with $TZ(\mathfrak{G})/Z(\mathfrak{G}) \not\subseteq \mathfrak{B}_1 \cup \mathfrak{B}_2$. By Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x = \langle T \rangle$ and if $\mathfrak{L} \subseteq \mathfrak{G}_x$ has order 5, then \mathfrak{L} normalizes $\mathfrak{G}_x \cap \mathfrak{G} = \langle T \rangle$. Thus $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{U}/\mathfrak{U}\mathfrak{G} \subseteq \bar{\mathfrak{F}}_5$ centralizes the involution vector corresponding to T . Since $C_{\mathfrak{B}}(\bar{\mathfrak{L}}_1) = \mathfrak{B}_2$ and $C_{\mathfrak{B}}(\bar{\mathfrak{L}}_2) = \mathfrak{B}_1$ we see by our choice of T that $\bar{\mathfrak{L}} \neq \bar{\mathfrak{L}}_1$ or $\bar{\mathfrak{L}}_2$. But then $\bar{\mathfrak{L}}$ acts irreducibly on \mathfrak{B}_1 and \mathfrak{B}_2 so by the Jordan-Holder Theorem, $C_{\mathfrak{B}}(\bar{\mathfrak{L}}) = \langle 1 \rangle$, a contradiction.

Now let $|\bar{\mathfrak{F}}_5| = 5$. By the preceding lemma $\bar{\mathfrak{F}}$ is abelian. Since the irreducible nonprincipal representations of $\bar{\mathfrak{F}}_5$ over $GF(2)$ have degree 4 we see that either $\bar{\mathfrak{F}}$ is irreducible or it has two irreducible constituents of dimension 4. Thus $\bar{\mathfrak{F}}$ has two generators and $\bar{\mathfrak{F}}$ is abelian of type (5), (3, 5) or (3, 3, 5). Hence

$$|\bar{\mathfrak{G}}| \leq 3^2 |GL(2, 3)| \cdot 5 \cdot 4 = 8640 < 10^4,$$

a contradiction.

Thus $\bar{\mathfrak{F}} = \bar{\mathfrak{F}}_3$ is elementary abelian. If $|\bar{\mathfrak{F}}_3| \leq 3^2$, then

$$|\bar{\mathfrak{G}}| \leq 3^2 |GL(2, 3)| = 432 < 10^4,$$

a contradiction. If $|\bar{\mathfrak{F}}_3| = 3^3$, then $|\bar{\mathfrak{G}}|$ divides both $|\bar{\mathfrak{F}}_3| |GL(3, 3)| = 2^5 \cdot 3^6 \cdot 13$ and $|Sp(8, 2)| = 2^{16} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 17$ so $|\bar{\mathfrak{G}}|$ divides $2^5 \cdot 3^5 = 7776 < 10^4$, a contradiction. Since the Sylow 3-subgroup of $Sp(8, 2)$ is nonabelian of order 3^5 this leaves only $|\bar{\mathfrak{F}}_3| = 3^4$.

Let \mathfrak{C} be a 3-subgroup of \mathfrak{G} with $\mathfrak{C}\mathfrak{U}\mathfrak{G}/\mathfrak{U}\mathfrak{G} = \bar{\mathfrak{F}}_3$. Clearly $\mathfrak{C}' \subseteq \mathfrak{U}$. The action of \mathfrak{C} on \mathfrak{B} is completely reducible since $q \neq 3$ and since $\dim \mathfrak{B} = 2^4$ is not divisible by 3 it follows that \mathfrak{C}' is in the kernel of some constituent so \mathfrak{C}' has a fixed point in \mathfrak{B}^* . Since \mathfrak{U} acts semi-regularly, $\mathfrak{C}' = \langle 1 \rangle$. Now \mathfrak{C} is abelian and $\mathfrak{C}/(\mathfrak{C} \cap \mathfrak{U})$ is abelian of type (3, 3, 3, 3). Thus \mathfrak{C} contains a subgroup of type (3, 3, 3, 3) and hence $3^3 \mid |\mathfrak{G}_x|$.

Now $\mathfrak{G} \cong \mathfrak{Q}\mathfrak{Q}\mathfrak{Q}\mathfrak{Q}$ so it is clear that the automorphism group of \mathfrak{G} contains $\bar{\mathfrak{K}} = \bar{\mathfrak{F}} \times (\bar{\mathfrak{F}} \sim \bar{\mathfrak{F}})$ where $|\bar{\mathfrak{F}}| = 3$. This group is a Sylow 3-subgroup of $Sp(8, 2)$ and hence is a Sylow 3-subgroup of $\text{Aut } \mathfrak{G}$. We describe it more precisely. Write $\mathfrak{G} = \mathfrak{G}_0\mathfrak{G}_1\mathfrak{G}_2\mathfrak{G}_3$ where each $\mathfrak{G}_i \cong \mathfrak{Q}$. Then $\bar{\mathfrak{K}}$ has an elementary abelian subgroup $\bar{\mathfrak{N}}$ of index 3 with

$\bar{\mathfrak{N}} = \bar{\mathfrak{L}}_0 \bar{\mathfrak{L}}_1 \bar{\mathfrak{L}}_2 \bar{\mathfrak{L}}_3$. Here $\bar{\mathfrak{L}}_i$ acts nontrivially on \mathfrak{E}_i and centralizes the remaining \mathfrak{E}_j . Every element of $\bar{\mathfrak{R}} - \bar{\mathfrak{N}}$ normalizes \mathfrak{E}_0 and cyclically permutes $\mathfrak{E}_1, \mathfrak{E}_2$ and \mathfrak{E}_3 . Let $\mathfrak{W} = \mathfrak{W}_0 \oplus \mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3$ be the corresponding decomposition of \mathfrak{W} .

Let T be a noncentral involution of \mathfrak{G} . Then there exists $x \in \mathfrak{B}^*$ by Lemma 1.5 with $\mathfrak{E}_x = \langle T \rangle$. Since $3^3 \mid |\mathfrak{G}_x|$ let \mathfrak{L} be a subgroup of \mathfrak{G}_x of order 3^3 . Then \mathfrak{L} normalizes $\mathfrak{G}_x \cap \mathfrak{G} = \mathfrak{E}_x = \langle T \rangle$. Since $\mathfrak{L} \cap \mathfrak{A}\mathfrak{G} = \langle 1 \rangle$, \mathfrak{L} acts faithfully on \mathfrak{G} . Thus a suitable conjugate $\bar{\mathfrak{L}}$ of \mathfrak{L} in $\text{Aut } \mathfrak{G}$ is contained in $\bar{\mathfrak{R}}$ and clearly $\bar{\mathfrak{L}}$ also centralizes an involution vector of \mathfrak{W} . Let $W = W_0 + W_1 + W_2 + W_3 \in \mathfrak{W}$ with $W_i \in \mathfrak{W}_i$. Then we see easily that W is an involution vector if and only if either none or two of the W_i are zero. Suppose two of the W_i are zero. Then clearly $C_{\bar{\mathfrak{R}}}(W) \subseteq \bar{\mathfrak{N}}$ and then $|C_{\bar{\mathfrak{R}}}(W)| \leq 3^2$. If none of the W_i are zero, then $C_{\bar{\mathfrak{R}}}(W) \cap \bar{\mathfrak{N}} = \langle 1 \rangle$ so $|C_{\bar{\mathfrak{R}}}(W)| \leq 3$. This contradicts the fact that $|\bar{\mathfrak{L}}| = 3^3$ and $\bar{\mathfrak{L}}$ fixes an involution vector.

LEMMA 5.7. $\mathfrak{G} = \text{iso II}$ does not occur.

Proof. Here $i(\mathfrak{W}) = 7 \cdot 17$ by Lemma 1.3. Hence only $\bar{\mathfrak{G}}_7$ and $\bar{\mathfrak{G}}_{17}$ can be nontrivial. If $\bar{\mathfrak{G}}_7 \neq \langle 1 \rangle$ then since $7^2 \nmid |Sp(8, 2)|$, $|\bar{\mathfrak{G}}_7| = 7$. But the nonprincipal irreducible representations of this group over $GF(2)$ all have degree 3. Since $3 \nmid \dim \mathfrak{W}$ we have a contradiction. Then $\bar{\mathfrak{G}} = \bar{\mathfrak{G}}_{17}$ has order 1 or 17 and $|\bar{\mathfrak{G}}| \leq 17 \cdot 16 < 10^4$, a contradiction.

We have therefore shown in this section that if \mathfrak{G} is solvable then $m = 3$ and 4 do not occur.

6. **Theorem B.** The following assumption holds throughout this section.

ASSUMPTION. Group \mathfrak{G} acts faithfully on vector space \mathfrak{B} of order q^n , q a prime, and acts half-transitively but not semiregularly on \mathfrak{B}^* . Further \mathfrak{G} is primitive as a linear group and \mathfrak{G} is solvable.

Let $\mathcal{S}(q^n)$ denote the group of all semilinear transformations on $GF(q^n)$ of the form $x \mapsto ax^\sigma$ where $a \in GF(q^n)^*$ and σ is a field automorphism. Thus $\mathcal{S}(q^n)$ is the stabilizer in the permutation group $\mathcal{S}(q^n)$ of the point 0.

LEMMA 6.1. Let $\bar{\mathfrak{G}} = F(\mathfrak{G})$ and set $\mathfrak{A} = Z(C_{\bar{\mathfrak{G}}}(\Phi(\bar{\mathfrak{G}})))$. Then \mathfrak{A} is a normal cyclic subgroup of $\bar{\mathfrak{G}}$

(i) If $\mathfrak{A} = C_{\bar{\mathfrak{G}}}(\Phi(\bar{\mathfrak{G}}))$, then with suitable identification we have $\bar{\mathfrak{G}} \subseteq \mathcal{S}(q^n)$.

(ii) If $\mathfrak{A} \neq C_{\bar{\mathfrak{G}}}(\Phi(\bar{\mathfrak{G}}))$, then $C_{\bar{\mathfrak{G}}}(\Phi(\bar{\mathfrak{G}})) = \mathfrak{A}\mathfrak{G}$ where \mathfrak{G} is a group of type $E(2, m)$ and $\mathfrak{G} \triangle \bar{\mathfrak{G}}$. Moreover $m = 1$ or 2.

(iii) In the above if $m = 1$ and $4 \nmid |\mathfrak{A}|$, then either $\mathfrak{G} \subseteq \mathcal{S}(q^n)$ or $q^n = 3^2, 7^2$ or 11^2 .

Proof. Let \mathfrak{F}_p be the normal Sylow p -subgroup of \mathfrak{F} . By Theorem A \mathfrak{F}_p is cyclic for $p > 2$ and \mathfrak{F}_2 is a group of symplectic type. Since $\mathfrak{A} = Z(C_{\mathfrak{F}}(\Phi(\mathfrak{F})))$ is a normal abelian subgroup of a primitive group it is cyclic.

From the structure of 2-groups of symplectic type we see that if $\mathfrak{A} = C_{\mathfrak{F}}(\Phi(\mathfrak{F}))$, then \mathfrak{F}_2 is either cyclic or maximal class of order at least 16. Now $\mathfrak{F} = \mathfrak{A}\mathfrak{F}_2$ so $C_{\mathfrak{G}}(\mathfrak{A})/Z(\mathfrak{F})$ acts faithfully on \mathfrak{F}_2 . Since $\text{Aut } \mathfrak{F}_2$ is a 2-group and $Z(\mathfrak{F}) \subseteq \mathfrak{A}$ we see that $C_{\mathfrak{G}}(\mathfrak{A})$ is a normal nilpotent subgroup of \mathfrak{G} and hence $C_{\mathfrak{G}}(\mathfrak{A}) \subseteq \mathfrak{F}$. This yields easily $C_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{A}$. By Proposition 1.2 of [5] we see that $\mathfrak{G} \subseteq \mathcal{S}(q^n)$ and (i) follows.

Suppose $\mathfrak{A} \neq C_{\mathfrak{F}}(\Phi(\mathfrak{F}))$. Then as we pointed out in § 1, $C_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{C}$ where \mathfrak{C} is a group of type $E(2, m)$ and $\mathfrak{C} \triangle \mathfrak{G}$. By Theorem A and the results of § 5, $m = 1$ or 2 .

Let $m = 1$ and suppose $4 \nmid |\mathfrak{A}|$. Then $\mathfrak{F}_2 \cong \mathfrak{D}$ or \mathfrak{Q} . If $\mathfrak{F}_2 \cong \mathfrak{D}$ then \mathfrak{F} has a characteristic cyclic subgroup \mathfrak{B} of index 2. Since $\text{Aut } \mathfrak{D}$ is a 2-group, the above argument yields $\mathfrak{G} \subseteq \mathcal{S}(q^n)$ again. If $\mathfrak{F}_2 \cong \mathfrak{Q}$, then by Proposition 1.10 of [5] $q^n = 3^2, 7^2$ or 11^2 . This completes the proof.

We assume now that $\mathfrak{A} \neq C_{\mathfrak{F}}(\Phi(\mathfrak{F}))$.

LEMMA 6.2. *Let $\mathfrak{B} = C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}\mathfrak{C}$. Then $O_2(\mathfrak{B}) = \langle 1 \rangle$, \mathfrak{B} acts faithfully on $\mathfrak{C}/Z(\mathfrak{C})$ and $\mathfrak{B} \subseteq \text{Sp}(2m, 2)$.*

Proof. Let $\mathfrak{L}/\mathfrak{A}\mathfrak{C} = O_2(\mathfrak{B})$. Since \mathfrak{A} is central in \mathfrak{L} and $\mathfrak{L}/\mathfrak{A}$ is a 2-group, we see that \mathfrak{L} is a normal nilpotent subgroup of \mathfrak{G} and hence $\mathfrak{L} \subseteq \mathfrak{F}$. Now $\Phi(\mathfrak{F}) \subseteq \mathfrak{A}$ and $C_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{C}$. Hence

$$\mathfrak{L} \subseteq C_{\mathfrak{F}}(\mathfrak{A}) \subseteq C_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{C}$$

so $\mathfrak{L} = \mathfrak{A}\mathfrak{C}$ and $O_2(\mathfrak{B}) = \langle 1 \rangle$.

Let $\mathfrak{H} = C_{\mathfrak{G}}(\mathfrak{A})$ and let $\mathfrak{K} = C_{\mathfrak{B}}(\mathfrak{B})$ where $\mathfrak{B} = \mathfrak{C}/Z(\mathfrak{C})$. We have of course $\mathfrak{K} \supseteq \mathfrak{A}\mathfrak{C}$. First \mathfrak{K} centralizes $O_2(\mathfrak{F}) \subseteq \mathfrak{A}$. If $\mathfrak{F}_2 = O_2(\mathfrak{F})$, then since clearly $[\mathfrak{F}_2 : \mathfrak{A}_2\mathfrak{C}] = 2$, where $\mathfrak{A}_2 = \mathfrak{A} \cap \mathfrak{F}_2$, we see that \mathfrak{K} stabilizes the chain $\mathfrak{F}_2 \supseteq \mathfrak{A}_2\mathfrak{C} \supseteq \mathfrak{A}_2 \supseteq \langle 1 \rangle$. Thus $\mathfrak{K}/C_{\mathfrak{K}}(\mathfrak{F})$ is a 2-group. Since $\mathfrak{K} \supseteq Z(\mathfrak{F})$, $C_{\mathfrak{K}}(\mathfrak{F}) = Z(\mathfrak{F})$ and hence $\mathfrak{K}/Z(\mathfrak{F})$ is a 2-group. But $Z(\mathfrak{F}) \subseteq \mathfrak{A}$ and \mathfrak{A} is central in \mathfrak{K} so \mathfrak{K} is a normal nilpotent subgroup of \mathfrak{G} and $\mathfrak{K} \subseteq \mathfrak{F}$. This yields easily $\mathfrak{K} = \mathfrak{A}\mathfrak{C}$ and thus $\mathfrak{B} = \mathfrak{H}/\mathfrak{K}$ acts faithfully on \mathfrak{B} . It now follows immediately that $\mathfrak{B} \subseteq \text{Sp}(2m, 2)$.

LEMMA 6.3. *Let $\mathfrak{A} = \langle A \rangle$ and let ζ be an eigenvalue of A with $GF(q)(\zeta) = GF(q^r)$. Then*

- (i) $C_{\mathfrak{G}}(\mathfrak{A}) \subseteq GL(n/r, q^r)$, $|\mathfrak{A}| \mid (q^r - 1)$
- (ii) $\mathfrak{G}/C_{\mathfrak{G}}(\mathfrak{A})$ is cyclic of order dividing r .
- (iii) $n = w2^m r$ for some integer w .

Proof. Parts (i) and (ii) follow from Lemma 1.1 of [5]. Now all irreducible constituents of \mathfrak{G} are faithful and the same is clearly true if we view $\mathfrak{G} \subseteq GL(n/r, q^r)$. Thus n/r is divisible by 2^m , the degree of the nonlinear absolutely irreducible representations of \mathfrak{G} .

LEMMA 6.4. *If $m = 1$ and $4 \mid |\mathfrak{A}|$, then $q^n = 5^2$ or 17^2 .*

Proof. We can assume that $|Z(\mathfrak{G})| = 4$ so $\mathfrak{G} \cong 3\mathfrak{D}$. By the Reduction Lemma and Lemma 3.4, $q = 3, 5$ or 17 . Set $\mathfrak{H} = C_{\mathfrak{G}}(\mathfrak{A})$. Then by the above $\mathfrak{H}/\mathfrak{A}\mathfrak{H} = \mathfrak{B}$ is contained isomorphically in $Sp(2, 2) = SL(2, 2) \cong \text{Sym}_3$. Since $O_2(\mathfrak{B}) = \langle 1 \rangle$, $|\mathfrak{B}| = 1, 3$ or 6 .

Suppose $|\mathfrak{B}| = 1$. Now $2 \mid |\mathfrak{G}_x|$ so we can apply Lemma 3.1 with $p = 2$. Note that $\mathfrak{G}/\mathfrak{A}\mathfrak{G}$ is cyclic of order dividing r and $k = n/r = 2w$. If r is odd, then $\lambda_1 \leq 3$, $\lambda_2 = 0$ so by Lemma 3.1, (ii) and (iii), we have $q^r < 6$ so $r = 1$. If r is even, then $\lambda_1 \leq 3$, $\lambda_2 \leq 4$ so we get easily $q^{r/2} \leq 5$ and hence $r = 2$. Now \mathfrak{G} has precisely three normal abelian subgroups of type $(2, 2)$. Since $\mathfrak{G}/\mathfrak{A}\mathfrak{G}$ is a 2-group one of these three abelian groups will be normal in \mathfrak{G} , a contradiction since \mathfrak{G} is primitive. Thus $|\mathfrak{B}| = 3$ or 6 .

Suppose $3 \mid |\mathfrak{G}_x|$. We again apply Lemma 3.1. If $3 \nmid r$ then $\lambda_1 \leq 4$, $\lambda_2 = 0$ while if $3 \mid r$, then $\lambda_1 \leq 4$ and we see easily that $\lambda_2 \leq 9$. Let $3 \nmid r$ so by Lemma 3.1 we have $q^r < 8$. Since $4 \mid q^r - 1$, $q^r = 5$ and then by Lemma 3.1 (i) we have $k = 2$ and $n = 2$. But $3 \nmid q - 1$ so no element of $GL(2, 5)$ of order 3 can have a nonzero fixed point, a contradiction. Let $3 \mid r$. Then Lemma 3.1, (ii) and (iii), yields $q^{r/3} < 4$ so $q^r = 3^3$. This is a contradiction since $4 \nmid (3^3 - 1)$. Now $3 \mid |\mathfrak{G}|$ so we see also that $q \neq 3$ and thus $q = 5$ or 17 . We assume that $q^n \neq 5^2$ or 17^2 and derive a contradiction.

Suppose first that r is odd. We apply Lemma 3.1 with $p = 2$. Then $\lambda_1 \leq 9$, $\lambda_2 = 0$ so we have $q^r < 18$. Thus $q^r = 5$ or 17 and $r = 1$. By Lemma 1.5, there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{G} = \langle 1 \rangle$. Since $r = 1$ and $3 \nmid |\mathfrak{G}_x|$ we have $|\mathfrak{G}_x| = 2$. Hence by Lemma 1.9, $I(\mathfrak{G}) = q^{n/2} + 1$. Now \mathfrak{A} is central in \mathfrak{G} and cyclic so each involution of $\mathfrak{G}/\mathfrak{A}$ corresponds to at most two noncentral involutions of \mathfrak{G} . Thus

$$q^{n/2} + 1 = I(\mathfrak{G}) \leq 2 \cdot 9 = 18$$

so $q^n = 5^2$ or 17^2 , a contradiction.

Now let r be even. We have easily $\lambda_1 \leq 9$, $\lambda_2 \leq 10$. Thus if $k > 2$ then Lemma 3.1 (iii) yields $q^r = 5^2$ and then by Lemma 3.1 (i)

with $k > 2$ we have a contradiction. Thus $k = 2$ and by Lemma 3.1 (ii), $q^r + 1 \leq 18 + 10(q^{r/2} + 1)$ so $q^{r/2} < 13$. Since r is even $q^r = 5^2$. By Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{C} = \langle 1 \rangle$ and hence since $3 \nmid |\mathfrak{G}_x|$ we have $|\mathfrak{G}_x| = 2$ or 4 .

Suppose $|\mathfrak{G}_x| = 4$. Since $[\mathfrak{G} : \mathfrak{G}] = 2$ where $\mathfrak{G} = C(\mathfrak{A})$ we see that $2 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{B}^*$. Clearly \mathfrak{G} acts irreducibly on \mathfrak{B} so by Lemma 3.1 applied to \mathfrak{G} with $p = 2$ we have $\lambda_1 \leq 9$, $\lambda_2 = 0$ so $25 = q^r < 18$, a contradiction. Thus $|\mathfrak{G}_x| = 2$.

Now here $n = kr = 4$. By Lemma 1.9, we have $I(\mathfrak{G}) = 1 + q^{n/2} = 26$. Let \mathfrak{L} be a Sylow 3-subgroup of \mathfrak{G} . Since $3 \nmid |\mathfrak{G}_x|$, \mathfrak{L} is cyclic and acts semiregularly so $|\mathfrak{L}| \mid 5^4 - 1$ and $|\mathfrak{L}| = 3$. Since $3 \mid |\mathfrak{B}|$ we have $\mathfrak{L} \cap \mathfrak{A}\mathfrak{C} = \langle 1 \rangle$. Now \mathfrak{L} permutes by conjugation the noncentral involutions of \mathfrak{G} and since $3 \nmid I(\mathfrak{G})$ we see that \mathfrak{L} centralizes a noncentral involution of \mathfrak{G} . The group $\mathfrak{G}/\mathfrak{A}\mathfrak{C}$ acts on $\mathfrak{B} = \mathfrak{C}/Z(\mathfrak{C})$. If the action is faithful then clearly $\mathfrak{G}/\mathfrak{A}\mathfrak{C} \subseteq \text{Sym}_4$. Since subgroups of order 3 of Sym_4 are self-centralizing we have a contradiction. Hence the action is not faithful so say $\mathfrak{R}/\mathfrak{A}\mathfrak{C}$ is the kernel with $\mathfrak{R} > \mathfrak{A}\mathfrak{C}$. Now $\mathfrak{G}/\mathfrak{A}\mathfrak{C}$ does act faithfully so $[\mathfrak{R} : \mathfrak{A}\mathfrak{C}] = 2$. Note that $\mathfrak{R} \trianglelefteq \mathfrak{G}$. Also $3 \nmid |\mathfrak{A}|$ and $|\mathfrak{A}| \mid 5^2 - 1$ implies \mathfrak{A} is a 2-group and hence \mathfrak{R} is a 2-group. Since \mathfrak{G} is primitive, \mathfrak{R} is of symplectic type. Moreover $Z(\mathfrak{C}) \trianglelefteq \mathfrak{G}$, $|Z(\mathfrak{C})| = 4$ and $4 \mid q - 1$. Hence $Z(\mathfrak{C})$ is central in \mathfrak{G} so \mathfrak{R} must be the central product of a cyclic group with a nonabelian group of order 8. Now $\mathfrak{R} \subseteq \mathfrak{F}$ and since $[\mathfrak{F} : \mathfrak{A}\mathfrak{C}] \leq 2$ we have $\mathfrak{R} = \mathfrak{F}$. Then $\Phi(\mathfrak{F})$ is central in \mathfrak{F} and $\mathfrak{F} = C_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{C}$, a contradiction. This completes the proof of the lemma.

LEMMA 6.5. *If $m = 2$, then $q^n = 3^4$.*

Proof. By the Reduction Lemma and Lemmas 4.3, 4.4 and 4.5 we have $q = 3$ and $\mathfrak{C} \cong \mathfrak{DQ}$. Hence $4 \nmid |\mathfrak{A}|$. We consider $\bar{\mathfrak{R}} = F(\mathfrak{G}/\mathfrak{A}\mathfrak{C})$. By Lemma 6.2, $\bar{\mathfrak{R}}_2 = O_2(\bar{\mathfrak{R}}) = \langle 1 \rangle$. Suppose $\bar{\mathfrak{R}}_3 = O_3(\bar{\mathfrak{R}}) \neq \langle 1 \rangle$. Since $q = 3$, a Sylow 3-subgroup of \mathfrak{G} has a fixed point in \mathfrak{B}^* and hence by half-transitivity \mathfrak{G}_x contains a Sylow 3-subgroup of \mathfrak{G} for all $x \in \mathfrak{B}^*$. Let T be a noncentral involution of \mathfrak{C} . By Lemma 1.5 there exists $x \in \mathfrak{B}^*$ with $\mathfrak{C} = \langle T \rangle$. Now we can find 3-subgroup \mathfrak{L} of \mathfrak{G}_x such that $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{C}/\mathfrak{A}\mathfrak{C} = \bar{\mathfrak{R}}_3$. Since \mathfrak{L} normalizes $\mathfrak{C} \cap \mathfrak{G}_x = \mathfrak{C}_x$ we see that $\bar{\mathfrak{R}}_3$ centralizes the involution vector corresponding to T . Thus $\bar{\mathfrak{R}}_3$ centralizes all the involution vectors of $\mathfrak{B} = \mathfrak{C}/Z(\mathfrak{C})$ so by Lemma 6.3, $\bar{\mathfrak{R}}_3 = \langle 1 \rangle$.

Now $\mathfrak{B} \subseteq Sp(4, 2)$ and $|Sp(4, 2)| = 2^4 \cdot 3^2 \cdot 5$. Since $\bar{\mathfrak{R}} = O_6(\mathfrak{B})$ by the above we have $|\bar{\mathfrak{R}}| = 1$ or 5 and hence $|\mathfrak{B}| \leq 20$ and $|\mathfrak{G}/\mathfrak{A}\mathfrak{C}| \leq 16 \cdot 20 = 320$. We use Lemma 3.1 with $p = 2$. Note that $k = n/r \geq 2^m = 4$ so Lemma 3.1 (iii) always applies. Certainly $\lambda_2 \leq 320$. From the

structure of $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{A}$ we see that $\lambda_1 \leq 15 + 5 \cdot 4 = 35$. Hence

$$q^r < 2(\lambda_1 + \lambda_2) = 710.$$

Since $q = 3$, this yields $r \leq 5$. However if $r = 5$, then $[\mathfrak{G} : \bar{\mathfrak{G}}]$ is odd so $\lambda_2 = 0$ and then $q^r < 2\lambda_1 = 70$, a contradiction. Thus $r \leq 4$.

Since $r \leq 4$ we see that $\bar{\mathfrak{R}}$ is a Sylow 5-subgroup of $\mathfrak{G}/\mathfrak{A}\mathfrak{C}$. Hence if $5 \mid |\mathfrak{G}_x|$, then as in the preceding argument with $\bar{\mathfrak{R}}_3$ we conclude that $\bar{\mathfrak{R}}$ fixes all involution vectors of $\mathfrak{B} = \mathfrak{G}/Z(\mathfrak{G})$ and thus $\bar{\mathfrak{R}} = \langle 1 \rangle$. This certainly contradicts $5 \mid |\mathfrak{G}_x|$. Hence $5 \nmid |\mathfrak{G}_x|$. Let T be a noncentral involution of \mathfrak{G} . We show that $|C_{\mathfrak{B}}(T)| \leq q^{n/2}$. This is certainly the case if $T \in \mathfrak{A}\mathfrak{C}$. Let $T \in \mathfrak{G} - \mathfrak{A}\mathfrak{C}$. Then $|\bar{\mathfrak{R}}| = 5$ since $\mathfrak{G}/\mathfrak{A}\mathfrak{C} \neq \langle 1 \rangle$. Clearly there exists $K \in \mathfrak{G}$ so that the image of $\langle T, T^K \rangle$ in $\mathfrak{G}/\mathfrak{A}\mathfrak{C}$ contains $\bar{\mathfrak{R}}$. Since $5 \nmid |\mathfrak{G}_x|$ we see that $C_{\mathfrak{B}}(T) \cap C_{\mathfrak{B}}(T^K) = \{0\}$. Thus the result follows here. Finally if $T \in \mathfrak{G} - \mathfrak{G}$, then there exists $A \in \mathfrak{A}$ with $\langle T, T^A \rangle \cap \mathfrak{A} \neq \langle 1 \rangle$. Since \mathfrak{A} acts semiregularly the result follows.

We show that r is not even. If r is even, then $r = 2$ or 4 . If $r = 2$ then $|\mathfrak{A}| \mid q^r - 1$ and $q^r - 1 = 8$. Since $4 \nmid |\mathfrak{A}|$ we have $|\mathfrak{A}| = 2$ and $|\mathfrak{A}| \mid q - 1$. This violates the definition of r and hence $r = 4$. Here $|\mathfrak{A}| \mid q^r - 1$ and $q^r - 1 = 2^4 \cdot 5$ so $|\mathfrak{A}| \mid 10$. Since $|\mathfrak{A}| \leq 10$ each involution of $(\mathfrak{G} - \mathfrak{G})/\mathfrak{A}$ comes from at most 10 of $\mathfrak{G} - \mathfrak{G}$. Thus

$$I(\mathfrak{G}) \leq 2 \cdot 35 + 10 \cdot 320 = 3270.$$

Since $2 \mid |\mathfrak{G}_x|$ we have $\mathfrak{B} = \bigcup C_{\mathfrak{B}}(T)$ over involutions T and hence

$$q^n = |\mathfrak{B}| \leq I(\mathfrak{G})q^{n/2} \leq 3270q^{n/2}$$

so $q^{n/2} \leq 3270$. Thus $n < 16$. But $r = 4$ and $n \geq 2^m r = 16$ so we have a contradiction. Thus r is odd.

Since r is odd, all involutions of \mathfrak{G} are contained in \mathfrak{G} . Now \mathfrak{A} is cyclic and central in \mathfrak{G} so each involution of $\mathfrak{G}/\mathfrak{A}$ comes from at most two of \mathfrak{G} . Hence $I(\mathfrak{G}) \leq 2 \cdot 35 = 70$ and since $2 \mid |\mathfrak{G}_x|$ we have

$$q^n = |\mathfrak{B}| \leq I(\mathfrak{G})q^{n/2} \leq 70q^{n/2}$$

or $q^{n/2} \leq 70$. Since $4 \mid n$ we have $n = 4$ and thus $r = 1$. This completes the proof of the lemma.

Combining Lemmas 6.1, 6.4 and 6.5 we obtain

THEOREM 6.6. *Let \mathfrak{G} act faithfully on vector space \mathfrak{B} of order q^n and let \mathfrak{G} act half-transitively but not semiregularly on \mathfrak{B}^* . If \mathfrak{G} is primitive as a linear group and if \mathfrak{G} is solvable, then \mathfrak{G} satisfies one of the following.*

- (i) $\mathfrak{G} \subseteq \mathcal{S}(q^n)$.
- (ii) $q^n = 3^2, 5^2, 7^2, 11^2, 17^2$ or 3^4 .

The proof of the main theorem now follows easily.

Proof of Theorem B. Let \mathfrak{G} be the given solvable 3/2-transitive permutation group and assume that \mathfrak{G} is not a Frobenius group. By Theorem 10.4 of [11], \mathfrak{G} is primitive. Let \mathfrak{B} be a minimal normal subgroup of \mathfrak{G} . Since \mathfrak{G} is solvable, \mathfrak{B} is elementary abelian of order q^n . Since \mathfrak{G} is primitive, \mathfrak{B} is transitive and hence regular. If α is a point being permuted, then by Theorem 11.2 of [11], \mathfrak{G}_α is an automorphism group of \mathfrak{B} which acts half-transitively but not semiregularly on \mathfrak{B}^* . By Theorems 1.1 and 6.6 we have $\mathfrak{G}_\alpha = \mathcal{S}_0(q^{n/2})$, $\mathfrak{G}_\alpha \subseteq \mathcal{S}(q^n)$ or $q^n = 3^2, 5^2, 7^2, 11^2, 17^2, 3^4$. Note that the exception of Theorem 1.1 of degree 2^6 is a subgroup of $\mathcal{S}(2^6)$. Since $\deg \mathfrak{G} = q^n$ and $\mathfrak{G} = \mathfrak{B}\mathfrak{G}_\alpha$, the result follows.

7. Theorem C. We can now obtain several easy corollaries.

COROLLARY 7.1. *Let \mathfrak{G} be a solvable 3/2-transitive permutation group. Then for all points $\alpha \neq \beta$ the stabilizers $\mathfrak{G}_{\alpha\beta}$ are isomorphic. In fact if $q^n \neq 3^2$, then $\mathfrak{G}_{\alpha\beta}$ is cyclic, while if $q^n = 3^2$, then $\mathfrak{G}_{\alpha\beta} \subseteq \text{Sym}_3$.*

Proof. The result is clear if \mathfrak{G} is a Frobenius group, $\mathfrak{G} \subseteq \mathcal{S}(q^n)$ or $\mathfrak{G} = \mathcal{S}_0(q^n)$. Thus we need only consider the exceptions. Here \mathfrak{G}_α acts on \mathfrak{B} and $\mathfrak{G}_{\alpha\beta}$ is the stabilizer of $\beta \in \mathfrak{B}^*$. Suppose $q^n = 5^2, 7^2, 11^2$ or 17^2 . Since we see easily that $|\mathfrak{G}_\alpha|$ is prime to q it follows by complete reducibility that $\mathfrak{G}_{\alpha\beta}$ has a faithful 1-dimensional representation and hence is cyclic. Suppose $q^n = 3^2$. Since $\mathfrak{G}_\alpha \cong \mathfrak{G} \cong \mathfrak{D}$ we see that \mathfrak{G}_α is transitive on \mathfrak{B}^* . Also $\mathfrak{G}_\alpha/\mathfrak{G} \subseteq \text{Sym}_3$ and $\mathfrak{G}_{\alpha\beta} \cap \mathfrak{G} = \langle 1 \rangle$ so the result follows here. Finally let $q^n = 3^4$ so that $\mathfrak{G} \triangle \mathfrak{G}_\alpha$ with $\mathfrak{G} \cong \mathfrak{D}\mathfrak{D}$. Then $\mathfrak{G} = \mathcal{O}_2(\mathfrak{G}_\alpha)$. If $\mathfrak{G} = \mathfrak{G}_\alpha$ then $|\mathfrak{G}_{\alpha\beta}| = 2$. If $\mathfrak{G}_\alpha > \mathfrak{G}$ then as we have seen $5 \mid |\mathfrak{G}_\alpha/\mathfrak{G}|$. This implies that \mathfrak{G}_α acts transitively on \mathfrak{B}^* . The result now follows by Lemma 2.4 of [5].

COROLLARY 7.2. *Let \mathfrak{G} be a solvable linear group acting on $GF(q)$ -vector space \mathfrak{B} . Suppose \mathfrak{G} acts half-transitively on \mathfrak{B}^* . If $q \neq 2$ and $|\mathfrak{G}|$ is even, then \mathfrak{G} has a central involution.*

Proof. The result is well known if \mathfrak{G} acts semi-regularly and obvious in all of the remaining cases with the exception of $\mathfrak{G} \subseteq \mathcal{S}(q^n)$. Here the argument of Step 1 of the proof of Proposition 2.7 of [8] yields the result.

Finally we consider the transitive extensions of these exceptional 3/2-transitive groups.

Proof of Theorem C. Let \mathfrak{G} be a 5/2-transitive permutation group on the set Ω and assume that \mathfrak{G} is not a Zassenhaus group. Let $\infty, 0 \in \Omega$ and assume that \mathfrak{G}_∞ is solvable. Thus \mathfrak{G}_∞ is a solvable 3/2-transitive group which is not a Frobenius group. If $\mathfrak{G}_\infty \subseteq \mathcal{S}(q^n)$ or $\mathfrak{G}_\infty = \mathcal{S}_0(q^{n/2})$ then by the results of [8], $\bar{\Gamma}(q^n) < \mathfrak{G} \subseteq \Gamma(q^n)$. Hence we need only consider the exceptional groups. We show that these have no transitive extensions.

Set $\mathfrak{H} = \mathfrak{G}_{\infty,0}$ so that $\mathfrak{G}_\infty = \mathfrak{H}\mathfrak{V}$ where \mathfrak{V} is a regular normal elementary abelian subgroup of order q^n . Let Z denote the central involution of \mathfrak{H} . Then Z fixes 0 and ∞ and moves all the rest. Since \mathfrak{G} is doubly transitive we can find a suitable conjugate T of Z with $T = ((0, \infty)) \dots$. Thus T normalizes \mathfrak{H} . By Lemma 1.3 of [8], $|\mathfrak{H}| \geq (q^n - 1)/2$. If $q^n = 17^2$, then by Lemma 3.5

$$96 = |\mathfrak{H}| \geq (17^2 - 1)/2.$$

a contradiction.

We will use results of § 3 and § 4 about these exceptional groups which were not explicitly stated. Let $\mathfrak{G} = O_2(\mathfrak{H})$ so that T normalizes \mathfrak{G} . Suppose T fixes the point α . Since T centralizes Z we see that $(\alpha Z)T = \alpha TZ = \alpha Z$ so T also fixes $\beta = \alpha Z$ and these must be the two points of Ω fixed by T . Since T is conjugate to Z and Z is central in $\mathfrak{G}_{\infty,0}$ we see that T is central in $\mathfrak{G}_{\alpha\beta}$. Thus T centralizes $\mathfrak{H}_{\alpha\beta}$. Note that $\mathfrak{H}_{\alpha\beta} = \mathfrak{H}_\alpha = \mathfrak{H}_\beta$ since $\alpha Z = \beta$. Conversely let T centralize $H \in \mathfrak{H}$. Then $(\alpha H)T = \alpha TH = \alpha H$ so $\alpha H = \alpha$ or β . Hence $H \in \langle Z, \mathfrak{H}_\alpha \rangle$ and hence $C_{\mathfrak{H}}(T) = \langle Z, \mathfrak{H}_\alpha \rangle$.

Suppose $3 \nmid |\mathfrak{H}_x|$ for $x \in \mathfrak{B}^*$. This implies easily that $q^n = 3^2$ or 7^2 and $\mathfrak{G} \cong \Omega$. Since \mathfrak{G} acts semiregularly on \mathfrak{B}^* , $C_{\mathfrak{G}}(T) = \langle Z \rangle$ and thus T acts nontrivially on $\mathfrak{G}/Z(\mathfrak{G})$. Let \mathfrak{J} be a subgroup of \mathfrak{H}_α of order 3. Then $\langle T, \mathfrak{J} \rangle$ is cyclic of order 6 and acts faithfully on $\mathfrak{G}/Z(\mathfrak{G})$, a contradiction. Thus $|\mathfrak{H}_x|$ is a cyclic 2-group. Note that if $q^n = 3^2$, then $3 \nmid |\mathfrak{H}|$ so clearly $\mathfrak{H} \subseteq \mathcal{S}(3^2)$ and \mathfrak{G}_∞ is not exceptional.

Set $\mathfrak{K} = \mathfrak{G} \cap \text{Alt } \Omega$. Since $\mathfrak{K} \supseteq \mathfrak{V}$, $T, Z \in \mathfrak{K}$ is doubly transitive and $\mathfrak{K}_{\infty,0}$ has a central involution. Also $[\mathfrak{G} : \mathfrak{K}] \leq 2$. Let $q^n = 7^2$ or 11^2 . Then $|\mathfrak{H}_x| \mid q - 1$ so clearly $|\mathfrak{H}_x| = 2$. If H is a noncentral involution of \mathfrak{H} then H moves $q^2 - q$ points and hence H is a product of $q(q - 1)/2$ transpositions. Thus with $q = 7$ or 11 , $H \notin \mathfrak{K}$ and therefore \mathfrak{K} is a Zassenhaus group. Since $\mathfrak{K}_{\infty,0}$ has a central involution the results of [12] yield $\mathfrak{K} \subseteq \mathcal{S}(q^2)$ and hence $\mathfrak{K}_{\infty,0}$ has a normal Sylow 3-subgroup, a contradiction. This leaves only $q^n = 5^2$ and 3^4 .

Let $q^n = 5^2$. Suppose $H \in \mathfrak{H}$ has order 4 and fixes a point of \mathfrak{B}^* .

Since H and H^2 fix the same set of points here, we see that H is a product of $(5^2 - 5)/4 = 5$ 4-cycles. Thus $H \notin \mathfrak{R}$. Now \mathfrak{G}_∞ is exceptional so $3 \nmid |\mathfrak{G}_{\infty 0}|$ and hence by the above remarks $|\mathfrak{R}_{\infty 0}| = 16 \cdot 3 = 48$. Thus $|\mathfrak{R}| = 26 \cdot 25 \cdot 48$. Let \mathfrak{P} be a Sylow 13-subgroup of \mathfrak{R} . Then $[\mathfrak{R} : \mathfrak{P}] = 2 \cdot 25 \cdot 48 \equiv 8 \pmod{13}$. If $\mathfrak{N} = N_{\mathfrak{R}}(\mathfrak{P})$, then by Sylow's theorem, $[\mathfrak{N} : \mathfrak{P}] \equiv 8 \pmod{13}$. We see easily that \mathfrak{P} has two orbits of size 13. If \mathfrak{A} is an abelian subgroup of \mathfrak{R} containing \mathfrak{P} , then either \mathfrak{A} has two orbits and then $\mathfrak{A} = \mathfrak{P}$ or \mathfrak{A} is transitive. In the latter case \mathfrak{A} is regular so if $A \in \mathfrak{A}$ has order 2, then A is a product of 13 transpositions and $A \notin \text{Alt } \Omega$, a contradiction. Hence $\mathfrak{A} = \mathfrak{P}$ and $\mathfrak{P} = C_{\mathfrak{R}}(\mathfrak{P})$. Thus $\mathfrak{N}/\mathfrak{P} \subseteq \text{Aut } \mathfrak{P}$ so $[\mathfrak{N} : \mathfrak{P}] \mid 12$. Since $[\mathfrak{N} : \mathfrak{P}] \equiv 8 \pmod{13}$, we have a contradiction.

Finally let $q^n = 3^4$ so that \mathfrak{G} has degree $3^4 + 1 = 2 \cdot 41$. Now $|\mathfrak{G}| \geq (q^n - 1)/2 = 40$ so we cannot have $\mathfrak{G} \cong \mathfrak{D}\Omega$. Hence we must have $5 \mid |\mathfrak{G}|$ so \mathfrak{G} is transitive on \mathfrak{P}^* and we thus see easily that \mathfrak{R} is triply transitive. Now $|\mathfrak{G}_x| = 2, 4$ or 8 so write $|\mathfrak{R}_x| = 2 \cdot 2^s$ where $2^s = 1, 2$ or 4 . Then

$$|\mathfrak{R}| = 82(82 - 1)(82 - 2) \cdot 2 \cdot 2^s.$$

Let \mathfrak{P} be a Sylow 41-subgroup of \mathfrak{R} so that $[\mathfrak{R} : \mathfrak{P}] \equiv 8 \cdot 2^s \pmod{41}$. Hence if $\mathfrak{N} = N_{\mathfrak{R}}(\mathfrak{P})$, then $[\mathfrak{N} : \mathfrak{P}] \equiv 8 \cdot 2^s \pmod{41}$. As in the $q^n = 5^2$ case we see easily that \mathfrak{P} is self-centralizing so $\mathfrak{N}/\mathfrak{P} \subseteq \text{Aut } \mathfrak{P}$ and $[\mathfrak{N} : \mathfrak{P}] \mid 40$. Since $2^s \leq 4$ this yields $2^s = 1$ and $[\mathfrak{N} : \mathfrak{P}] = 8$.

The fact that $2^s = 1$ implies that $\mathfrak{G} \cong \mathfrak{D}\Omega$ is normal in $\mathfrak{R}_{\infty 0}$ and $[\mathfrak{R}_{\infty 0} : \mathfrak{G}] = 5$. Since $\mathfrak{N}/\mathfrak{P}$ is cyclic, let $\mathfrak{L} = \langle L \rangle$ be a subgroup of \mathfrak{N} of order 8. \mathfrak{L} permutes the two orbits of \mathfrak{P} . If it fixes each then L clearly has fixed points in each orbit. Thus some conjugate of L is contained in $\mathfrak{R}_{\infty 0}$, a contradiction since $\mathfrak{G} \cong \mathfrak{D}\Omega$ has period 4. Thus \mathfrak{L} interchanges the two orbits. This implies easily that L is a product of ten 8-cycles and one transposition. Hence L is an odd permutation, a contradiction. This completes the proof of the theorem.

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