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# A SPECIAL DEFORMATION OF THE METRIC WITH NO NEGATIVE SECTIONAL CURVATURE OF A RIEMANNIAN SPACE

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**The main results of this paper can be stated as follows. Let $M_1$, $M_2$ bc two big open submanifolds of the Riemannian manifolds $(R_1^2, h_1)$ and $(R_2^2, h_2)$, respectively. The submanifolds $M_1, M_2$ with the metrics $h_1/M_1$ and $h_2/M_2$, respectively, have positive constant sectional curvature. We have constructed a special I-parameter family of Riemannian metrics $d(t)$ on $M_1 \times M_2$ which is the deformation of the product metric $h_1/M_1 \times h_2/M_2$ and it has strictly positive sectional curvature. In other words, we have proved that $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature with respect to the parameter $t$ for $t = 0$ and for any plane which is spanned by $X \in (M_1)_p$ and $Y \in (M_2)_p$ is strictly positive.**

Let $S^2$ be a two-dimensional sphere with the canonical metric $g$ whose sectional curvature is positive constant. Consider the product of two manifolds $S^2 \times S^2$. It is not known, ([1], p. 287), ([4], p, 171), ([11], p. 106), if there exists a deformation of the metric $g \times g$ with strictly positive sectional curvature.

Let $\boldsymbol{R}^2$ be a two-dimensional Euclidean space with the metric $h$ induced from the canonical metric $g$ of $S^2$. It is obvious that the Riemannian manifold $\boldsymbol{R}^2$ with the metric $h$ has constant sectional curvature. Consider two such Riemannian manifolds $(\boldsymbol{R}_1^2, h_1)$, $(\boldsymbol{R}_2^2, h_2)$. The space $\boldsymbol{R}_1^2 \times \boldsymbol{R}_2^2$ with the metric $h_1 \times h_2$ has no negative sectional curvature. I do not know if there is a deformation of the metric $h_1 \times h_2$ whose sectional curvature is strictly positive.

1. Let $\boldsymbol{R}^2$ be a Euclidean plane which is referred to a coordinate system $(u_1, u_2)$ on which we obtain a metric defined by

$$h_1 = \{h_{11} = 1, h_{12} = h_{21} = 0, h_{22} = \sin^2 u_1\} ,$$

whose sectional curvature is positive constant 1.

Consider an open Riemannian submanifold $M_1$ of the Riemannian manifold $(\boldsymbol{R}_1^2, h_1)$ defined by

$$M_1 = \left\{(u_1, u_2) \in \boldsymbol{R}_1^2 : 0 < u_1 < \frac{\pi}{2}, -\infty < u_2 < \infty\right\} ,$$

whose metric is $h_1/M_1$.

Let $\boldsymbol{R}_2^2$ be also another Euclidean plane referred to a coordinate system $(u_3, u_4)$ on which we take a metric defined by

$$h_2 = \{h_{33} = 1,\ h_{34} = h_{43} = 0,\ h_{44} = \sin^2 u_3\} .$$

We also consider an open Riemannian submanifold $M_2$ of $\boldsymbol{R}_2^2$ defined by

$$M_2 = \left\{(u_3, u_4) \in \boldsymbol{R}^2 : 0 < u_3 < \frac{\pi}{2},\ -\infty < u_4 < \infty\right\},$$

whose metric is $h_2/M_2$.

Let $M_1 \times M_2$ be the product manifold of $M_1$, $M_2$ which is defined by

$$M_1 \times M_2 = \left\{(u_1, u_2, u_3, u_4) \in \boldsymbol{R}_1^2 \times \boldsymbol{R}_2^2 : 0 < u_1 < \frac{\pi}{2},\right.$$

$$\left. -\infty < u_2 < \infty,\ 0 < u_3 < \frac{\pi}{2},\ -\infty < u_4 < \infty\right\}.$$

On the manifold $M_1 \times M_2$ we get a special 1-parameter family of Riemannian metrics defined by

$$(1.1) \quad d(t) = \begin{cases} d_{11} = 1 + tf_1,\ d_{22} = \sin^2 u_1(1+tf_2) , \\ d_{33} = 1 + t\varphi_1,\ d_{44} = \sin^2 u_3(1+t\varphi_2),\ d_{ij} = 0, \text{ if } i \neq j , \end{cases}$$

where

$f_1 = f_1(u_3, u_4)$, $f_2 = f_2(u_3, u_4)$, $\varphi_1 = \varphi_1(u_1, u_2)$, $\varphi_2 = \varphi_2(u_1, u_2)$, $-\varepsilon < t < \varepsilon$, $\varepsilon$ is a small positive number.

It is obvious that $d(0) = h_1/\mathrm{M}_1 \times h_2/M_2$.

2. Let $P$ be any point of $M_1 \times M_2$. As is known, the sectional curvature of a plane spanned two vectors $X$, $Y$ of the tangent space $(M_1 \times M_2)_P$ is given by

$$\sigma(X, Y)(t) = -\frac{\langle R(X, Y)X, Y\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}.$$

If we apply Taylor's expansion theorem for the function $\sigma(X, Y)(t)$, we get

$$\sigma(X, Y)(t) = \sigma(X, Y)(0) + \sigma_t'(X, Y)(0)\frac{t}{1} + \sigma_t''(X, Y)(0)\frac{t^2}{2!} + \cdots .$$

From the above formula we conclude that the sign of $\sigma(X, Y)(t)$ depends on the sign of $\sigma(X, Y)(0)$, if $t$ is a small positive number and $\sigma(X, Y)(0) \neq 0$, but if $\sigma(X, Y) = 0$, then its sign depends on $t\sigma_t'(X, Y)(0)$.

As is known ([1], p. 287), $\sigma(X, Y)(0) = 0$, if $X \in (M_1)_P$ and $Y \in (M_2)_P$. In this case we estimate $\sigma(X, Y)(t)$ which is given by the formula

$$(2.1) \qquad \sigma(X, Y)(t) = -\frac{A(t)}{B(t)},$$

where

$$\begin{aligned} A(t) = \langle R(X, Y)X, Y\rangle &= R_{1313}(X^1)^2(Y^3)^2 + R_{1414}(X^1)^2(Y^4)^2 \\ &+ R_{2323}(X^2)^2(Y^3)^2 + R_{2424}(X^2)^2(Y^4)^2 + 2R_{1323}X^1X^2(Y^3)^2 \\ &+ 2R_{1314}(X^1)^2Y^3Y^4 + 2R_{2324}(X^2)^2Y^3Y^4 + 2R_{1424}X^1X^2(Y^4)^2 \\ &+ 2(R_{1324} + R_{1423})X^1X^2Y^3Y^4 . \end{aligned} \tag{2.2}$$

$$B(t) = \{d_{11}(X^1)^2 + d_{22}(X^2)^2\}\{d_{33}(Y^3)^2 + d_{44}(Y^2)^2\} > 0 , \tag{2.3}$$

because, in this case, $\langle X, Y\rangle = 0$.

From relation (2.1), we obtain

$$\sigma(X, Y)(0) = -\frac{A(0)}{B(0)} = 0 ,$$

or

$$A(0) = 0 . \tag{2.4}$$

If we differentiate the same relation (2.1) with respect to $t$, we obtain

$$\sigma'_t(X, Y)(0) = -\frac{A'(0)B(0) - A(0)B'(0)}{B^2(0)} ,$$

which, by virtue of (2.4), takes the form

$$\sigma'_t(X, Y)(0) = -\frac{A'(0)}{B(0)} . \tag{2.5}$$

From the formula (2.2), we obtain

$$\begin{aligned} A'(0) &= R'_{1313}(0)(X^1)^2(Y^3)^2 + R'_{2323}(0)(X^2)^2(Y^3)^2 + R'_{1414}(0)(X^1)^2(Y^4)^2 \\ &+ R'_{2424}(0)(X^2)^2(Y^4)^2 + 2R'_{1323}(0)X^1X^2(Y^3)^2 + 2R'_{1314}(0)(X^1)^2Y^3Y^4 \\ &+ 2R'_{2324}(0)(X^2)^2Y^3Y^4 + 2R'_{1424}(0)X^1X^2(Y^4)^2 \\ &+ 2\{R'_{1324}(0) + R'_{1423}(0)\}X^1X^2Y^3Y^4 . \end{aligned} \tag{2.6}$$

We shall estimate the coefficients of the Riemannian tensor which appear in the formula (2.6). As is known, $R_{ijkl}$ is given by ([18], p. 18)

$$\begin{aligned} R_{ijkl} &= \frac{1}{2}\left\{\frac{\partial^2 d_{ik}}{\partial u_j \partial u_l} + \frac{\partial^2 d_{jl}}{\partial u_i \partial u_k} - \frac{\partial^2 d_{jk}}{\partial u_i \partial u_l} - \frac{\partial^2 d_{il}}{\partial u_j \partial u_k}\right\} \\ &- d_{rs}\left\{\Gamma^r_{jk}\Gamma^s_{il} - \Gamma^r_{jl}\Gamma^s_{ik}\right\} , \end{aligned} \tag{2.7}$$

where $\Gamma^r_{jk}$, $\Gamma^s_{il}$, $\Gamma^r_{jl}$, $\Gamma^s_{ik}$ are the Christoffel symbols of second kind.

From (1.1) and (2.7), if we make the calculations, we obtain

$$R_{1313} = \frac{t}{2}\left(\frac{\partial^2 f_1}{\partial u_3^2} + \frac{\partial^2 \varphi_1}{\partial u_1^2}\right) - \frac{t^2}{4}\left\{\frac{(\partial f_1/\partial u_3)^2}{1+tf_1} + \frac{(\partial\varphi_1/\partial u_1)^2}{1+t\varphi_1}\right\},$$

$$R_{1414} = \frac{t}{2}\left(\frac{\partial^2 f_1}{\partial u_4^2} + \sin^2 u_3 \frac{\partial^2 \varphi_2}{\partial u_1^2} + \frac{\sin 2u_3(\partial f_1/\partial u_3)}{2(1+t\varphi_1)}\right) - \frac{t^2}{4}\left\{\frac{(\partial f_1/\partial u_4)^2}{1+tf_1} + \frac{\sin^2 u_3(\partial\varphi_2/\partial u_1)^2}{1+tf_2} - \frac{\sin 2u_3(\partial f_1/\partial u_3)\varphi_2}{1+t\varphi_1}\right\},$$

$$R_{2323} = \frac{t}{2}\left(\sin^2 u_1 \frac{\partial^2 f_2}{\partial u_3^2} + \frac{\partial^2 \varphi_1}{\partial u_2^2} + \frac{\sin 2u_1(\partial\varphi_1/\partial u_1)}{2(1+tf_1)}\right) - \frac{t^2}{4}\left\{\frac{\sin^2 u_1(\partial f_2/\partial u_3)^2}{1+tf_2} + \frac{(\partial\varphi_1/\partial u_2)^2}{1+t\varphi_1} - \frac{\sin 2u_1 f_2(\partial\varphi_1/\partial u_1)}{1+tf_1}\right\},$$

$$R_{2424} = \frac{t}{2}\left(\sin^2 u_1 \frac{\partial^2 f_2}{\partial u_4^2} + \sin^2 u_3 \frac{\partial^2 \varphi_2}{\partial u_2^2} + \frac{\sin 2u_1 \sin^2 u_3(\partial\varphi_2/\partial u_1)}{2(1+tf_1)} + \frac{\sin 2u_3 \sin^2 u_1(\partial f_2/\partial u_3)}{2(1+t\varphi_1)}\right) - \frac{t^2}{4}\left\{\frac{\sin^2 u_1(\partial f_2/\partial u_4)^2}{1+tf_2} + \frac{\sin^2 u_3(\partial\varphi_2/\partial u_3)^2}{1+t\varphi_2} - \frac{\sin 2u_1 \sin^2 u_3 f_2(\partial\varphi_2/\partial u_1)}{1+tf_1} - \frac{\sin 2u_3 \sin^2 u_1 \varphi_2(\partial f_2/\partial u_3)}{1+t\varphi_1}\right\}, \tag{2.8}$$

$$R_{1323} = \frac{t}{2}\left(\frac{\partial^2\varphi_1}{\partial u_1\partial u_2} - 2\frac{\cos u_1}{\sin u_1}\frac{\partial\varphi_1}{\partial u_2}\right) - \frac{t^2}{4}\frac{(\partial\varphi_1/\partial u_1)(\partial\varphi_1/\partial u_2)}{1+t\varphi_1},$$

$$R_{1314} = \frac{t}{2}\left(\frac{\partial^2 f_1}{\partial u_3\partial u_4} - 2\frac{\cos u_3}{\sin u_3}\frac{\partial f_1}{\partial u_4}\right) - \frac{t^2}{4}\frac{(\partial f_1/\partial u_3)(\partial f_1/\partial u_4)}{1+tf_1},$$

$$R_{2324} = \frac{t}{2}\sin^2 u_1\left(\frac{\partial^2 f_2}{\partial u_3\partial u_4} - \frac{\cos u_3}{\sin u_3}\frac{\partial f_2}{\partial u_4}\right) - \frac{t^2}{4}\frac{\sin^2 u_1(\partial f_2/\partial u_3)(\partial f_2/\partial u_4)}{1+tf_2}, \tag{2.9}$$

$$R_{1424} = \frac{t}{2}\sin^2 u_3\left(\frac{\partial^2\varphi_2}{\partial u_1\partial u_2} - \frac{\cos u_1}{\sin u_1}\frac{\partial\varphi_2}{\partial u_2}\right) - \frac{t^2}{4}\frac{\sin^2 u_3(\partial\varphi_2/\partial u_1)(\partial\varphi_2/\partial u_2)}{1+t\varphi_2}.$$

$$R_{1324} = R_{1423} = 0\,. \tag{2.10}$$

If we choose the functions $\varphi_1$, $f_1$, $f_2$, $\varphi_2$ such that they satisfy the partial differential equations

$$(2.11)\quad \begin{aligned} &\frac{\partial^2\varphi_1}{\partial u_1\partial u_2} - 2\frac{\cos u_1}{\sin u_1}\frac{\partial\varphi_1}{\partial u_2} = 0\ ,\\ &\frac{\partial^2 f_1}{\partial u_3\partial u_4} - 2\frac{\cos u_3}{\sin u_3}\frac{\partial f_1}{\partial u_4} = 0\ ,\\ &\frac{\partial^2 f_2}{\partial u_3\partial u_4} - \frac{\cos u_3}{\sin u_3}\frac{\partial f_2}{\partial u_4} = 0\ ,\\ &\frac{\partial^2\varphi_2}{\partial u_1\partial u_2} - \frac{\cos u_1}{\sin u_1}\frac{\partial\varphi_2}{\partial u_2} = 0\ ,\end{aligned}$$

then the formulas (2.9) take the form

$$(2.12)\quad \begin{aligned} R_{1323} &= -\frac{t^2}{4}\frac{(\partial\varphi_1/\partial u_1)(\partial\varphi_1/\partial u_2)}{1+t\varphi_1}\ ,\\ R_{1314} &= -\frac{t^2}{4}\frac{(\partial f_1/\partial u_3)(\partial f_1/\partial u_4)}{1+tf_1}\ ,\\ R_{2324} &= -\frac{t^2}{4}\frac{\sin^2 u_1(\partial f_2/\partial u_3)(\partial f_2/\partial u_4)}{1+tf_2}\ ,\\ R_{1424} &= -\frac{t^2}{4}\frac{\sin^2 u_3(\partial\varphi_2/\partial u_1)(\partial\varphi_2/\partial u_2)}{1+t\varphi_2}\ .\end{aligned}$$

From the relations (2.8) and (2.12) we obtain

$$(2.13)\quad \begin{aligned} R'_{1313}(0) &= \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial u_3^2} + \frac{\partial^2\varphi_1}{\partial u_1^2}\right),\\ R'_{1414}(0) &= \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial u_4^2} + \sin^2 u_3\frac{\partial^2\varphi_2}{\partial u_1^2} + \frac{\sin 2u_3}{2}\frac{\partial f_1}{\partial u_3}\right)\ ,\\ R'_{2323}(0) &= \frac{1}{2}\left(\frac{\partial^2\varphi_1}{\partial u_1^2} + \sin^2 u_1\frac{\partial^2 f_2}{\partial u_3^2} + \frac{\sin 2u_1}{2}\frac{\partial\varphi_1}{\partial u_1}\right)\ ,\\ R'_{2424}(0) &= \frac{1}{2}\left(\sin^2 u_1\frac{\partial^2 f_2}{\partial u_4^2} + \sin^2 u_3\frac{\partial^2\varphi_2}{\partial u_2^2} + \frac{\sin 2u_1\sin^2 u_3}{2}\frac{\partial\varphi_2}{\partial u_1}\right.\\ &\qquad \left. + \frac{\sin 2u_3\sin^2 u_1}{2}\frac{\partial f_2}{\partial u_3}\right)\ .\end{aligned}$$

$$(2.14)\qquad R'_{1323}(0) = R'_{1314}(0) = R'_{2324}(0) = R'_{1424}(0) = 0\ .$$

The first partial differential equation of (2.11) can be written

$$\frac{\partial^2\varphi_1}{\partial u_1\partial u_2} - \frac{\partial}{\partial u_1}\log\sin^2 u_1\frac{\partial\varphi_1}{\partial u_2} = 0\ ,$$

or

$$\frac{\partial^2\varphi_1/\partial u_1\partial u_2}{\partial\varphi_1/\partial u_2} = \frac{\partial}{\partial u_1}\log\sin^2 u_1\ ,$$

or

$$\frac{\partial\varphi_1}{\partial u_2} = Z(u_2)\sin^2 u_1\ ,$$

whose general solution is

$$\varphi_1 = V_1(u_2)\sin^2 u_1 + T_1(u_1) , \tag{2.15}$$

where $V_1(u_2)$ and $T_1(u_1)$ are arbitrary functions of $u_2$ and $u_1$, respectively.

We can find the general solutions of the rest of partial differential equations (2.11) in the same way. The general solutions of these equations are

$$\begin{aligned} f_1 &= \sin^2 u_3\lambda_1(u_4) + \mu_1(u_3) , \\ \varphi_2 &= \sin u_1 V_2(u_2) + T_2(u_1) , \\ f_2 &= \sin u_3\lambda_2(u_4) + \mu_2(u_3) , \end{aligned} \tag{2.16}$$

where $\lambda_1(u_4)$, $\mu_1(u_3)$, $V_2(u_2)$, $T_2(u_1)$, $\lambda_2(u_4)$, $\mu(u_3)$ are arbitrary functions of $u_4$, $u_3$, $u_2$, $u_1$, $u_4$, $u_3$, respectively.

The formulas (2.13) by virtue of (2.15) and (2.16) take the form

$$\begin{aligned} R'_{1313}(0) &= \frac{1}{2}\Big\{2\cos 2u_1 V_1(u_2) + T_1''(u_1)\Big\} + \frac{1}{2}\Big\{2\cos 2u_3\lambda_1(u_4) \\ &\quad + \mu_1''(u_3)\Big\} , \\ R'_{1414}(0) &= \frac{1}{2}\Big\{\sin^2 u_3(\lambda_1''(u_4) + T_2''(u_1)) + \frac{\sin^2 2u_3}{2}\lambda_1(u_4) \\ &\quad + \frac{\sin 2u_3}{2}\mu_1'(u_3) - \sin^2 u_3 \sin u_1 V_2(u_2)\Big\} , \\ R'_{2323}(0) &= \frac{1}{2}\Big\{\sin^2 u_1(\mu_2''(u_3) + V_1''(u_2)) + \frac{\sin^2 2u_1}{2} V_1(u_2) \\ &\quad + \frac{\sin 2u_1}{2} T_1'(u_1) - \sin^2 u_1 \sin u_3\lambda_2(u_4)\Big\} , \\ R'_{2424}(0) &= \frac{\sin^2 u_1 \sin u_3}{2}\Big\{\lambda_2''(u_4) + \cos u_3\mu_2'(u_3) + \cos^2 u_3\lambda_2(u_4) \\ &\quad + \frac{\sin u_1 \sin^2 u_3}{2}\Big\{V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2)\Big\} . \end{aligned} \tag{2.17}$$

The relation (2.6) by means of (2.10) and (2.14) takes the form

$$\begin{aligned} A'(0) &= R'_{1313}(0)(X^1)^2(Y^3)^2 + R'_{2323}(0)(X^2)^2(Y^3)^2 + R'_{1414}(0)(X^1)^2(Y^4)^2 \\ &\quad + R'_{2424}(0)(X^2)^2(Y^4)^2 . \end{aligned} \tag{2.18}$$

In order that $\sigma'(X, Y)(0) = -A'(0)/B(0)$ be positive on the Riemannian manifold $M_1 \times M_2$, it must be

$$A'(0) < 0 . \tag{2.19}$$

From the formula (2.18) we conclude that (2.19) is valid when we have

$$R'_{1313}(0) < 0 \ , \quad R'_{1414}(0) < 0 \ , \quad R'_{2323}(0) < 0 \ , \quad R'_{2424}(0) < 0 \ ,$$

which, by virtue of (2.17), take the form

$$\frac{1}{2}\{2\cos 2u_1 V_1(u_2) + T_1''(u_1)\} + \frac{1}{2}\{2\cos 2u_3\lambda_1(u_4) + \mu_1''(u_3)\} < 0 \ ,$$

$$\frac{1}{2}\Big\{\sin^2 u_3(\lambda_1''(u_4) + T_2''(u_1)) + \frac{\sin^2 2u_3}{2}\lambda_1(u_4) + \frac{\sin 2u_3}{2}\mu_1'(u_3) - \sin^2 u_3 \sin u_1 V_2(u_2)\Big\} < 0 \ ,$$

$$\frac{1}{2}\Big\{\sin^2 u_1(\mu_2''(u_3) + V_1''(u_2)) + \frac{\sin^2 2u_1}{2}V_1(u_2) + \frac{\sin 2u_1}{2}T_1'(u_1) - \sin^2 u_1 \sin u_3\lambda_2(u_4)\Big\} < 0 \ ,$$

$$\frac{\sin^2 u_1 \sin u_3}{2}\Big\{\lambda_2''(u_4) + \cos u_3\mu_2'(u_3) + \cos^2 u_3\lambda_2(u_4)\Big\} + \frac{\sin u_1 \sin^2 u_3}{2}\Big\}V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2)\Big\} < 0 \ ,$$

which must be valid on the Riemannian manifold $M_1 \times M_2$.

The above inequalities hold if we have

$$\text{(2.20)} \quad \begin{aligned} &2\cos 2u_1 V_1(u_2) + T_1''(u_1) < 0 \ , \\ &\sin^2 u_1(\mu_2''(u_3) + V_1''(u_2)) + \frac{\sin^2 2u_1}{2}V_1(u_2) + \frac{\sin 2u_1}{2}T_1'(u_1) \\ &\qquad - \sin^2 u_1 \sin u_3\lambda_2(u_4) < 0 \ , \\ &\lambda_2''(u_4) + \cos u_3\mu_2'(u_3) + \cos^2 u_3\lambda_2(u_4) < 0 \ , \end{aligned}$$

$$\text{(2.21)} \quad \begin{aligned} &2\cos 2u_3\lambda_1(u_4) + \mu_1''(u_3) < 0 \ , \\ &\sin^2 u_3(\lambda_1''(u_4) + T_2''(u_1)) + \frac{\sin^2 2u_3}{2}\lambda_1(u_4) + \frac{\sin 2u_3}{2}\mu_1'(u_3) \\ &\qquad - \sin^2 u_3 \sin u_1 V_2(u_2) < 0 \ , \\ &V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2) < 0 \ . \end{aligned}$$

The inequalities (2.21) are similar to the inequalities (2.20); for this reason we shall only study the inequalities (2.20).

The factor $\cos 2u_1$ changes sign when $0 < u_1 < 2/\pi$; from this and from the fact that $V_1(u_2)$ and $V_1''(u_2)$ must have constant sign and bounded when $-\infty < u_2 < \infty$, we conclude that $V_1(u_2)$ must be a constant negative number $-\alpha$.

From the above remark, the inequalities (2.20) take the form

$$- 2\alpha \cos 2u_1 + T_1''(u_1) < 0 ,$$

$$\text{(2.22)} \quad \sin^2 u_1 \mu_2''(u_3) - \alpha\frac{\sin^2 2u_1}{2} + \frac{\sin 2u_1}{2} T_1'(u_1) - \sin^2 u_1 \sin u_3 \lambda_2(u_4) < 0 ,$$

$$\lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4) < 0 .$$

In order for the second and the third inequalities of (2.22) to be valid, the function $\lambda_2(u_4)$ must be a positive constant number $\beta$.

Therefore the above inequalities become

$$- 2\alpha \cos 2u_1 + T_1''(u_1) < 0 ,$$

$$\text{(2.23)} \quad \sin^2 u_1 \mu_2''(u_3) - \frac{\alpha \sin^2 2u_1}{2} + \frac{\sin 2u_1}{2} T_1'(u_1) - \beta \sin^2 u_1 \sin u_3 < 0 ,$$

$$\mu_2'(u_3) + \beta \cos u_3 < 0 .$$

If the functions $T_1(u_1)$, $\mu_2(u_3)$ are chosen such that

$$T_1'(u_1) < 0 , \quad \max\{T_1''(u_1)\} < - 2\alpha, \quad 0 < u_1 < \frac{\pi}{2} ,$$

$$\max\{\mu_2'(u_3)\} < - \beta , \quad \mu_2''(u_3) < 0 , \quad 0 < u_3 < \frac{\pi}{2} ,$$

then the inequalities (2.23) hold.

We also conclude that if the functions $\lambda_1(u_4)$, $V_2(u_2)$, $\mu_1(u_3)$, $T_2(u_1)$ satisfy the conditions

$$\lambda_1(u_4) = - \gamma , \quad V_2(u_2) = \delta ,$$

$$\mu_1'(u_3) < 0 , \quad \max\{\mu_1''(u_3)\} < - 2\gamma , \quad 0 < u_3 < \frac{\pi}{2} ,$$

$$\max\{T_2'(u_1)\} < - \delta , \quad T_2''(u_1) < 0, \quad 0 < u_1 < \frac{\pi}{2} ,$$

then the inequalities (2.21) hold.

Therefore, if the functions $\varphi_1, f_1, \varphi_2, f_2$ have the form

$$\text{(2.24)} \quad \begin{aligned} \varphi_1 &= - \alpha \sin^2 u_1 + T_1(u_1) , & \alpha > 0 , \\ f_1 &= - \gamma \sin^2 u_3 + \mu_1(u_3) , & \gamma > 0 , \\ \varphi_2 &= \delta \sin u_1 + T_2(u_1) , & \delta > 0 , \\ f_2 &= \beta \sin u_3 + \mu_2(u_3) , & \beta > 0 , \end{aligned}$$

such that the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$ and $\mu_2(u_3)$ satisfy the conditions

$$(2.25)\quad \begin{aligned} &T_1'(u_1) < 0\,, \quad \max\{T_1''(u_1)\} < -2\alpha\,, \quad 0 < u_1 < \frac{\pi}{2}\,, \\ &\max\{\mu_2'(u_3)\} < -\beta\,, \quad \mu_2''(u_3) < 0\,, \quad 0 < u_3 < \frac{\pi}{2}\,, \\ &\mu_1'(u_3) < 0\,, \quad \max\{\mu_1''(u_3)\} < -2\gamma\,, \quad 0 < u_3 < \frac{\pi}{2}\,, \\ &\max\{T_2'(u_1)\} < -\delta\,, \quad T_2''(u_1) < 0\,, \quad 0 < u_1 < \frac{\pi}{2}\,, \end{aligned}$$

then $\sigma_t'(X, Y)(0) > 0$ for $X \in (M_1)_P$, $Y \in (M_2)_P$.

Hence we have the following theorem.

THEOREM. *Let $M_1$, $M_2$ be two Riemannian spaces with positive constant sectional curvature defined in* §1. *If we consider a special* 1-*parameter family of Riemannian metrics $d(t)$ on $M_1 \times M_2$ defined by* (1.1) *where the functions $f_1, f_2, \varphi_1, \varphi_2$ have the form* (2.24) *in which the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$* and *$\mu_2(u_3)$ must satisfy the conditions* (2.25), *then $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature of any plane spanned by $X \in (M_1)_P$ and $Y \in (M_2)_P$ with respect to $t$ for $t = 0$ is strictly positive.*

From the above, we conclude that if the parameter $t$ is positive and small enough, then the corresponding Riemannian metric $d(t)$ defined by (1.1) on $M_1 \times M_2$, where the functions $f_1, f_2, \varphi_1, \varphi_2$ have the form (2.24) in which the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$ and $\mu_2(u_3)$ must satisfy the conditions (2.25), has strictly positive sectional curvature.

**3.** We can extend the manifold $M_1 \times M_2$ to a manifold

$$N_1 \times N_2 \supset M_1 \times M_2$$

such that there is a deformation of another product metric on $N_1 \times N_2$ which has strictly positive sectional curvature.

This method can be stated as follows. On the Euclidean plane $\boldsymbol{R}_2^2$ we obtain a metric which is given by

$$\omega_1 = \left\{\omega_{11} = 1\,, \quad \omega_{12} = \omega_{21} = 0\,, \quad \omega_{22} = \sin^2 \frac{u_1}{n}\right\},$$

where $n$ is an integer $> 1$. The sectional curvature of this metric is $1/n^2$.

Now, consider an open Riemannian submanifold $N_1$ of the Riemannian manifold $(\boldsymbol{R}_1^2, \omega_1)$ defined by

$$N_1 = \{(u_1, u_2) \in \boldsymbol{R}_1^2 : 0 < u_1 < n\frac{\pi}{2}\,, -\infty < u_2 < \infty\}\,,$$

whose metric is $\omega_1/N_1$.

Similarly, on the Euclidean plane $\boldsymbol{R}_2^2$, we obtain a metric which is given by

$$\omega_2 = \left\{\omega_{33} = 1 \;, \quad \omega_{34} = \omega_{43} = 0 \;, \;\; \omega_{44} = \sin^2 \frac{u_3}{n}\right\} ,$$

whose sectional curvature is $1/n^2$.

Let $N_2$ be an open Riemannian submanifold of the Riemannian manifold $(\boldsymbol{R}_2^2, \omega_2)$ which is defined by

$$N_2 = \{(u_3, u_4) \in \boldsymbol{R}_2^2 : 0 < u_3 < n\frac{\pi}{2} \;, \quad -\infty < u_4 < \infty\} \;,$$

whose metric is $\omega_2/N_2$.

We consider the product manifold $N_1 \times N_2$ of $N_1$, $N_2$ defined by

$$N_1 \times N_2 = \{(u_1, u_2, u_3, u_4) \in \boldsymbol{R}_1^2 \times \boldsymbol{R}_2^2 : 0 < u_1 < n\frac{\pi}{2} \;, \quad -\infty < u_2 < \infty \;,$$

$$0 < u_3 < n\frac{\pi}{2} \;, \quad -\infty < u_4 < \infty\} \;.$$

It is obvious that $(N_1 \times N_2) \supset (M_1 \times M_2)$ and with the same technique as in §2 we can prove that there is a deformation of the metric $\omega_1/N_1 \times \omega_2 / N_2$ which has strictly positive sectional curvature on the manifold $N_1 \times N_2$.

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