

# Pacific Journal of Mathematics

**A NONIMBEDDING THEOREM OF ASSOCIATIVE ALGEBRAS**

ERNEST LESTER STITZINGER

## A NONIMBEDDING THEOREM OF ASSOCIATIVE ALGEBRAS

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**Let  $A$  and  $B$  be associative algebras and define the Frattini subalgebra of  $A$ ,  $\phi(A)$ , to be the intersection of all maximal subalgebras of  $A$  if maximal subalgebras of  $A$  exist and as  $A$  otherwise. Conditions on  $B$  will be found such that  $B$  cannot be an ideal of  $A$  contained in  $\phi(A)$ .**

Hobby in [2] has shown that a nonabelian group  $G$  cannot be the Frattini subgroup of any  $p$ -group if the center of  $G$  is cyclic. Chao in [1] has shown that a nonabelian Lie algebra  $L$  can not be the Frattini subalgebra of any nilpotent Lie algebra if the center of  $L$  is one dimensional. In this note, we find a similiar result in the theory of associative algebras. However, in this case, it is not necessary to place any restrictions on the containing algebra.

Let  $A$  be an associative algebra over a field  $F$  and let  $B$  be an ideal of  $A$ . If  $x \in A$ , then  $x$  induces an endomorphism of the additive group of  $B$  by  $L_x(b) = xb$  for all  $b \in B$ . Let  $E(B, A)$  be the collection of all endomorphisms of this type. Then  $E(B, A)$  is a subspace of the vector space of all linear transformations from  $B$  into  $B$  and is an associative algebra under the compositions  $L_x + L_y = L_{x+y}$ ,  $\alpha L_x = L_{\alpha x}$  and  $L_x L_y = L_{xy}$  for all  $x, y \in A$  and all  $\alpha \in F$ . Clearly  $E(B, B)$  is an ideal of  $E(B, A)$ . If  $C$  is an ideal of  $A$  contained in  $B$ , then let  $E(B, A, C) = \{E \in E(B, A); E(c) = 0 \text{ for all } c \in C\}$ . Then  $E(B, A, C)$  is an ideal of  $E(B, A)$  and  $E(B, A)/E(B, A, C)$  is isomorphic to  $E(C, A)$ . Note that the mapping from  $A$  onto  $E(B, A)$  which assigns to each  $a \in A$  the element  $L_a$  is an algebra homomorphism. We define the right annihilating series of  $B$  inductively. Let  $r_1(B) = \{c \in B; bc = 0 \text{ for all } b \in B\}$  and let  $r_j(B)$  be the ideal of  $B$  such that  $r_j(B)/r_{j-1}(B) \cong r_1(B)/r_{j-1}(B)$  for  $j > 1$ . Since  $B$  is an ideal in  $A$ ,  $r_i(B)$  is an ideal in  $A$  for all  $i$ .

The following lemma is immediate.

**LEMMA.** *If  $A$  and  $A'$  are associative algebras and  $\pi$  is a homomorphism from  $A$  onto  $A'$ , then  $\pi(\phi(A)) \subseteq \phi(\pi(A))$ . Furthermore, if the kernel of  $\pi$  is contained in  $\phi(A)$ , then  $\pi(\phi(A)) = \phi(\pi(A))$ .*

**THEOREM.** *Let  $B$  be an associative algebra such that  $\dim r_1(B) = 1$  and  $\dim r_2(B) = k$  where  $1 < k < \infty$ . Then  $B$  cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

*Proof.* Suppose that to the contrary  $B$  is an ideal contained in the Frattini subalgebra of the associative algebra  $A$ . Then

$$E(B, B) \subseteq \phi(E(B, A)) .$$

For if  $T$  is the mapping from  $A$  onto  $E(B, A)$  defined by  $T(a) = L_a$  for all  $a \in A$ , then, by the lemma,

$$E(B, B) = T(B) \subseteq T(\phi(A)) \subseteq \phi(T(A)) = \phi(E(B, A)) .$$

Let  $z_1, \dots, z_k$  be a basis for  $r_2(B)$  such that  $z_k$  is a basis  $r_1(B)$ . For notational convenience, let  $r_i = r_i(B)$  for all  $i$ . Let  $\pi$  be the natural homomorphism from  $E(B, A)$  onto  $E(r_2, A)$ . Since

$$\begin{aligned} E(B, B) + E(B, A, r_2)/E(B, A, r_2) &\simeq E(B, B)/E(B, A, r_2) \cap E(B, B) \\ &= E(B, B)/E(B, B, r_2) \simeq E(r_2, B) \end{aligned}$$

it follows that

$$E(r_2, B) \simeq \pi(E(B, B)) \subseteq \pi(\phi(E(B, A))) \subseteq \phi(E(r_2, A)) .$$

We now show that  $E(r_2, B) \not\subseteq \phi(E(r_2, A))$  by showing that  $E(r_2, B)$  is complemented in  $E(r_2, A)$ . For  $i = 1, \dots, k - 1$ , define linear transformations  $e_i$  from  $r_2$  onto  $r_1$  by

$$e_i(z_j) = \begin{cases} \delta_{ij}z_k & \text{for } j = 1, \dots, k - 1 \\ 0 & \text{for } j = k \end{cases}$$

where  $\delta_{ij}$  is the Kronecker delta. Let  $S = ((e_1, \dots, e_{k-1}))$ . We claim that  $S = E(r_2, B)$ . Since  $r_1 = ((z_k))$  and  $B \cdot r_2 \subseteq r_1, E(r_2, B) \subseteq S$ . To show that  $S = E(r_2, B)$ , we shall show that  $\dim E(r_2, B) = k - 1 = \dim S$ . For each  $x \in B, L_x$  induces a linear transformation from  $r_2$  into  $r_1 \simeq F$ , where  $F$  is the ground field. Therefore, we may consider each  $L_x, x \in B$  as a linear functional on  $r_2$ . That is,  $E(r_2, B) \subseteq (r_2)^*$  where  $(r_2)^*$  is the dual space of  $r_2$ . Consequently,  $\dim E(r_2, B) = \dim r_2 - \dim r_2^B$  where  $r_2^B = \{z \in r_2; L_x(z) = 0 \text{ for all } x \in B\}$ . Clearly  $r_2^B = r_1$ . Then, since  $\dim r_2 = k$  and  $\dim r_1 = 1, \dim E(r_2, B) = k - 1$  and  $S = E(r_2, B)$ .

We now show that  $S$  is complemented in  $E(r_2, A)$ . Let

$$M = \{E \in E(r_2, A); E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij}z_j, \lambda_{ij} \in F, i = 1, \dots, k - 1$$

and  $E(z_k) = \lambda_k z_k, \lambda_k \in F\}$ .  $M$  is clearly a subalgebra of  $E(r_2, A)$  and  $M \cap S = 0$ . We claim that  $M + S = E(r_2, A)$ . Let  $E \in E(r_2, A)$ . Then  $E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij}z_j + \lambda_{ik}z_k$  for  $i = 1, \dots, k - 1$  and  $E(z_k) = \lambda_k z_k$ . However  $E = E - \sum_{i=1}^{k-1} \lambda_{ik}e_i + \sum_{i=1}^{k-1} \lambda_{ik}e_i$  where  $E - \sum_{i=1}^{k-1} \lambda_{ik}e_i \in M$  and  $\sum_{i=1}^{k-1} \lambda_{ik}e_i \in S$ . Therefore  $M + S = E(r_2, A)$ . We claim that  $M \neq 0$ . If  $M = 0$ , then  $E(r_2, A) = E(r_2, B)$  which contradicts

$$E(r_2, B) \subseteq \phi(E(r_2, A)) \subset E(r_2, A).$$

Consequently,  $S$  is complemented in  $E(r_2, A)$ , contradicting  $S \subseteq \phi(E(r_2, A))$ . This contradiction establishes the result.

**COROLLARY.** *Let  $B$  be a finite dimensional nontrivial nilpotent associative algebra with  $\dim r_1(B) = 1$ . Then  $B$  cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

#### REFERENCES

1. C. Y. CHAO, *A nonimbedding theorem of nilpotent Lie algebras*, Pacific J. Math. **22** (1967), 231-234.
2. C. Hobby, *The Frattini subgroup of a  $p$ -group*, Pacific J. Math. **10** (1960), 209-212.

Received January 23, 1969. This paper is part of the author's doctoral dissertation at the University of Pittsburgh.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$ 8.00; single issues, \$ 3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$ 4.00 per volume; single issues \$ 1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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