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**SINGULAR PERTURBATION OF LINEAR PARTIAL
DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS**

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Let $P_j(z, \varepsilon)$ be a polynomial in z and ε with complex coefficients, where z is in E^m and $\varepsilon > 0$ is a small parameter. Let $L_\varepsilon = \sum_{j=0}^l P_{l-j}(\delta_x, \varepsilon)(\delta_t)^j$ be a polynomial in δ_t, δ_x and ε , which is not divisible by the square of a similar nonconstant polynomial. We shall assume that $P_0(z, \varepsilon) = \varepsilon$ and $P_1(z)$ is independent of ε .

In this paper we shall show that under certain conditions the solution $u_\varepsilon(t, x)$ of $L_\varepsilon(u) = f_\varepsilon(t, x)$ converges to the solution $u_0(t, x)$ of $L_0(u) = f_0(t, x)$.

Let $(t, x) = (t, x_1, x_2, \dots, x_m)$ be a point in $R \times E^m$ where $0 \leq t \leq T$, and x in E^m , and E^m denotes an m -dimensional Euclidean space. Let also C_0^∞ be the set of all infinitely times continuously differentiable complex valued functions on E^m with compact support. For any u in C_0^∞ , let the norm $\|u\|_p$ be defined for any integer $p < 0$ as follows:

$$(1) \quad \int_{E^m} \sum_{|\varphi| \leq p} |\partial_{x_1}^{\varphi_1} \cdots \partial_{x_m}^{\varphi_m} u|^2 dx = \|u\|_p^2 \quad (|\varphi| = \varphi_1 + \cdots + \varphi_m).$$

It is easy to see that the space C_0^∞ with the norm (1) gives a Hilbert space, which we shall call an H_p -space. We may also notice that $H_p \supset H_q$ and $\|u\|_p \leq \|u\|_q$ if $p < q$. If for each φ in H_p we denote by $\hat{\varphi}$ the Fourier transform of φ

$$\hat{\varphi}(z) = [1/(2\pi)^{m/2}] \int_{E^m} \exp(-ix \cdot z) \varphi(x) dx$$

where,

$$x, z = \sum_1^m x_i z_i$$

then the norm defined in (1) will be equivalent to the norm

$$(2) \quad \|\varphi\|_p^2 = \int_{E^m} (1 + |z|^2)^{p/2} |\hat{\varphi}(z)|^2 dz = \|\hat{\varphi}\|_p^2.$$

Notice that H_p with respect to the norm defined in (2) is the set of all complex valued measurable functions such that $\|\varphi\|_p < \infty$.

Let D^k be any differential operator with respect to x with constant coefficients of order $k \leq p$. Then D^k is a bounded linear operator which maps H_p into H_{p-k} .

DEFINITION 1. Let $\varphi(t)$ be a variable element of H_p depending on a real parameter t in a finite interval $J = [0, T]$. We say that $\varphi(t)$ is H_p -continuous in t in J , if the mapping t in $J \rightarrow \varphi(t)$ in H_p is continuous; That is, $t \rightarrow t_0$ in the interval J implies $\varphi(t) \rightarrow \varphi(t_0)$ in H_p . We also maintain that $\varphi(t)$ is H_p -differentiable at $t = t_0$, if there exist a function $g(t)$ in H_p such that

$$(t - t_0)^{-1}[\varphi(t) - \varphi(t_0)] \rightarrow g(t_0)$$

in H_p as $t \rightarrow t_0$, then we denote $g(t)$ by $\varphi'(t) = (d/dt)\varphi(t)$.

If D^k is a differential operator in x in E^m with constant coefficients of order k and $\varphi(t)$ is H_p -continuous in t , then $D^k\varphi(t)$ is H_{p-k} continuous, and if $\varphi(t)$ is H_p -differentiable in t then $D^k\varphi(t)$ is H_{p-k} differentiable in t and

$$(d/dt)[D^k\varphi(t)] = D^k[(d/dt)\varphi(t)].$$

Mr. Nagumo in [1] considered a system of linear partial differential equations for an r -vector function with parameter $\varepsilon > 0$,

$$(3) \quad L_\varepsilon(u) = \sum_{j=0}^l P_j(\partial_x, \varepsilon)(\partial_t)^j u = f_\varepsilon(t, x)$$

where $P_j(z, \varepsilon)$ are $r \times r$ matrices of polynomials in (z, ε) with constant coefficients and P_l is free from ∂_x such that $\det [P_l(\varepsilon)] \neq 0$ for $\varepsilon = 0$.

Here we are concerned with the case of one equation for one complex valued function $u(t, x)$ containing the parameter $\varepsilon > 0$

$$(4) \quad L_\varepsilon(u) = \sum_{j=0}^l P_j(\partial_x, \varepsilon)(\partial_t)^j u = f_\bullet(t, x)$$

with the following assumptions:

(0) $L_\varepsilon(\sigma)$ be a polynomial in $\sigma, \partial x$ and ε which is not divisible by the square of a similar nonconstant polynomial for $0 \leq \varepsilon \leq \varepsilon_0$ and $f_\varepsilon(t, x)$ is H_p -continuous. $P_j(z, \varepsilon)$ are polynomials in $(z, \varepsilon) = (z_1, \dots, z_m, \varepsilon)$ with constant coefficients such that $P_l(\varepsilon) \equiv \varepsilon$ and $P_{l-1}(z)$ is independent of ε .

System (4) is certainly a special case of system (3). Restricting ourselves to this special case, we will prove a stability theorem somewhat different from that of Mr. Nagumo [1]. Mr. Nagumo proved the convergence of the weak solution to $u_0(t, x)$; where as we shall prove the convergence of the solution $u_\varepsilon(t, x)$ to $u_0(t, x)$.

DEFINITION 2. We say that equation (4) is an H_p -stable equation for $\varepsilon \rightarrow 0$ in $0 \leq t \leq T$ with respect to a particular solution $u_0(t, x)$ of (4) for $\varepsilon = 0$ if and only if $u_\varepsilon(t) \rightarrow u_0(t)$ in H_p for $0 \leq t \leq T$ provided that

$$(5) \quad f_\varepsilon(t, x) \rightarrow f_0(t, x)$$

in H_p for $0 \leq t \leq T$ and $u_\varepsilon(t, x)$ is a solution of the partial differential equation (4) such that

$$(6) \quad \begin{aligned} & (i) \quad \partial_t^{j-1} u_\varepsilon(0) \rightarrow \partial_t^{j-1} u_0(0) \text{ in } H_p \quad (j = 1, \dots, l-1). \\ & (ii) \quad \text{There exists a function } F(x) \text{ in } H_p \text{ such that} \\ & \quad |\partial_t^{l-1} \hat{u}_\varepsilon(0, z)| \leq |\hat{F}(z)| \text{ for all small } \varepsilon > 0. \end{aligned}$$

As in [1] we associate the partial differential equation (4) with the ordinary differential equation

$$(7) \quad \sum_{j=0}^l P_j(iz, \varepsilon) (d/dt)^j y = 0.$$

Let $Y_j(t, z, \varepsilon)$ be the solution of the ordinary differential equation (7) with the initial conditions

$$(8) \quad \partial_t^{k-1} Y_j(0, z, \varepsilon) = \delta_{jk} \quad (\delta_{jk} \text{ is the Kronecker delta}).$$

We state here a well known result

THEOREM 1. *Let $P(z)$ be a polynomial in z (z in E^m) with complex coefficients. If $S = \{z \text{ such that } P(z) = 0\}$ then S is measurable and has E^m measure zero unless $P(z)$ is identically zero.*

The proof is simple. One approach is to use mathematical induction on m and Fubini's theorem.

COROLLARY. *There exist an $\varepsilon_0 > 0$ such that for each ε in $0 \leq \varepsilon \leq \varepsilon_0$ then assumption (0) implies that the polynomial equation*

$$(9) \quad \sigma^l + P_{l-1} \sigma^{l-1} + \dots + P_0(iz, \varepsilon) = 0$$

has distinct roots except for z belonging to a set of E^m measure zero.

Proof. Notice that the assumption (0) implies that $D(z, \varepsilon)$ the discriminant of equation (9) is not identically zero. $D(z, \varepsilon)$ is a polynomial of $(z_1, \dots, z_m, \varepsilon)$. Let us write $D(z, \varepsilon)$ as a product of irreducible polynomials in z and ε over the field of complex coefficients. If one or several of the factors do not depend on z explicitly, then they are polynomials in ε ; in fact they are linear. All of these have at most finitely many positive zeros, say $\varepsilon_1, \dots, \varepsilon_n$. Let $\varepsilon_0 = \min(\varepsilon_1, \dots, \varepsilon_n)$; then for $\varepsilon \leq \varepsilon_0$ we can write $D(z, \varepsilon)$ as a product of irreducible polynomials in z and ε none of which vanishes identically. Now by Theorem 1 the zeros of such polynomials for each ε are set of E^m measure zero.

Let $Y_i(t, z, \varepsilon)$ ($i = 1, 2, \dots, l$) be the solution of the ordinary dif-

ferential equation (7) with the initial conditions (8). If $\sigma_1, \dots, \sigma_l$ are the distinct roots of equation (9) then we can write

$$(10) \quad Y_i(t, z, \varepsilon) = \sum_{j=1}^l \alpha_j^i \exp(\sigma_j t) .$$

Here α_j^i are constants to be computed by using the initial conditions (8). Let $V(\sigma_1, \dots, \sigma_l)$ be the Vandermonde determinant of $\sigma_1, \dots, \sigma_l$, i.e., $V(\sigma_1, \dots, \sigma_l) = \pi_{q>p}(\sigma_q - \sigma_p)$. Denote by V_i^j the determinant obtained from V by cancelling the i -th column and the j -th row.

THEOREM 2. $V_i^j = \pi_{qp, q \neq j \neq p}(\sigma_q - \sigma_p)E_{j,i}$ where $E_{j,i}$ is the coefficient of the i -th power of σ_j in the expression

$$(\sigma_1 - \sigma_j) \cdots (\sigma_{j-1} - \sigma_j)(\sigma_{j+1} - \sigma_j) \cdots (\sigma_l - \sigma_j)(-1)^j .$$

The proof is simple. Just write $V(\sigma_1, \dots, \sigma_l)$ in two ways; first as a polynomial in σ_j , and second as a product of linear terms then equate the coefficients of σ_j in the two expressions.

Then initial conditions (8) and further use of Vandermonde determinant give the following result

$$(11) \quad Y_i(t, z, \varepsilon) = (-1)^{i-1} \left[\sum_{j=1}^l E_{j, (i-1)} \exp(\sigma_j t) / A_j \right]$$

where,

$$(12) \quad A_j = (\sigma_l - \sigma_j) \cdots (\sigma_{j-1} - \sigma_j)(\sigma_{j+1} - \sigma_j) \cdots (\sigma_1 - \sigma_j) .$$

Since the preceding result can be computed easily, we shall omit the details.

THEOREM 3. *If $\sigma_1, \dots, \sigma_l$ are the roots of equation (9) and Y_1, Y_2, \dots, Y_l are the solutions of the ordinary differential equation (7) with the initial conditions (8). Then for each $\sigma_j \neq 0 (j = 1, \dots, l)$ we have*

$$(13) \quad \sigma_j^{l-1} \sum_{i=1}^l (Y_i / \sigma_j^{l-1}) = \exp(\sigma_j t) .$$

Proof. The initial conditions (8) shows that the identity (13) is valid for $t = 0$. Furthermore take the 1st, 2nd, \dots , $(l-1)$ -th derivatives of both sides of the identity with respect to t and each time apply the initial conditions (8) we get the validity of the identity for $t = 0$. Since the right side of equation (13) is a solution of the ordinary differential equation (7) therefore the identity (13) is valid for all t in $0 \leq t \leq T$.

THEOREM 4. *For each fixed z in E^m consider $Y_j(t, z, \varepsilon)$ a function of t and ε only and assume that there exists a number $M_j(z)$ ($j = 1, \dots, l$) independent of both t and ε such that*

$$|Y_j(t, z, \varepsilon)| \leq M_j(z)$$

for $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq t \leq T$. Then the roots of equation (9) have for each z in E^m a real part bounded from above as $\varepsilon \rightarrow 0$.

Proof. Let z_0 be a fixed point in E^m . Let $\mu = \varepsilon\sigma$ when σ is a root of equation (9) then, equation (9) becomes

$$(14) \quad \mu^l + P_{l-1}(iz_0)\mu^{l-1} + \varepsilon P_2(iz, \varepsilon)\mu^{l-2} + \dots + \varepsilon^{l-1}P_0(iz, \varepsilon) = 0.$$

Now assume first that $P_{l-1}(iz_0) \neq 0$ then for $\varepsilon = 0$ equation (14) becomes

$$(15) \quad \mu^l + P_{l-1}(iz_0)\mu^{l-1} = 0 = \mu^{l-1}(\mu + P_{l-1}(iz_0)).$$

Here we have one simple root $\mu_1 = -P_{l-1}(iz_0)$, if we call this root $\mu_1(0)$ then we can write

$$\mu_1(\varepsilon) = \mu_1(0) + \sum_{i=1}^{\infty} q_{1i}\varepsilon^i, \text{ so } \mu_1(\varepsilon) \rightarrow \mu_1(0) \text{ as } \varepsilon \rightarrow 0.$$

Therefore we can write the simple root $\sigma_1(z_0, \varepsilon)$ of equation (14) as follows,

$$\sigma_1 = (1/\varepsilon)\mu_1(\varepsilon) = (-P_{l-1}(iz_0)/\varepsilon) + \sum_{i=1}^{\infty} q_{1i}(z_0)\varepsilon^{i-1}.$$

Hence $\text{Real } \sigma_1(z_0, \varepsilon) = -\text{Real } P_{l-1}(iz_0)/\varepsilon + \text{Real } q_{11}(z_0) + o(\varepsilon)$. Therefore if $\text{Real } P_{l-1}(iz_0) \geq 0$ then obviously $\text{Real } \sigma_1(\varepsilon)$ is bounded from above. Now suppose that $\text{Real } P_{l-1} < 0$. Then $|\sigma_1(\varepsilon)| \geq \text{Real } (\sigma_1(\varepsilon)) \rightarrow \infty$ in turn implies that for $t > 0$ we get

$$\left| \sum_{j=1}^l [Y_j(t, z_0, \varepsilon)/\sigma^{l-j}] \right| M_j(z_0)$$

as $\varepsilon \rightarrow 0$ for some number $M_j(z_0)$ independent of t and ε . This is so because of the hypothesis of the theorem. Now we use Theorem 3. Then identity (13) shows for small

$$\varepsilon > 0, 2|P_{l-1}| \geq |-P_{l-1}(iz_0 + 0(\varepsilon))| \geq (1/2)|P_{l-1}(iz_0)|$$

since $\text{Real } P_{l-1}(iz_0) < 0$ and as $t > 0$,

$$\begin{aligned} [\exp(t \text{Re } \sigma_1)/\sigma_1^{l-1}] &\geq [\exp(2'_2\varepsilon|P_{l-1}|)\varepsilon^{l-1}/(4|P_{l-1}(iz_0)|)^{l-1}] \\ &\geq (t/4^l l! \varepsilon) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0 \end{aligned}$$

utilizing only the $(l-1)$ th term of the Taylor series of the exponential function.

But the above result contradicts the boundedness of expression (13) for small $\varepsilon > 0$. Consequently $\text{Real } P_{l-1}(iz_0) \geq 0$ and hence $\text{Real } \sigma_1(\varepsilon)$ is bounded from above as $0 < \varepsilon \leq \varepsilon_0$.

In order to prove the result for the remaining of the roots of equation (9) we shall give a reasoning which is incidentally does not utilize the condition $P_{l-1}(iz_0) \neq 0$. Equation (15) has a root $\mu(0) = 0$ of multiplicity $(l - 1)$. From the Puiseux series expansion we deduce that the $(l - 1)$ roots $\mu(\varepsilon)$ of equation (14) other than μ_1 will split into r groups of $m_1, \dots, m_r, 1 \leq m_1 \leq \dots \leq m_r$ and $\sum_1^r m_i = l - 1$ as follows: each root $\mu_{0j}, \dots, \mu_{m_{j-1}j}$ can be written as $\mu_{\eta j} = \sum_{i=1}^{\infty} q_{\eta j i} x^i$ where,

$$x = (\varepsilon)^{1/m_j} \text{ and } \eta = 0, 1, \dots, m_{j-1}.$$

Notice that the above series converges for sufficiently small x . We shall here and later understand by $(\varepsilon)^{1/m_j}$ the positive m_j -th root of $\varepsilon > 0$. Let $\mu_{\eta j}$ be any one of the $(l - 1)$ roots of equation (14) which tend to zero as ε tends to zero, then we can write the corresponding roots σ_η of equation (9) as follows

$$(16) \quad (\mu_{\eta}/\varepsilon) = \sigma_\eta(x, z_0) = (1/X^m j \left(\sum_{i=1}^{\infty} q_{\eta j i} x^i \right)).$$

Now to simplify notations, let us drop indices j, η of the root $\mu_{\eta j}$ once we are dealing with only one root. Put $q\eta_{ji} = q_i$ and $m_j = m$. Assume that q_k is the first nonzero coefficient in equation (16) if there is any, and q_s is the first nonzero coefficient that has a nonzero real part, if there is any, in the expression $\mu = \sum q_i x^i$. Evidently, if $s \geq m$ then, $\text{Real } \sigma(x, z_0) = x^{s-m} \text{Real } q_s + 0(x)$, and this is bounded from above as $x \rightarrow 0$. If there is no s , then $\sigma(x, z_0) \leq 0$. Let $s < m$, notice that $k \leq s \leq m$. Then we can write σ as follows,

$$\sigma(x) = x^{k-m}(q_k + \dots + q_s x^{s-k} + q_{s+1} x^{s-k+1} + \dots)$$

and

$$\text{Real } \sigma(x) = x^{s-m}(\text{Real } q_s + 0(x)).$$

If $\text{Real } q_s < 0$ then $\text{Real } \sigma(x) \rightarrow -\infty$ as $x \rightarrow 0$ implying $\text{Real } \sigma \leq 0$ as $\varepsilon \rightarrow 0$. Suppose finally $\text{Real } q_s > 0$, i.e., $\text{Real } \sigma(x) \rightarrow +\infty$ as $x \rightarrow 0$. Then for small $x > 0$, $2|q_k| \geq |q_k + 0(x)|$ and $\text{Real } q_s + 0(x) \geq \text{Re } q_s/2$, and

$$\begin{aligned} \exp(t \text{Re } \sigma)/|\sigma|^{l-1} &\geq \exp(t \text{Re } q_s x^{s-m})/2^{l-1} |q_k|^{l-1} x^{(k-m)(l-1)} \\ &> (t^j/j! 2^j) x^{j(s-m)} (\text{Real } q_s)^j/2^{l-1} |q_k|^{l-1} x^{(k-m)(l-1)}. \end{aligned}$$

Here we utilize only the $(j + 1)$ -th term of the Taylor series of the exponential function where j is the smallest integer greater than zero

such that $(m - k)(l - 1) + j(s - m) < 0$. Therefore

$$\exp(t \operatorname{Re} \sigma) / |\sigma|^{l-1} > t^j (\operatorname{Re} q_s)^j x^{(m-k)(l-1) + j(s-m)} / j! 2^{j+l-1} |q_k|^{l-1} \rightarrow \infty$$

as $x \rightarrow 0$ because of the assumption on j .

Now in order to prove that $\operatorname{Re} \sigma$ is bounded from above we use identity (13) once more.

In order to complete the proof of the theorem we should assume that $P_{l-1}(iz_0) = 0$. Notice that equation (14) for $P_{l-1} = 0$ becomes

$$\mu^l + \varepsilon P_{l-2}(iz_0, \varepsilon) + \cdots + \varepsilon^{l-1} P_0(iz_0, \varepsilon) = 0.$$

This means that $\mu = 0$ is a root of multiplicity l , not $l - 1$, of equation (15). So by Puiseux series expansion we can write the roots

$$\begin{aligned} \sigma_{1\eta_1} &= (1/\varepsilon) \left[\sum_{j=1}^{\infty} \beta_{1j} \exp(2\pi i \eta_1 j / m_1) \varepsilon^{j/m_1} \right] & \eta_1 &= 0, \dots, m_1 - 1 \\ &\vdots & & \\ \sigma_{r\eta_r} &= (1/\varepsilon) \left[\sum_{j=1}^{\infty} \beta_{rj} \exp(2\pi i \eta_r j / m_r) \varepsilon^{j/m_r} \right] & \eta_r &= 0, \dots, m_r - 1. \end{aligned}$$

Now we can carry on the same proof as before for any root. The proof of Theorem 4 is completed.

In what follows there will be for each ε certain exceptional sets of z of measure zero for which our conclusion do not apply. In order to be able to draw inferences as $\varepsilon \rightarrow 0, \varepsilon > 0$ we wish to be able to disregard these sets. Now let the notion $\varepsilon \rightarrow 0$ be henceforth interpreted as meaning “ ε tends to zero through an arbitrary sequence of positive numbers.” Then all of the corresponding exceptional sets will still be a countable union of sets of measure zero and accordingly has itself measure zero.

THEOREM 5. *Assume in equation (9) that $P_{l-1}(iz_0)$ does not vanish identically. Also assume that for $0 \leq t \leq T, 0 < \varepsilon \leq \varepsilon_0$, there are numbers $M(z)$ and $M_j(z) (j = 1, \dots, l)$ independent of both t and ε such that,*

$$|Y_j(t, z, \varepsilon)| \leq M_j(z) \text{ for } 0 < \varepsilon \leq \varepsilon_0.$$

Then for almost all z in E^m we have

- (i) $|Y_l(t, z, \varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$ and for $0 \leq t \leq T$.
- (ii) $|(1/\varepsilon) Y_l(t, z, \varepsilon)| \leq M(z)$ for $0 \leq t \leq T$ and as $\varepsilon \rightarrow 0$.

Proof. Here we use Theorem 1 in order to be able to assume $P_{l-1}(iz_0) \neq 0$. By the corollary of Theorem 1, equation (9) has for fixed small ε and for almost all z in E^m distinct roots. By letting $\mu = \varepsilon \sigma$ the equation (9) becomes as we have seen before

$$\mu^l + P_{l-1}(iz_0)\mu^{l-1} + \varepsilon P_{l-1}(iz_0, \varepsilon)\mu^{l-2} + \dots + \varepsilon^{l-1}P_0(iz_0, \varepsilon) = 0.$$

So for $\varepsilon = 0$ the above equation will be,

$$\mu^{l-1}(\mu + P_{l-1}(iz_0)) = 0.$$

Therefore by Puiseux series expansion we can write the roots of equation (9) as follows,

$$\sigma_1 = (1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon)]$$

m_1 roots as follows,

$$\begin{aligned} \sigma_{1+\eta_1} &= (1/\varepsilon) \left[\sum_{j=1}^{\infty} \beta_{1j} \varepsilon^{j/m_1} \exp(2\pi i \eta_1 j / m_1) \right] & n_1 = 1, \dots, m_1 \\ &\vdots \end{aligned}$$

and m_r roots as follows,

$$\sigma_{-m_r+\eta_r} = (1/\varepsilon) \left[\sum_{j=1}^{\infty} \beta_{rj} \exp(2\pi i \eta_r j / m_r) \varepsilon^{j/m_r} \right] \quad \eta_r = 1, \dots, m_r.$$

Provided that $l = 1 + m_1 + \dots + m_r$ where $1 \leq m_1 \leq \dots \leq m_r$. Let us write $\gamma = [\sigma_1, \dots, \sigma_l]$. Notice that each σ in γ is a regular function in ε and hence the difference between any two of them is also regular in ε . If $\sigma_1 = (1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon)]$ then for any $j, 1 < j \leq l$ we have, since $P_{l-1}(iz_0) \neq 0$,

$$|\sigma_j - \sigma_1| = |(1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon) - 0(\varepsilon^R)]| \rightarrow \infty \text{ as } \varepsilon \rightarrow 0.$$

For $\sigma_j = (1/\varepsilon)[0(\varepsilon^R)]$, R 0 a rational number.

Notice that for each $i = 1, \dots, l, i \neq j$,

$$|\sigma_j - \sigma_i|$$

either; (1) goes to zero as ε tends to zero; (2) tends to some fixed number greater than zero or (3) goes to infinity as ε tends to zero. For any arbitrary σ in γ collect those, and only those, elements σ' of γ such that $|\sigma' - \sigma| \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then γ will split into disjoint subsets, namely:

$$(17) \quad \gamma = \gamma_1 U \gamma_2 \dots U \gamma_r \text{ and } \gamma_j \cap \gamma_k = \emptyset \text{ for } j \neq k,$$

which incidentally do not necessarily coincide with our previous grouping of the σ 's. According to this decomposition of γ and using identity (10) we can write

$$|Y_i(t, z, \varepsilon)| = \left| \sum_{\sigma_j \in \gamma_1} [\exp(t\sigma_j)/A_j] + \dots + \sum_{\sigma_j \in \gamma_r} [\exp(t\sigma_j)/A_j] \right|.$$

LEMMA 1. *In the above expression each summation tends to zero in absolute value as ε tends to zero.*

Proof. Denoting by $\sigma'_1, \dots, \sigma'_n$ the elements of γ_1 . Then,

$$\sum_{\sigma'_j \in n\gamma_1} [\exp(\sigma'_j t)/A_j] = \sum_{\delta'_j \in n\gamma_1} [\exp(\sigma'_j t)/\varphi]$$

$$\varphi = \prod_{\sigma_k \in n\gamma - \gamma_1} (\sigma_k - \sigma'_j) \prod_{\substack{\sigma_k \in n\gamma_1 \\ k \neq j}} (\sigma'_k - \sigma'_j).$$

Let $F(\sigma'_j) = \exp(t\sigma'_j)/\prod_{\sigma_k \in n\gamma - \gamma_1} (\sigma_k - \sigma'_j)$ and $F(\sigma'_j) = \alpha(\varepsilon)$. Now if γ_1 contains the root $\sigma_1 = (1/\varepsilon)[-P_{l-1}(iz_0) + 0(\varepsilon)]$ then it will contain only σ_1 . Since $P_1(iz_0) \neq 0$ it is easily shown that,

$$\exp(t\sigma_1)/A_1 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Now suppose that γ_1 does not contain σ_1 . Then $\prod_{\sigma_k \in n\gamma - \gamma_1} (\sigma_k - \sigma'_j)$ will have a factor $(\sigma_1 - \sigma'_j) = 0(\varepsilon)$ and hence tends to infinity as ε tends to zero while no factor of $\prod_{\sigma_k \in n\gamma - \gamma_1} (\sigma_k - \sigma'_j)$ tends to zero as ε tends to zero. Therefore,

$$\prod_{\sigma_k \in n\gamma - \gamma_1} (\sigma_k - \sigma'_j) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0.$$

This in turn implies that $\alpha(\varepsilon)$ tends to zero as ε tends to zero. Let $\alpha = \min(\lim_{\varepsilon \rightarrow 0} |\sigma_k - \sigma'_j|)$. The minimum is taken over all σ_k in $\gamma - \gamma_1$. Notice that by the definition of $\gamma_1, \delta > 0$. Now chose a circle C of radius $\delta/2$ about one of the points $\sigma'_1, \dots, \sigma'_n$ of γ_1 . Then for sufficiently small $\varepsilon > 0, C$ will contain those, and only those, σ'_j which belong to γ_1 . We may likewise assume that for any point w on the circumference of C , that $|w - \sigma'_j| > \delta/4, (j = 1, \dots, n)$. Let

$$I = (1/2\pi i) \int_C F(\eta) d\eta / \prod_{\delta'_k \in n\gamma_1} (\sigma'_k - \eta)$$

then,

$$(18) \quad I < 4^n \alpha(\varepsilon) / \delta^n \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

On the other hand, I equal the absolute value of the sum of the residues of the integrand at $\sigma'_1, \dots, \sigma'_n$. Notice that for each σ'_j the residue of I at σ'_j is $F(\sigma'_j)/\prod_{\sigma_k \neq \sigma'_j} (\sigma_k - \sigma'_j)$ and hence the sum of the residues of I at the $\sigma'_1, \dots, \sigma'_n$ is equal to

$$\sum_{\sigma'_j \in n\gamma_1} [\exp(t\sigma'_j)/A_j].$$

Hence by (18) the above expression tends to zero as ε tends to zero and this ends the proof of the lemma. Now in order to finish the proof of the theorem we just write,

$$|Y_i(t, z\varepsilon)| \leq \left| \sum_{\sigma_j i n \gamma_1} [\exp(t\sigma_j)/A_j] + \cdots + \sum_{\sigma_j i n \gamma_r} \exp(t\sigma_j)/A_j \right|.$$

Using Lemma 1 one sees for all positive $t \leq T$ that

$$|Y_i(t, z, \varepsilon)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Now for proving the second part of the theorem, one notices from the preceding discussion that $Y_i(t, z, \varepsilon) = 0(\varepsilon)$, i.e., there exist a number $M(z)$ such that $|Y_i(t, z, \varepsilon)/\varepsilon| \leq M(z)$.

LEMMA A. *Limit* $_{\varepsilon \rightarrow 0} |E_{1, (i-1)}/A_1| = 0$.

Proof. $|E_{1, (i-1)}/A_1| = |\sigma_2 \cdots \sigma_{l-i+1} + \cdots + \sigma_{i+1} \cdots \sigma_{l-1}/j = 2(\sigma_j - \sigma_1)|$ ($j = 1, \dots, l$). Notice that as ε tends to zero σ_1 tends to infinity while $\sigma_j(\varepsilon)$ tends to $\sigma_j(0)$ and hence this proves the lemma.

LEMMA B. *Limit* $_{\varepsilon \rightarrow 0} (E_{j, (i-1)}/\sigma_1) = E'_{j, (i-1)}$ where $E'_{j, (i-1)}$ is the coefficient of the $(i-1)$ th power of $\sigma_j(0)$ in the expression,

$$\prod_{\substack{k=2 \\ k \neq j}}^l [\sigma_k(0) - \sigma_j(0)](-1)^j.$$

Proof. Notice that $E_{j, (i-1)}$ is the sum of the product of σ taken $l - (i+1)$ at a time, i.e.,

$$E_{j, (i-1)} = [\sigma_1 \cdots \sigma_{l-i+1} + \cdots + \sigma_{i+1} \cdots \sigma_{l-1}](-1)^{j-1}(-1)^{i-1}.$$

Therefore,

$$E_{j, (i-1)}/\sigma_1 = \sigma_2 \cdots \sigma_{l-i+1} + \cdots + (\sigma_{i+1} \cdots \sigma_{l-1}/\sigma_1)(-1)^{i+j-2}.$$

Now it is easy to see that $E_{j, (i-1)}$ tends to $E'_{j, (i-1)}$ as ε tends to zero.

LEMMA C. *Limit* $_{\varepsilon \rightarrow 0} [E_{j, (i-1)} \exp(t\sigma_j)/A_j] = E'_{j, (i-1)} \exp(\sigma(0)t)/A'_j$ ($j = 2, \dots$) and $A'_j = k = 2[\sigma_k(0) - \sigma_j(0)]$.

Proof. Let us write

$$A_j = (\sigma_1 - \sigma_j) \prod_{\substack{k=2 \\ k \neq j}}^l (\sigma_k - \sigma_j) = \sigma_1 \prod_{\substack{k=2 \\ k \neq j}}^l (\sigma_k - \sigma_j) - \sigma_j \prod_{\substack{k=2 \\ k \neq j}}^l (\sigma_k - \sigma_j).$$

Therefore A_j/σ_1 tends to A'_j as ε tends to zero. Now we use Lemma B and the proof of Lemma C will be completed.

$$\text{Notice that } Y_i(t, z, 0) = \sum_{j=2}^l E'_{j, (i-1)} \exp(\sigma_j(0)t)/A'_j$$

is the solution of the ordinary differential equation $L_0(D)Y = 0$ with the initial conditions $\sigma_i^{k-1} Y_i(0, z, 0) = \delta_{jk} \mathbf{1} \leq j, k \leq l-1$. Now we may

sum the results of Lemmas A, B, and C in the following theorem.

THEOREM 6. *Let $Y_1(t, z, \varepsilon), \dots, Y_l(t, z, \varepsilon)$ be the solutions of the ordinary differential equation (7) with the initial conditions (8). Assume that; (1) $P_{l-1}(iz)$ is not identically zero. (2) assumption (0) on page (3). And finally (3) There exist numbers $M_i(z)$ and ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$ and for almost all z in E^m ,*

$$|Y_i(t, z, \varepsilon)| \leq M_i(z) \quad 1 \leq i \leq l, l > 2.$$

Let ε tend to zero on a sequence, then $Y_i(t, z, \varepsilon)$ tends to $Y_i(0, z, 0)$ for $0 \leq t \leq T, 1 \leq i \leq l-1$ and for almost all z in E^m . Where $Y_i(t, z, 0)$ are the solution of the ordinary differential equation $L_0(D)Y = 0$ with the initial conditions $\partial_t^{k-1} Y_i(0, z, 0) = \delta_{ik}, 1 \leq i, k \leq l-1$.

THEOREM 7. *Assuming all the hypothesis of Theorem (5) then for each z in E^m we have,*

$$|(1/\varepsilon)Y_l(t, z, \varepsilon) - (1/P_{l-1}(iz))Y_{l-1}(t, z, \varepsilon)| \rightarrow 0$$

as $\varepsilon \rightarrow 0$ and for all $0 \leq t \leq T$.

Proof.

$$\begin{aligned} & |\varepsilon^{-1}Y_l(t, z, \varepsilon) - P_{l-1}(iz)^{-1}Y_{l-1}(t, z, \varepsilon)| \\ & \leq \left| (\varepsilon A_l)^{-1} \exp(\sigma_1 t) + (A_l P_{l-1})^{-1} \exp(\sigma_1 t) \sum_{i=1}^l \sigma_i \right| \\ & + \left| \sum_{j=2}^l [(\varepsilon A_j)^{-1} \exp(\sigma_j t) + (P_{l-1} A_j)^{-1} \exp(\sigma_j t) \sum_{\substack{i=1 \\ i \neq j}}^l \sigma_i] \right|. \end{aligned}$$

It is clear that the first term of the above sum tends to zero as ε tends to zero. Before dealing with the second term we shall reduce it into a simpler form. Notice that $\sum_{i=1, i \neq j}^l \sigma_i = \sum_{i=1}^l \sigma_i - \sigma_j = -\varepsilon^{-1}P_{l-1}(iz) - \sigma_j$.

Then it is easy to see that

$$\begin{aligned} & \sum_{j=2}^l \left[(\varepsilon A_j)^{-1} \exp(\sigma_j t) + (A_j P_{l-1})^{-1} \exp(\sigma_j t) \sum_{\substack{i=1 \\ i \neq j}}^l \sigma_i \right] \\ & = - \sum_{j=2}^l (A_j P_{l-1})^{-1} \sigma_j \exp(\sigma_j t). \end{aligned}$$

Now if, $|\sigma_i - \sigma_j| \geq \delta > 0$ ($i, j = 2, \dots, l$) $i \neq j$ then it is clear that

$$\left| \sum_{j=2}^l (A_j P_{l-1})^{-1} \sigma_j \exp(\sigma_j t) \right| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

On the other hand, if for some i and $i \neq j$ we have $|\sigma_i - \sigma_j|$ tends

to zero as ε tends to zero then we use the residue theorem to prove that

$$\sum_{j=2} (A_j P_{l-1})^{-1} \sigma_j \exp(\tau_j t) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

in the same way used before. The proof of Theorem 7 is ended.

Now we arrive at the main theorem of this paper.

THEOREM 8. *Let the degree of $P_j(iz)$ be at most k in z ($j = 1, \dots, l$) and assume that $P_{l-1}(iz)$ not identically zero. Denote by $u_0(t)$ the $l-1$ times H_{p+k} continuously differentiable solution of the partial differential equation (4) for $\varepsilon = 0$ in $0 \leq t \leq T$. If there exist two constants $\varepsilon_0 > 0$ and C such that*

$$(19, i) \quad \sup_{z \in E^m} |Y_j(t, z, \varepsilon)| \leq C \text{ for } 0 \leq t \leq T \text{ and } 0 < \varepsilon \leq 0 \quad (j = 1, \dots, l)$$

$$(19, ii) \quad \sup_{z \in E^m} \int_0^T |\varepsilon^{-1} Y_l(t, z, \varepsilon)| dt \leq C \text{ for } 0 < \varepsilon \leq \varepsilon_0$$

where $y = Y_j$ the solutions of equation (7) with the initial conditions (8). Then equation (4) is an H_p -stable equation with respect to $u_0(t)$.

Proof. Let $u_\varepsilon(t, x)$ be l -times H_{p+k} continuously differentiable solution of the partial differential equation (4) with the initial conditions (6). Then from Theorem 2 in [1] we may write

$$u_\varepsilon(t, x) = \sum_{j=1}^l (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_j(t, z, \varepsilon) \partial_t^{j-1} \hat{u}_\varepsilon(0, z) dz \\ + 2\pi^{-m/2} \int_{E^m} \exp(ix, z) \int_0^t \varepsilon^{-1} Y(t - \tau, z\varepsilon) \hat{f}_\varepsilon(\tau, z) d\tau dz$$

and

$$u_0(t, x) = \sum_{j=1}^{l-1} (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_j(t, z, 0) \partial_t^{j-1} \hat{u}_0(0, z) dz \\ + (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) \int_0^t P_{l-1}^{-1} Y_l(t - \tau, z) \hat{f}_0(\tau, z) d\tau dz.$$

We have to prove that

$$\|u_\varepsilon(t, x) - u_0(t, x)\|_p \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Let us write

$$\|u_\varepsilon(t, x) - u_0(t, x)\|_p = M(x) + N(x) + Q(x) \\ M(x) = \left\| (2\pi)^{-m/2} \sum_{j=1}^l \int_{E^m} \exp(ix, z) Y_j(t, z, \varepsilon) \partial_t^{j-1} [\hat{u}_\varepsilon(0, z) - \hat{u}_0(0, z)] \right. \\ \left. + (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_{l-1}^{l-1} \hat{u}_\varepsilon(0, z) \right\|_p$$

$$\begin{aligned}
 N(x) &= \int_{E^m} |(1 + |z|^2)^{m/2} \int_0^t \varepsilon^{-1} Y_l(t - \tau, z, \varepsilon) \hat{g}(\tau, z) d\tau|^2 dz \\
 &\quad \times \hat{g}(\tau, z) = \hat{f}_\varepsilon(\tau, z) - \hat{f}_0(\tau, z) \\
 Q(x) &= \int_{E^m} (1 + |z|^2)^{m/2} \int_0^t (\varepsilon^{-1} Y_l(t - \tau, z, \varepsilon) - P_{l-1}^{-1} Y_{l-1}(t - \tau, z, 0)) \\
 &\quad \times \hat{f}_0(\tau, z) d\tau|^2 dz .
 \end{aligned}$$

To prove the convergence of $M(x)$ we proceed as follows,

$$\begin{aligned}
 M(x) &\leq \| (2\pi)^{-m/2} \sum_{j=1}^{l-1} \int_{E^m} \exp(ix, z) Y_j(t, z, \varepsilon) \partial_t^{j-1} [\hat{u}_\varepsilon(0, z) - \hat{u}_0(0, z)] \|_p \\
 &\quad + \| (2\pi)^{-m/2} \int_{E^m} \exp(ix, z) Y_l(t, z, \varepsilon) \partial_t^{l-1} \hat{u}_\varepsilon(0, z) \|_p .
 \end{aligned}$$

Using the condition (19, i) and Ascoli's theorem we conclude

$$M(x) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 .$$

Condition (19, ii) and Ascoli's theorem imply that

$$N(x) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 .$$

For $Q(x)$ we proceed as follows. Notice that Theorem 6 shows that

$$(1/P_{l-1}(iz)) Y_{l-1}(t, z, \varepsilon) \rightarrow (1/P_{l-1}(iz)) Y_{l-1}(t, z, 0)$$

and Theorem 7 shows that

$$|\varepsilon^{-1} Y_l(t, z, \varepsilon) - P_{l-1}^{-1} Y_{l-1}(t, z, \varepsilon)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 .$$

Therefore

$$|\varepsilon^{-1} Y_l(t, z, \varepsilon) - P_{l-1}^{-1}(iz) Y_{l-1}(t, z, 0)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 .$$

Consequently, using Ascoli's theorem once more we get

$$Q(x) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 .$$

This ends the proof of Theorem 8.

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