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A SPECIAL DEFORMATION OF THE METRIC WITH NO NEGATIVE SECTIONAL CURVATURE OF A RIEMANNIAN SPACE

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The main results of this paper can be stated as follows. Let M_1 , M_2 be two big open submanifolds of the Riemannian manifolds (R_1^2, h_1) and (R_2^2, h_2) , respectively. The submanifolds M_1, M_2 with the metrics h_1/M_1 and h_2/M_2 , respectively, have positive constant sectional curvature. We have constructed a special I-parameter family of Riemannian metrics d(t) on $M_1 \times M_2$ which is the deformation of the product metric $h_1/M_1 \times h_2/M_2$ and it has strictly positive sectional curvature. In other words, we have proved that $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature with respect to the parameter t for t=0 and for any plane which is spanned by $X \in (M_1)_p$ and $Y \in (M_2)_p$ is strictly positive.

Let S^2 be a two-dimensional sphere with the canonical metric g whose sectional curvature is positive constant. Consider the product of two manifolds $S^2 \times S^2$. It is not known, ([1], p. 287), ([4], p, 171), ([11], p. 106), if there exists a deformation of the metric $g \times g$ with strictly positive sectional curvature.

Let \mathbb{R}^2 be a two-dimensional Euclidean space with the metric h induced from the canonical metric g of S^2 . It is obvious that the Riemannian manifold \mathbb{R}^2 with the metric h has constant sectional curvature. Consider two such Riemannian manifolds (\mathbb{R}^2, h_1) , (\mathbb{R}^2, h_2) . The space $\mathbb{R}^2_1 \times \mathbb{R}^2_2$ with the metric $h_1 \times h_2$ has no negative sectional curvature. I do not know if there is a deformation of the metric $h_1 \times h_2$ whose sectional curvature is strictly positive.

1. Let R^2 be a Euclidean plane which is referred to a coordinate system (u_1, u_2) on which we obtain a metric defined by

$$h_{\scriptscriptstyle 1} = \{h_{\scriptscriptstyle 11} = 1, \, h_{\scriptscriptstyle 12} = h_{\scriptscriptstyle 21} = 0, \, h_{\scriptscriptstyle 22} = \sin^2\!u_{\scriptscriptstyle 1} \}$$
 ,

whose sectional curvature is positive constant 1.

Consider an open Riemannian submanifold M_1 of the Riemannian manifold (R_1^2, h_1) defined by

$$extbf{ extit{M}}_{\scriptscriptstyle 1} = \left\{ (u_{\scriptscriptstyle 1}, \ u_{\scriptscriptstyle 2}) \in extbf{ extit{R}}_{\scriptscriptstyle 1}^{\scriptscriptstyle 2} : 0 < u_{\scriptscriptstyle 1} < rac{\pi}{2} \ ext{,} \ -\infty < u_{\scriptscriptstyle 2} < \infty
ight\} ext{,}$$

whose metric is h_1/M_1 .

Let R_2^2 be also another Euclidean plane referred to a coordinate system (u_3, u_4) on which we take a metric defined by

$$h_2 = \{h_{33} = 1, h_{34} = h_{43} = 0, h_{44} = \sin^2 u_3\}$$
.

We also consider an open Riemannian submanifold M_2 of R_2^2 defined by

$$M_{\scriptscriptstyle 2} = \left\{ (u_{\scriptscriptstyle 3}, \, u_{\scriptscriptstyle 4}) \in {\it R}^{\scriptscriptstyle 2} : 0 < u_{\scriptscriptstyle 3} < rac{\pi}{2} \; , \; -\infty < u_{\scriptscriptstyle 4} < \infty
ight\} \, ,$$

whose metric is h_2/M_2 .

Let $M_1 \times M_2$ be the product manifold of M_1 , M_2 which is defined by

$$egin{aligned} extbf{ extit{M}}_{\scriptscriptstyle 1} imes extbf{ extit{M}}_{\scriptscriptstyle 2} &= \left\{ (u_{\scriptscriptstyle 1},\, u_{\scriptscriptstyle 2},\, u_{\scriptscriptstyle 3},\, u_{\scriptscriptstyle 4}) \in extbf{ extit{R}}_{\scriptscriptstyle 1}^{\scriptscriptstyle 2} imes extbf{ extit{R}}_{\scriptscriptstyle 2}^{\scriptscriptstyle 2} : 0 < u_{\scriptscriptstyle 1} < rac{\pi}{2} \;, \ & \ -\infty < u_{\scriptscriptstyle 2} < \infty , \; 0 < u_{\scriptscriptstyle 3} < rac{\pi}{2} \;, \; -\infty < u_{\scriptscriptstyle 4} < \infty
ight\} \,. \end{aligned}$$

On the manifold $M_1 \times M_2$ we get a special 1-parameter family of Riemannian metrics defined by

$$(1.1) \quad d(t) = egin{cases} d_{\scriptscriptstyle 11} = 1 + t f_{\scriptscriptstyle 1}, \; d_{\scriptscriptstyle 22} = \sin^2 u_{\scriptscriptstyle 1} (1 + t f_{\scriptscriptstyle 2}) \; , \ d_{\scriptscriptstyle 33} = 1 + t arphi_{\scriptscriptstyle 1}, \; d_{\scriptscriptstyle 44} = \sin^2 u_{\scriptscriptstyle 3} (1 + t arphi_{\scriptscriptstyle 2}), \; d_{ij} = 0, ext{ if } i
eq j \; , \end{cases}$$

where

 $f_1 = f_1(u_3, u_4), f_2 = f_2(u_3, u_4), \varphi_1 = \varphi_1(u_1, u_2), \varphi_2 = \varphi_2(u_1, u_2), -\varepsilon < t < \varepsilon,$ ε is a small positive number.

It is obvious that $d(0) = h_1/M_1 \times h_2/M_2$.

2. Let P be any point of $M_1 \times M_2$. As is known, the sectional curvature of a plane spanned two vectors X, Y of the tangent space $(M_1 \times M_2)_P$ is given by

$$\sigma(X, Y)(t) = -\frac{\langle R(X, Y)X, Y \rangle}{||X||^2||Y||^2 - \langle X, Y \rangle^2}.$$

If we apply Taylor's expansion theorem for the function $\sigma(X, Y)(t)$, we get

$$\sigma(X, Y)(t) = \sigma(X, Y)(0) + \sigma'_t(X, Y)(0) \frac{t}{1} + \sigma''_t(X, Y)(0) \frac{t^2}{2!} + \cdots$$

From the above formula we conclude that the sign of $\sigma(X, Y)(t)$ depends on the sign of $\sigma(X, Y)(0)$, if t is a small positive number and $\sigma(X, Y)(0) \neq 0$, but if $\sigma(X, Y) = 0$, then its sign depends on $t\sigma'_t(X, Y)(0)$.

As is known ([1], p. 287), $\sigma(X, Y)(0) = 0$, if $X \in (M_1)_P$ and $Y \in (M_2)_P$. In this case we estimate $\sigma(X, Y)(t)$ which is given by the formula

(2.1)
$$\sigma(X, Y)(t) = -\frac{A(t)}{B(t)},$$

where

$$(2.2) \begin{array}{c} A(t) = \left< R(X,\,Y)X,\,Y \right> = R_{\scriptscriptstyle 1313}(X^{\scriptscriptstyle 1})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 3})^{\scriptscriptstyle 2} + R_{\scriptscriptstyle 1414}(X^{\scriptscriptstyle 1})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 4})^{\scriptscriptstyle 2} \\ \qquad + R_{\scriptscriptstyle 2323}(X^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 3})^{\scriptscriptstyle 2} + R_{\scriptscriptstyle 2424}(X^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 4})^{\scriptscriptstyle 2} + 2R_{\scriptscriptstyle 1323}X^{\scriptscriptstyle 1}X^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 3})^{\scriptscriptstyle 2} \\ \qquad + 2R_{\scriptscriptstyle 1314}(X^{\scriptscriptstyle 1})^{\scriptscriptstyle 2}Y^{\scriptscriptstyle 3}Y^{\scriptscriptstyle 4} + 2R_{\scriptscriptstyle 2324}(X^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}Y^{\scriptscriptstyle 3}Y^{\scriptscriptstyle 4} + 2R_{\scriptscriptstyle 1424}X^{\scriptscriptstyle 1}X^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 4})^{\scriptscriptstyle 2} \\ \qquad + 2(R_{\scriptscriptstyle 1324} + R_{\scriptscriptstyle 1423})X^{\scriptscriptstyle 1}X^{\scriptscriptstyle 2}Y^{\scriptscriptstyle 3}Y^{\scriptscriptstyle 4} \; . \end{array}$$

$$(2.3) B(t) = \{d_{\scriptscriptstyle 11}(X^{\scriptscriptstyle 1})^{\scriptscriptstyle 2} + d_{\scriptscriptstyle 22}(X^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}\} \{d_{\scriptscriptstyle 33}(Y^{\scriptscriptstyle 3})^{\scriptscriptstyle 2} + d_{\scriptscriptstyle 44}(Y^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}\} > 0 \ ,$$

because, in this case, $\langle X, Y \rangle = 0$.

From relation (2.1), we obtain

$$\sigma(X, Y)(0) = -\frac{A(0)}{B(0)} = 0$$
,

or

$$A(0) = 0.$$

If we differentiate the same relation (2.1) with respect to t, we obtain

$$\sigma_t'(X,\;Y)(0) = -\;rac{A'(0)B(0) - A(0)B'(0)}{B^2(0)}\;$$
 ,

which, by virtue of (2.4), takes the form

(2.5)
$$\sigma'_t(X, Y)(0) = -\frac{A'(0)}{B(0)}.$$

From the formula (2.2), we obtain

$$(2.6) \begin{array}{c} A'(0) = R'_{1313}(0)(X^{1})^{2}(Y^{3})^{2} + R'_{2323}(0)(X^{2})^{2}(Y^{3})^{2} + R'_{1414}(0)(X^{1})^{2}(Y^{4})^{2} \\ \qquad + R'_{2424}(0)(X^{2})^{2}(Y^{4})^{2} + 2R'_{1323}(0)X^{1}X^{2}(Y^{3})^{2} + 2R'_{1314}(0)(X^{1})^{2}Y^{3}Y^{4} \\ \qquad + 2R'_{2324}(0)(X^{2})^{2}Y^{3}Y^{4} + 2R'_{1424}(0)X^{1}X^{2}(Y^{4})^{2} \\ \qquad + 2\{R'_{1324}(0) + R'_{1423}(0)\}X^{1}X^{2}Y^{3}Y^{4} \; . \end{array}$$

We shall estimate the coefficients of the Riemannian tensor which appear in the formula (2.6). As is known, R_{ijkl} is given by ([18], p. 18)

$$(2.7) \begin{array}{c} R_{ijkl} = \frac{1}{2} \left\{ \frac{\partial^2 d_{ik}}{\partial u_j \partial u_l} + \frac{\partial^2 d_{jl}}{\partial u_i \partial u_k} - \frac{\partial^2 d_{jk}}{\partial u_i \partial u_l} - \frac{\partial^2 d_{il}}{\partial u_j \partial u_k} \right\} \\ - d_{rs} \left\{ \Gamma^r_{jk} \Gamma^s_{il} - \Gamma^r_{jl} \Gamma^r_{ik} \right\} , \end{array}$$

where Γ_{jk}^r , Γ_{il}^s , Γ_{jl}^s , Γ_{ik}^s are the Christoffel symbols of second kind. From (1.1) and (2.7), if we make the calculations, we obtain

$$R_{1313} = \frac{t}{2} \left(\frac{\partial^{2} f_{1}}{\partial u_{3}^{2}} + \frac{\partial^{2} \varphi_{1}}{\partial u_{1}^{2}} \right) - \frac{t^{2}}{4} \left\{ \frac{(\partial f_{1}/\partial u_{3})^{2}}{1 + tf_{1}} + \frac{(\partial \varphi_{1}/\partial u_{4})^{2}}{1 + t\varphi_{1}} \right\},$$

$$R_{1414} = \frac{t}{2} \left(\frac{\partial^{2} f_{1}}{\partial u_{4}^{2}} + \sin^{2} u_{3} \frac{\partial^{2} \varphi_{2}}{\partial u_{1}^{2}} + \frac{\sin 2u_{3}(\partial f_{1}/\partial u_{3})}{2(1 + t\varphi_{1})} \right)$$

$$- \frac{t^{2}}{4} \left\{ \frac{(\partial f_{1}/\partial u_{4})^{2}}{1 + tf_{1}} + \frac{\sin^{2} u_{3}(\partial \varphi_{2}/\partial u_{1})^{2}}{1 + tf_{2}} - \frac{\sin 2u_{3}(\partial f_{1}/\partial u_{3})\varphi_{2}}{1 + t\varphi_{1}} \right\},$$

$$R_{2333} = \frac{t}{2} \left(\sin^{2} u_{1} \frac{\partial^{2} f_{2}}{\partial u_{3}^{2}} + \frac{\partial^{2} \varphi_{1}}{\partial u_{2}^{2}} + \frac{\sin 2u_{1}(\partial \varphi_{1}/\partial u_{1})}{2(1 + tf_{1})} \right)$$

$$- \frac{t^{2}}{4} \left\{ \frac{\sin^{2} u_{1}(\partial f_{2}/\partial u_{3})^{2}}{1 + tf_{2}} + \frac{(\partial \varphi_{1}/\partial u_{2})^{2}}{1 + t\varphi_{1}} - \frac{\sin 2u_{1}f_{2}(\partial \varphi_{1}/\partial u_{1})}{1 + tf_{1}} \right\},$$

$$R_{2424} = \frac{t}{2} \left(\sin^{2} u_{1} \frac{\partial^{2} f_{2}}{\partial u_{4}^{2}} + \sin^{2} u_{3} \frac{\partial^{2} \varphi_{2}}{\partial u_{2}^{2}} + \frac{\sin 2u_{1}\sin^{2} u_{3}(\partial \varphi_{2}/\partial u_{1})}{2(1 + tf_{1})} \right\},$$

$$(2.8) + \frac{\sin 2u_{3}\sin^{2} u_{1}(\partial f_{2}/\partial u_{3})}{2(1 + t\varphi_{1})} - \frac{t^{2}}{4} \left\{ \frac{\sin^{2} u_{1}(\partial f_{2}/\partial u_{4})^{2}}{1 + tf_{2}} + \frac{\sin^{2} u_{3}(\partial \varphi_{2}/\partial u_{1})}{1 + t\varphi_{2}} - \frac{\sin 2u_{1}\sin^{2} u_{3}f_{2}(\partial \varphi_{2}/\partial u_{1})}{1 + tf_{1}} \right\},$$

$$R_{1332} = \frac{t}{2} \left(\frac{\partial^{2} \varphi_{1}}{\partial u_{1}\partial u_{2}} - 2 \frac{\cos u_{1}}{\sin u_{1}} \frac{\partial \varphi_{1}}{\partial u_{2}} \right) - \frac{t^{2}}{4} \frac{(\partial \varphi_{1}/\partial u_{1})(\partial \varphi_{1}/\partial u_{2})}{1 + t\varphi_{1}},$$

$$R_{1314} = \frac{t}{2} \left(\frac{\partial^{2} f_{1}}{\partial u_{2}\partial u_{4}} - 2 \frac{\cos u_{3}}{\sin u_{3}} \frac{\partial f_{1}}{\partial u_{4}} \right) - \frac{t^{2}}{4} \frac{(\partial f_{1}/\partial u_{3})(\partial f_{1}/\partial u_{4})}{1 + tf_{1}},$$

$$R_{2324} = \frac{t}{2} \sin^{2} u_{1} \left(\frac{\partial^{2} f_{2}}{\partial u_{2}\partial u_{4}} - \frac{\cos u_{3}}{\sin u_{3}} \frac{\partial f_{2}}{\partial u_{4}} \right) - \frac{t^{2}}{4} \frac{\sin^{2} u_{1}}{1 + tf_{1}},$$

$$R_{1424} = \frac{t}{2} \sin^{2} u_{3} \left(\frac{\partial^{2} \varphi_{2}}{\partial u_{1}\partial u_{2}} - \frac{\cos u_{1}}{\sin u_{1}} \frac{\partial \varphi_{2}}{\partial u_{2}} \right) - \frac{t^{2}}{4} \frac{\sin^{2} u_{1}(\partial f_{2}/\partial u_{3})}{1 + t\varphi_{2}}.$$

$$(2.10) R_{1324} = R_{1423} = 0.$$

If we choose the functions φ_1 , f_1 , f_2 , φ_2 such that they satisfy the partial differential equations

$$egin{aligned} rac{\partial^2arphi_1}{\partial u_1\partial u_2} - 2rac{\cos u_1}{\sin u_1} rac{\partialarphi_1}{\partial u_2} = 0 \;, \ rac{\partial^2f_1}{\partial u_3\partial u_4} - 2rac{\cos u_3}{\sin u_3} rac{\partial f_1}{\partial u_4} = 0 \;, \ rac{\partial^2f_2}{\partial u_3\partial u_4} - rac{\cos u_3}{\sin u_3} rac{\partial f_2}{\partial u_4} = 0 \;, \ rac{\partial^2arphi_2}{\partial u_1\partial u_2} - rac{\cos u_1}{\sin u_1} rac{\partialarphi_2}{\partial u_2} = 0 \;, \end{aligned}$$

then the formulas (2.9) take the form

$$egin{align} R_{_{1323}} &= \, -rac{t^2}{4} \, rac{(\partial arphi_1/\partial u_1)(\partial arphi_1/\partial u_2)}{1 + t arphi_1} \;, \ &R_{_{1314}} &= \, -rac{t^2}{4} \, rac{(\partial f_1/\partial u_3)(\partial f_1/\partial u_4)}{1 + t f_1} \;, \ &R_{_{2324}} &= \, -rac{t^2}{4} \, rac{\sin^2 u_1(\partial f_2/\partial u_3)(\partial f_2/\partial u_4)}{1 + t f_2} \;, \ &R_{_{1424}} &= \, -rac{t^2}{4} \, rac{\sin^2 u_3(\partial arphi_2/\partial u_1)(\partial arphi_2/\partial u_2)}{1 + t arphi_2} \,. \end{split}$$

From the relations (2.8) and (2.12) we obtain

$$egin{align*} R'_{:313}(0) &= rac{1}{2} igg(rac{\partial^2 f_1}{\partial u_3^2} + rac{\partial^2 arphi_1}{\partial u_1^2} igg), \ R'_{1414}(0) &= rac{1}{2} igg(rac{\partial^2 f_1}{\partial u_4^2} + \sin^2 u_3 rac{\partial^2 arphi_2}{\partial u_1^2} + rac{\sin 2u_3}{2} rac{\partial f_1}{\partial u_3} igg), \ (2.13) \quad R'_{2323}(0) &= rac{1}{2} igg(rac{\partial^2 arphi_1}{\partial u_1^2} + \sin^2 u_1 rac{\partial^2 f_2}{\partial u_3^2} + rac{\sin 2u_1}{2} rac{\partial arphi_1}{\partial u_1} igg), \ R'_{2424}(0) &= rac{1}{2} igg(\sin^2 u_1 rac{\partial^2 f_2}{\partial u_4^2} + \sin^2 u_3 rac{\partial^2 arphi_2}{\partial u_2^2} + rac{\sin 2u_1 \sin^2 u_3}{2} rac{\partial arphi_2}{\partial u_1} \\ &+ rac{\sin 2u_3 \sin^2 u_1}{2} rac{\partial f_2}{\partial u_3} igg). \ R'_{1323}(0) &= R'_{1314}(0) = R'_{2324}(0) = R'_{1424}(0) = 0 \ . \end{split}$$

The first partial differential equation of (2.11) can be written

$$rac{\partial^2 arphi_1}{\partial u_1 \partial u_2} - rac{\partial}{\partial u_1} \log \sin^2 u_1 rac{\partial arphi_1}{\partial u_2} = 0$$
 ,

or

(2.14)

$$rac{\partial^2 arphi_1/\partial u_1\partial u_2}{\partial arphi_1/\partial u_2} = rac{\partial}{\partial u_1}\log \sin^2 u_1$$
 ,

or

$$rac{\partial arphi_{\scriptscriptstyle 1}}{\partial u_{\scriptscriptstyle 2}} = Z(u_{\scriptscriptstyle 2}) \sin^{\scriptscriptstyle 2} u_{\scriptscriptstyle 1}$$
 ,

whose general solution is

$$(2.15) \varphi_1 = V_1(u_2) \sin^2 u_1 + T_1(u_1) ,$$

where $V_1(u_2)$ and $T_1(u_1)$ are arbitrary functions of u_2 and u_1 , respectively.

We can find the general solutions of the rest of partial differential equations (2.11) in the same way. The general solutions of these equations are

$$egin{align} f_1 &= \sin^2 u_3 \lambda_1(u_4) \,+\, \mu_1(u_3) \;, \ & arphi_2 &= \sin u_1 V_2(u_2) \,+\, T_2(u_1) \;, \ & f_2 &= \sin u_3 \lambda_2(u_4) \,+\, \mu_2(u_3) \;, \ \end{pmatrix}$$

where $\lambda_1(u_4)$, $\mu_1(u_3)$, $V_2(u_2)$, $T_2(u_1)$, $\lambda_2(u_4)$, $\mu(u_3)$ are arbitrary functions of u_4 , u_3 , u_2 , u_1 , u_4 , u_3 , respectively.

The formulas (2.13) by virtue of (2.15) and (2.16) take the form

$$R'_{1313}(0) = rac{1}{2} \Big\{ 2\cos 2u_1 V_1(u_2) + T''_1(u_1) \Big\} + rac{1}{2} \Big\{ 2\cos 2u_3 \lambda_1(u_4) + \mu''_1(u_3) \Big\} \;, \ R'_{1414}(0) = rac{1}{2} \Big\{ \sin^2 u_3(\lambda''_1(u_4) + T''_2(u_1)) + rac{\sin^2 2u_3}{2} \lambda_1(u_4) + rac{\sin 2u_3}{2} \mu'_1(u_3) - \sin^2 u_3 \sin u_1 V_2(u_2) \Big\} \;, \ (2.17) \ R'_{2323}(0) = rac{1}{2} \Big\{ \sin^2 u_1(\mu''_2(u_3) + V''_1(u_2)) + rac{\sin^2 2u_1}{2} V_1(u_2) + rac{\sin 2u_1}{2} T'_1(u_1) - \sin^2 u_1 \sin u_3 \lambda_2(u_4) \Big\} \;, \ R'_{2424}(0) = rac{\sin^2 u_1 \sin u_3}{2} \Big\{ \lambda''_2(u_4) + \cos u_3 \mu'_2(u_3) + \cos^2 u_3 \lambda_2(u_4) + rac{\sin u_1 \sin^2 u_3}{2} \Big\{ V''_2(u_2) + \cos u_1 T'_2(u_1) + \cos^2 u_1 V_2(u_2) \Big\} \;.$$

The relation (2.6) by means of (2.10) and (2.14) takes the form

$$(2.18) \quad \begin{array}{ll} A'(0) = R'_{1313}(0)(X^{\scriptscriptstyle 1})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 3})^{\scriptscriptstyle 2} + R'_{2323}(0)(X^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 3})^{\scriptscriptstyle 2} + R'_{1414}(0)(X^{\scriptscriptstyle 1})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 4})^{\scriptscriptstyle 2} \\ & + R'_{2424}(0)(X^{\scriptscriptstyle 2})^{\scriptscriptstyle 2}(Y^{\scriptscriptstyle 4})^{\scriptscriptstyle 2} \; . \end{array}$$

In order that $\sigma'(X, Y)(0) = -A'(0)/B(0)$ be positive on the Riemannian manifold $M_1 \times M_2$, it must be

$$(2.19) A'(0) < 0.$$

From the formula (2.18) we conclude that (2.19) is valid when we have

$$R'_{1313}(0) < 0$$
 , $R'_{1414}(0) < 0$, $R'_{2323}(0) < 0$, $R'_{2424}(0) < 0$,

which, by virtue of (2.17), take the form

$$egin{aligned} rac{1}{2} &\{2\cos 2u_1 V_1(u_2) + T_1''(u_1)\} + rac{1}{2} \{2\cos 2u_3 \lambda_1(u_4) + \mu_1''(u_3)\} < 0 \;, \ &rac{1}{2} \Big\{ \sin^2 u_3 (\lambda_1''(u_4) + T_2''(u_1)) + rac{\sin^2 2u_3}{2} \lambda_1(u_4) + rac{\sin 2u_3}{2} \mu_1'(u_3) \ &- \sin^2 u_3 \sin u_1 V_2(u_2) \Big\} < 0 \;, \ &rac{1}{2} \Big\{ \sin^2 u_1 (\mu_2''(u_3) + V_1''(u_2)) + rac{\sin^2 2u_1}{2} \; V_1(u_2) + rac{\sin 2u_1}{2} \; T_1'(u_1) \ &- \sin^2 u_1 \sin u_3 \lambda_2(u_4) \Big\} < 0 \;, \ &rac{\sin^2 u_1 \sin u_3}{2} \Big\{ \lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4) \Big\} \ &+ rac{\sin u_1 \sin^2 u_3}{2} \Big\{ V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2) \Big\} < 0 \;, \end{aligned}$$

which must be valid on the Riemannian manifold $M_1 imes M_2$. The above inequalities hold if we have

$$egin{aligned} 2\cos2u_{_1}V_{_1}(u_{_2}) + T_{_1}''(u_{_1}) &< 0 \;, \ &\sin^2u_{_1}(\mu_{_2}''(u_{_3}) + V_{_1}''(u_{_2})) + rac{\sin^22u_{_1}}{2} \; V_{_1}(u_{_2}) + rac{\sin2u_{_1}}{2} \; T_{_1}'(u_{_1}) \ &-\sin^2u_{_1}\sin u_{_3}\lambda_{_2}(u_{_4}) &< 0 \;, \ &\lambda_{_2}''(u_{_4}) + \cos u_{_3}\mu_{_2}'(u_{_3}) + \cos^2u_{_3}\lambda_{_2}(u_{_4}) &< 0 \;, \ &2\cos2u_{_3}\lambda_{_1}(u_{_4}) + \mu_{_1}''(u_{_3}) &< 0 \;, \ &\sin^2u_{_3}(\lambda_{_1}''(u_{_4}) + T_{_2}''(u_{_1})) + rac{\sin^22u_{_3}}{2} \; \lambda_{_1}(u_{_4}) + rac{\sin2u_{_3}}{2} \; \mu_{_1}'(u_{_3}) \ &-\sin^2u_{_3}\sin u_{_1}V_{_2}(u_{_2}) &< 0 \;, \ &V_{_2}''(u_{_2}) + \cos u_{_1}T_{_2}'(u_{_1}) + \cos^2u_{_1}V_{_2}(u_{_2}) &< 0 \;. \end{aligned}$$

The inequalities (2.21) are similar to the inequalities (2.20); for this reason we shall only study the inequalities (2.20).

The factor $\cos 2u_1$ changes sign when $0 < u_1 < 2/\pi$; from this and from the fact that $V_1(u_2)$ and $V_1''(u_2)$ must have constant sign and bounded when $-\infty < u_2 < \infty$, we conclude that $V_1(u_2)$ must be a constant negative number $-\alpha$.

From the above remark, the inequalities (2.20) take the form

$$-2\alpha\cos 2u_{_1}+T_{_1}^{"}(u_{_1})<0$$

$$egin{aligned} (2.22) & \sin^2 u_1 \mu_2''(u_3) - lpha rac{\sin^2 2u_1}{2} + rac{\sin 2u_1}{2} \ T_1'(u_1) - \sin^2 u_1 \sin u_3 \lambda_2(u_4) < 0 \ , \ & \lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4) < 0 \ . \end{aligned}$$

In order for the second and the third inequalities of (2.22) to be valid, the function $\lambda_2(u_4)$ must be a positive constant number β .

Therefore the above inequalities become

$$-2lpha\cos 2u_{_1}+\,T_{_1}^{\prime\prime}(u_{_1})< 0$$
 ,

$$egin{align} (2.23) & \sin^2 u_{_1}\mu_{_2}^{\prime\prime\prime}(u_{_3}) \, -rac{lpha\,\sin^2 2u_{_1}}{2} \, +rac{\sin 2u_{_1}}{2}\,\, T_{_1}^{\prime\prime}(u_{_1}) \, -\, eta\,\sin^2 u_{_1}\sin u_{_3} < 0 \; , \ & \mu_{_2}^{\prime\prime}(u_{_3}) \, +\, eta\,\cos\,u_{_3} < 0 \; . \end{split}$$

If the functions $T_1(u_1)$, $\mu_2(u_3)$ are chosen such that

$$T_1'(u_1) < 0$$
 , $\max\{T_1''(u_1)\} < -2lpha$, $0 < u_1 < rac{\pi}{2}$, $\max\{\mu_2'(u_3)\} < -eta$, $\mu_2''(u_3) < 0$, $0 < u_3 < rac{\pi}{2}$,

then the inequalities (2.23) hold.

We also conclude that if the functions $\lambda_1(u_4)$, $V_2(u_2)$, $\mu_1(u_3)$, $T_2(u_1)$ satisfy the conditions

$$\lambda_{_1}(u_{_4}) = -\ \gamma$$
 , $V_{_2}(u_{_2}) = \delta$, $\mu_{_1}'(u_{_3}) < 0$, $\max\{\mu_{_1}''(u_{_3})\} < -2\gamma$, $0 < u_{_3} < rac{\pi}{2}$, $\max\{T_{_2}'(u_{_1})\} < -\delta$, $T_{_2}''(u_{_1}) < 0$, $0 < u_{_1} < rac{\pi}{2}$,

then the inequalities (2.21) hold.

Therefore, if the functions $\varphi_1, f_1, \varphi_2, f_2$ have the form

$$egin{align} arphi_1 &= -lpha \sin^2 u_1 + T_1(u_1) \;, \quad lpha > 0 \;, \ &f_1 &= -\gamma \sin^2 u_3 + \mu_1(u_3) \;, \quad \gamma > 0 \;, \ &arphi_2 &= \delta \sin u_1 + T_2(u_1) \;, \qquad \delta > 0 \;, \ &f_2 &= eta \sin u_3 + \mu_2(u_3) \;, \qquad eta > 0 \;, \ \end{pmatrix}$$

such that the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$ and $\mu_2(u_3)$ satisfy the conditions

$$T_1'(u_1) < 0 \;, \quad \max\{T_1''(u_1)\} < -2lpha \;, \quad 0 < u_1 < rac{\pi}{2} \;, \ \max\{\mu_2'(u_3)\} < -eta \;, \quad \mu_2''(u_3) < 0 \;, \qquad 0 < u_3 < rac{\pi}{2} \;, \ (2.25)$$
 (2.25) $\mu_1'(u_3) < 0 \;, \quad \max\{\mu_1''(u_3)\} < -2\gamma \;, \quad 0 < u_3 < rac{\pi}{2} \;, \ \max\{T_2'(u_1)\} < -\hat{\sigma} \;, \quad T_2''(u_1) < 0 \;, \quad 0 < u_1 < rac{\pi}{2} \;, \$

then $\sigma'_t(X, Y)(0) > 0$ for $X \in (M_1)_P$, $Y \in (M_2)_P$. Hence we have the following theorem.

THEOREM. Let M_1 , M_2 be two Riemannian spaces with positive constant sectional curvature defined in §1. If we consider a special 1-parameter family of Riemannian metrics d(t) on $M_1 \times M_2$ defined by (1.1) where the functions f_1 , f_2 , φ_1 , φ_2 have the form (2.24) in which the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$ and $\mu_2(u_3)$ must satisfy the conditions (2.25), then $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature of any plane spanned by $X \in (M_1)_P$ and $Y \in (M_2)_P$ with respect to t for t = 0 is strictly positive.

From the above, we conclude that if the parameter t is positive and small enough, then the corresponding Riemannian metric d(t) defined by (1.1) on $M_1 \times M_2$, where the functions $f_1, f_2, \varphi_1, \varphi_2$ have the form (2.24) in which the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$ and $\mu_2(u_3)$ must satisfy the conditions (2.25), has strictly positive sectional curvature.

3. We can extend the manifold $M_1 \times M_2$ to a manifold

$$N_{\scriptscriptstyle 1} imes N_{\scriptscriptstyle 2} \supset M_{\scriptscriptstyle 1} imes M_{\scriptscriptstyle 2}$$

such that there is a deformation of another product metric on $N_1 \times N_2$ which has strictly positive sectional curvature.

This method can be stated as follows. On the Euclidean plane R_2^2 we obtain a metric which is given by

$$\omega_{\scriptscriptstyle 1} = \left\{ \omega_{\scriptscriptstyle 11} = 1 \; ext{,} \quad \omega_{\scriptscriptstyle 12} = \omega_{\scriptscriptstyle 21} = 0 \; ext{,} \quad \omega_{\scriptscriptstyle 22} = \sin^2rac{u_{\scriptscriptstyle 1}}{n}
ight\}$$
 ,

where n is an integer > 1. The sectional curvature of this metric is $1/n^2$.

Now, consider an open Riemannian submanifold N_1 of the Riemannian manifold (R_1^2, ω_1) defined by

$$N_{_{1}} = \{(u_{_{1}},\,u_{_{2}}) \in R_{_{1}}^{_{2}} : 0 < u_{_{1}} < n\,rac{\pi}{2}$$
 , $-\, \infty < u_{_{2}} < \infty\}$,

whose metric is ω_1/N_1 .

Similarly, on the Euclidean plane R_2^2 , we obtain a metric which is given by

$$\omega_{\scriptscriptstyle 2} = \left\{ \omega_{\scriptscriptstyle 33} = 1 \; , \;\;\; \omega_{\scriptscriptstyle 34} = \omega_{\scriptscriptstyle 43} = 0 \; , \;\; \omega_{\scriptscriptstyle 44} = \sin^2 rac{u_{\scriptscriptstyle 3}}{n}
ight\} \, ,$$

whose sectional curvature is $1/n^2$.

Let N_2 be an open Riemannian submanifold of the Riemannian manifold $(\mathbf{R}_2^2, \omega_2)$ which is defined by

$$N_{\scriptscriptstyle 2} = \{(u_{\scriptscriptstyle 3}, u_{\scriptscriptstyle 4}) \in \emph{\emph{R}}_{\scriptscriptstyle 2}^{\scriptscriptstyle 2} : 0 < u_{\scriptscriptstyle 3} < n \, rac{\pi}{2} \; , \quad - \, \circ \, < u_{\scriptscriptstyle 4} < \, \circ \circ \} \; ,$$

whose metric is ω_2/N_2 .

We consider the product manifold $N_1 \times N_2$ of N_1 , N_2 defined by

$$N_{\scriptscriptstyle 1} imes N_{\scriptscriptstyle 2} = \{(u_{\scriptscriptstyle 1}, u_{\scriptscriptstyle 2}, u_{\scriptscriptstyle 3}, u_{\scriptscriptstyle 4}) \in \pmb{R}_{\scriptscriptstyle 1}^{\scriptscriptstyle 2} imes \pmb{R}_{\scriptscriptstyle 2}^{\scriptscriptstyle 2} \colon 0 < u_{\scriptscriptstyle 1} < n \, rac{\pi}{2} \; , \quad - \, \infty < u_{\scriptscriptstyle 2} < \infty \; , \ 0 < u_{\scriptscriptstyle 3} < n \, rac{\pi}{2} \; , \quad - \, \infty < u_{\scriptscriptstyle 4} < \infty \} \; .$$

It is obvious that $(N_1 \times N_2) \supset (M_1 \times M_2)$ and with the same technique as in §2 we can prove that there is a deformation of the metric $\omega_1/N_1 \times \omega_2/N_2$ which has strictly positive sectional curvature on the manifold $N_1 \times N_2$.

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