TOPOLOGICAL RINGS WITH PROPERTY (Y)

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This paper continues the study of Jordan ideals of the symmetric elements, S, of a 2-torsion free semiprime ring, A, with $2A = A$. Previous results assuming proper involution, as well as the annihilator condition on certain right ideals in topological rings with property (Y), are shown to be too restrictive. Finally, it is shown that every Jordan ideal of S either contains an idempotent, or is contained in the annihilator of the socle.

We denote the involution by $x \to x^*$, and let $(a, b)_J = ab + b^*a^*$ and $(a, b)_L = ab - b^*a^*$ for all $a, b \in A$. In [1], we observed that $B = \{b \mid (b, a)_J \in U$, a Jordan ideal of S, for all $a \in A\}$ is a right ideal, that $SB \subseteqq B$, $U \circ U \subseteqq B$, and $\mathscr{L}(B)S \subseteqq \mathscr{L}(B)$. This is the notation of [1].

In [2], Herstein proves that if A is semiprime then there exists no nonzero left or right nil ideals of bounded index of nilpotency. In [1], we proved that if U is a Jordan ideal of S and $u^2 = \theta$ for all $u \in U$ then $U = (\theta)$. We now generalize this result to the analog of the Herstein result.

THEOREM 1. *Let U be a Jordan ideal of S. Assume that there exists an integer k such that $u^k = \theta$ for all $u \in U$, then $U = (\theta)$.*

The theorem is proved in [1] for $k = 2$. It is sufficient by induction argument to show that $u^{4N} = \theta$ for all $u \in U$ implies $u^{2N} = \theta$; for then, $u^4 = \theta$ for all $u \in U$ implies $U = (\theta)$, while for $k \geqq 5$, then 4 is a factor of $k, k + 1, k + 2$, or $k + 3$ and so $2N < k$ and by induction we are done. Now, $u \in U$, implies $u^{2N} \in B$ and so assuming $v^{4N} = \theta$ for all $v \in U$, we have for each $a \in A$, $\theta = u^{2N}(u^{2N}a + a^*u^{2N})^{4N} = u^{2N}(a^*u^{2N})^{4N}$ and therefore, $(u^{2N}a)^{4N+1} = \theta$ or $u^{2N} = \theta$.

We note next that if b and c are in B and $r \in A$, then both $rc^*b = (c, r)_J b - cr^*b \in B$ and $c^*b = (c + c^*)b - cb \in B$. We make use of these facts in the following theorem which is the main result of §2, [1].

THEOREM 2. $\mathscr{L}(B)$ *is a self-adjoint two-sided ideal in A.*

We need only show that $\mathscr{L}(B)$ is a right ideal. Using the above remark $(ab^*b \in B$ for all $b \in B, a \in A)$ we see that as

$$x((b, a)_J)^2 = \theta$$

for all $a \in A$, $b \in B$, $x \in \mathscr{L}(B)$, then

(I) $x(a^*b^*)^2 = \theta$.

For $x \in \mathscr{L}(B)$, $b \in B$, $xb^* = x(b + b^*) \in \mathscr{L}(B)$. By (I) and the fact that B^* is a left ideal, we have $(xb^*)[a^*(xb^*)]^2 = \theta$ for all $a \in A$. Thus, xb^*A is nil of bounded index 3, and so $xb^* = \theta$. As B^* is a left ideal, $xab^* = \theta$ for all $a \in A$. This means $x^*b = (b^*x)^* = \theta^* = \theta$. Therefore, $\mathscr{L}(B)$ is self-adjoint and hence is a two-sided ideal.

LEMMA 3. $\mathscr{L}(B) \cap B = (\theta)$.

$\mathscr{L}(B)$ being two-sided implies that $\mathscr{L}(B) \cap B$ is a nilpotent right ideal and so is (θ), A being a semiprime ring.

It now follows immediately that.

LEMMA 4. $U \cap \mathscr{L}(B) = (\theta)$.

One observes that bSb^* and uSu are in $B \cap U$ for all $b \in B$ and $u \in U$. We make use of these facts to prove.

THEOREM 5. $\mathscr{L}(B) = \mathscr{L}(B \cap S) = \mathscr{L}(B \cap U) = \mathscr{L}(U)$.

The proper order of set inclusions for the first three of these sets are obvious. Thus the theorem is proved if we show $\mathscr{L}(U) \subseteqq \mathscr{L}(B)$ and $\mathscr{L}(B \cap U) \subseteqq \mathscr{L}(U)$.

Let $x \in \mathscr{L}(U)$ and $b \in B$ then $xbxbx = xbx(bx + x^*b^*) - x(bxx^*b^*) = \theta$. Therefore, $\mathscr{L}(U)b$ is a nil left ideal of index 3, leading to the conclusion that $\mathscr{L}(U) \subseteqq \mathscr{L}(B)$.

Similarly, if $x \in \mathscr{L}(B \cap U)$ and $u \in U$ then

$$xuxuxu = x[u(x + x^*)u]xu - xu(x^*ux)u = \theta ,$$

which shows that $\mathscr{L}(B \cap U) \subset \mathscr{L}(U)$.

We draw as an immediate consequence,

COROLLARY 6. $\mathscr{L}(S) = (\theta)$.

Now, define

DEFINITION 7. $U_J = \{s \in S \mid (s, u)_J = \theta \text{ for all } u \in U\}$.

We wish to conclude that

THEOREM 8. U_J is a Jordan ideal of S and $U_J = \mathscr{L}(U) \cap S$.

Let $s \in U_J, u \in U$ then $(s, u)_J = \theta = (s, u^2)_J$. The left equality yields $su = -us$ and so $2su^2 = \theta$, or by hypothesis on A, $su^2 = \theta$. Thus,

$$usu = \theta .$$

Now, let $s \in U_J, u \in B \cap U$, and $a \in A$. Noting that $suasu = s(u, a)_J su$ we conclude that

$$(sua)^3 = s[(u, a)_J s(u, a)_J]sua = \theta .$$

Thererfore, $s \in \mathscr{L}(B \cap U) = \mathscr{L}(U)$. Thus, $U_J \subseteqq \mathscr{L}(U) \cap S$.

However, $\mathscr{L}(U) \cap S \subseteqq U_J$ is immediate. As the equality holds we conclude that U_J is a Jordan ideal.

Two immediate corollaries are important for later theorems.

COROLLARY 9. *If* $U_J = (\theta)$, *then* $\mathscr{L}(U) = (\theta)$.

As $\mathscr{L}(U)$ is self-adjoint, the hypothesis implies $\mathscr{L}(U) \subseteqq K$ and so the desired conclusion.

COROLLARY 10. $T = \{t \in S \mid (t, s)_J \in U \text{ for all } s \in S\}$ *is a Jordan ideal and* $T_J = U_J = (B \cap S)_J$.

It follows immediately that T is a Jordan ideal of S and that $T_J \subseteqq (B \cap S)_J$. By Theorem 8,

$$U_J = \mathscr{L}(U) \cap S = \mathscr{L}(B \cap S) \cap S = (B \cap S)_J.$$

Let $r \in (B \cap S)_J$ and $t \in T$ then

$$(r, t)_J \in (B \cap S)_J \cap U \subseteqq \mathscr{L}(U) \cap U = (\theta) .$$

Thus, $r \in T_J$ and the corollary is proved.

2. **Topological rings with property (Y).** In [1] we defined these rings. Because of the results of the previous section, we now modify that definition by removing the condition of proper involution and by defining the annihilator condition for Jordan ideals based wholly on elements in S.

DEFINITION 11. A semiprime topological ring A with involution $(x \to x^*)$ is called a topological ring with property (Y) if, and only if,

(1) $A = 2A$, any net $\{2x_\alpha\} \to \theta$ implies $\{x_\alpha\} \to \theta$, and A^3 is dense in A (the latter is true if the socle is dense in A);

(2) the involution is continuous;

(3) $U_J = (\theta)$, for a closed Jordan ideal U of S, if, and only if, $U = S$.

A part of our previous definition had $\mathscr{L}(U) = (\theta)$ if, and only if, $U = S$ for closed Jordan ideals U of S. We now see that Theorem 8 and Corollary 9 yield this equivalent formulation. By Corollary 6, $\mathscr{L}(S) = (\theta)$ in a semiprime ring. Hence, we see that the restriction (3) really asserts that for proper closed Jordan ideals U, $U_J \neq (\theta)$.

Although we have introduced the notion of a topological ring with property (Y), our next theorem is stated for less restrictive A since this form of the theorem is used in the next section.

THEOREM 12. *Let A be a semiprime topological ring with continuous involution, $2A = A$, any net $\{2x_\alpha\} \to \theta$ implies $\{x_\alpha\} \to \theta$, and $U_J = (\theta)$ for a closed Jordan ideal, U, of S if, and only if, $U = S$. Let I be a self-adjoint ideal of A with $\mathscr{L}(I) = (\theta)$. Then $A^3 \subseteq \bar{I}$.*

Let V be the additive subgroup of S generated by the elements $x + x^*$ where $x \in I$. As $(x + x^*, s)_J = (x, s)_J + (x^*, s)_J$ we have V as a Jordan ideal of S. Clearly, $I \subseteq B_V = \{b \mid ba + a^*b^* \in V\}$. Thus, $\mathscr{L}(V) = \mathscr{L}(B_V) \subseteq \mathscr{L}(I) = (\theta)$. Hence, $S = \bar{V} \subseteq \overline{I \cap S} \subseteq \bar{I}$. This means that $a + a^*$ and aa^* are in $\bar{I}$ for all $a \in A$. Therefore, $a(a + a^*) \in \bar{I}$, concluding that $a^2 \in \bar{I}$ for all $a \in A$. Therefore, $A^3 \subseteq \bar{I}$.

As a corollary we have

COROLLARY 13. *Let A a topological ring with property (Y). Let I be a closed, self-adjoint two-sided ideal of A with $\mathscr{L}(I) = (\theta)$. Then, $I = A$.*

THEOREM 14. *Let U be a closed Jordan ideal of S in a topological ring with property (Y) then $B \oplus \mathscr{L}(B) = H$ is two-sided and hence, dense in A.*

H is a right ideal. We wish to show that it is a left ideal and also self-adjoint. Then, Theorem 12 will apply.

Let $W = (\mathscr{L}(B) \cap S) \oplus B \cap S$. Now, $x \in W_J$ implies $x \in (B \cap S)_J$. Thus by Corollary 10 and Theorem 8, $x \in \mathscr{L}(B) \cap S$ and so $x^2 = \theta$. Therefore, W_J is a nil Jordan ideal and by Theorem 1, $\mathscr{L}(W) = (\theta)$. Thus, $\bar{W} = S$. As $W \subseteq H \cap S$, we conclude that $\bar{H} \cap S = S$. Therefore, $a^2 = (a + a^*)a - a^*a \in \bar{H}$ for all $a \in A$. Linearizing this we have for all a and c in A

$$ac + ca \in \bar{H} .$$

Hence, if $a \in A$, $b \in B$ then there exists a net $b_\alpha + d_\alpha \to ab + ba$ with

$b_\alpha \in B$, $d_\alpha \in \mathscr{L}(B)$. Thus,

$$(b_\alpha - ba) + d_\alpha \to ab$$

and so $ab \in \bar{H}$. As $\mathscr{L}(B)$ is two-sided, we conclude that $\bar{H}$ is a left ideal. However, we have just observed that $a^2 \in \bar{H}$ for all $a \in A$. By the hypothesis on A we can conclude, as before, that $A = \bar{H}$.

The density of H in Theorem 14 leads to the same sequence of theorems as Theorems 3, 4, and 5 of [1]. We summarize these results as

THEOREM 15. *Let U be a closed Jordan ideal of S in a topological ring A with property (Y) then*
  (i) B *is a self-adjoint two-sided ideal of A with $U \subseteqq B$.*
  (ii) $C = U + \overline{[\{(u, a)_L \mid u \in U, a \in A\}]}$ *is a closed two-sided ideal of A and $C \cap S = U$.*

COROLLARY 16. $A = S \oplus \overline{[S, S] + S \circ K}$.

Let $U = S$ in Theorem 15, then C has the form of the right side. By Corollary 6, $\mathscr{L}(S) = (\theta)$ and hence $\mathscr{L}(S + [S, S] + S \circ K) = (\theta)$. Corollary 13 then implies the conclusion.

COROLLARY 17. $A = S \oplus \overline{S \circ K}$.

We see that $(S \circ S)_J = (\theta)$ and from this that $S \circ S$ is dense in S. A generator of $[S, S]$ has the form $st - ts$ where s and t are in S. As $t \in \overline{S \circ S}$ we can find a net $y_\alpha = \sum_{i=1}^{n_\alpha} (u_{\alpha_i}, v_{\alpha_i})_J \to t$ with

$$u_{\alpha_i}, v_{\alpha_i} \in S \; .$$

Now,

$$sy_\alpha - y_\alpha s \to st - ts \; .$$

Consider a summand,

$$[s, (u_\alpha, v_\alpha)_J] = [s, u_\alpha] \circ v_\alpha + u_\alpha \circ [s, v_\alpha]$$

and note that the right side is in $S \circ K$. Therefore, $[S, S] \subseteqq \overline{S \circ K}$.

We follow the outline of [4] in showing a decomposition for topological rings with property (Y) which have dense socles (such rings have A^3 dense in A). Let eA be a minimal right ideal, then e^*A also have this property. Let $\hat{I}$ be the intersection of all closed self-adjoint two-sided ideals containing eA. Now, $e \in \hat{I}$ and so $e^* \in \hat{I}$, therefore, $A(e + e^*)A \subseteqq \hat{I}$, and so equality holds. Let L be a non-

zero closed self-adjoint ideal of A contained in $\hat{I}$ then $\hat{I} = L$. This follows since either $eA \cap L = (\theta)$ or $eA \cap L = eA$. The latter yields immediately $L = \hat{I}$. While $eA \cap L = (\theta)$ implies that e and hence $e^* \in \mathscr{L}(L)$. Thus, $\hat{I} \subseteq \mathscr{L}(L)$ and this leads to a contradiction. Now since e and e^* belong to the socle, ζ, then $\hat{I} = \overline{A(e + e^*)A} \subseteq \zeta \cap \hat{I}$. Now A is a semiprime ring so the socle of $\hat{I}, \zeta_I$, is $\zeta \cap \hat{I}$. Therefore, $\hat{I}$ has dense socle.

This argument forms the basis of the following theorem.

THEOREM 18. *If A is a topological ring with property (Y) and dense socle, then A is a topological direct sum of two-sided ideals $\hat{I}$ each of which is an involutionally simple and closed ring with dense socle. Moreover, the symmetric elements in $\hat{I}$ form a simple closed Jordan ideal.*

We have shown that each $\hat{I}$ is a closed self-adjoint ideal and involutionally minimal closed in A. Now, in a semiprime ring it is known that every nonzero ideal of $\hat{I}$ contains a nonzero ideal of A, and therefore, $\hat{I}$ is involutionally closed simple as a ring. The density of the socle together with the minimal property of $\hat{I}$ guarantees the direct topological sum. We need only see that $U = \{s \in I \mid s^* = s\}$, the set of symmetric elements of $\hat{I}$, is simple Jordan. Let $G \neq (\theta)$ be a closed Jordan ideal of U. As $U \oplus U_J$ is dense in S then $G \circ S = G \circ (\overline{U + U_J}) \subseteq \overline{G \circ U} \subseteq G$. That is, G is a closed Jordan ideal of A. Letting E be the closure of the subgroup generated by

$$\{(a, g)_L \mid a \in A, g \in G\},$$

we conclude from Theorem 15 that $G + E$ is an involutionally closed, nonzero ideal of $\hat{I}$. Thus, $G = U$ completing the argument.

Finally, in this section we exhibit a topological ring with property (Y) which is not an annihilator ring. Consider the ring of polynomials in noncommuting indeterminates over C, the complex numbers, modulo the ideal generated by $xy - yx + 1$. This ring is known to be a simple ring, of characteristic 0, with identity and involution which, furthermore, is an integral domain (not a division ring). The ring with discrete topology becomes a topological ring with property (Y) as S, because of simplicity, is a simple Jordan ideal. It is not an annihilator ring since it is an integral domain.

It should be noted that Theorem 15 holds for the annihilator rings of [1] as well.

S-Dual rings. Following in the vein of dual rings, as defined in

[3], we define a ring A to be an *S-dual ring* if, and only if, A is a semiprime topological ring with the following property:

  (i) $A = 2A$, any net $\{2x_\alpha\} \to \theta$ implies $\{x_\alpha\} \to \theta$;
  (ii) A has continuous involution;
  (iii) $(U_J)_J = U$ for a closed Jordan ideal U of S.

We note quickly that $S_J = (\theta)$ by Corollary 6 and Theorem 8. Thus, $(S_J)_J = S$ and so if A^3 is dense in A then we see S-dual implies property (Y).

THEOREM 19. *Let A be an S-dual ring and U be a closed Jordan ideal of S. Then, B is a two-sided self-adjoint ideal and A^3 is contained in $\overline{B + \mathscr{L}(B)}$.*

By Corollary 10, $U = B \cap S$. One readily observes that $V = \{b + b^* \mid b \in B\}$ is a Jordan ideal of S. Let $x \in U_J$. Then, by Theorems 5 and 8, $x \in \mathscr{L}(B)$. For $b \in B$, $xb^* = (bx)^* \in (B \cap \mathscr{L}(B))^* = (\theta)$. Thus, $x \in V_J$ or $U_J \subseteqq V_J$. As, $U \subseteqq V$, we conclude that $\bar{V} = U$. Thus, for all $b \in B$, $b + b^* \in U \subseteqq B$, and so, $b^* \in B$. Thus, B is self-adjoint and hence a two-sided ideal. $H = B \oplus \mathscr{L}(B)$ then has the property that $\mathscr{L}(H) = (\theta)$. Applying Theorem 12 we have the desired conclusion.

COROLLARY 20. *Let A be an S-dual ring. Then every closed Jordan ideal, U, of S is the intersection of S with a closed two-sided ideal, I, of A.*

The proof is immediate upon setting $I = B$ in Theorem 19.

Idempotents related to Jordan ideals.

THEOREM 21. *Let A be a semiprime ring, let U be a Jordan ideal of S. Then either $U \subset \zeta^a$ (the annihilator of the socle) or U contains an idempotent.*

Let $U \neq (\theta)$. If e is a minimal idempotent then $eB = eA$ or $eB = (\theta)$. If the latter is true for all e then $B \subseteqq \zeta^a$. Also, $\zeta \subseteqq \mathscr{L}(B) = \mathscr{L}(U)$ and therefore $U \subseteqq \zeta^a$. Assume that $B \nsubseteqq \zeta^2$ then it is well known that B contains a minimal idempotent e. Now, if e is symmetric then $(e, (1/2)e)_J = e \in U$ and the proof is complete. Thus, assume e is not in S, then $e^*B = e^*A$ (else $e^* \in \mathscr{L}(B)$ and as the latter is self-adjoint, $e \in \mathscr{L}(B) \cap B = (\theta)$) as this ideal is minimal. Also, $(e + e^*)e \in B$ and so $e^*e \in B$. Thus, either $e^*eA = e^*A$ or $e^*e = \theta$.

Case 1. $e^*eA = e^*A$. This implies that $e^*e \neq \theta$. Let

$$T = \{t \in e^*A \mid e^*et = \theta\} \,. \quad \text{Then,} \quad I = e^*A \text{ or } T = (\theta).$$

Subcase 1. If $T = e^*A$, then $e^*ee^{**} = \theta$. Now suppose $ee^*A \neq (\theta)$, then $ee^*A = eA$ and so $(\theta) = e^*eA = e^*A$, a contradiction. Therefore, $ee^* = \theta$. Summarizing we have e and $e^*e \in B$, $ee^* = \theta$. Thus,

(II) $(e + e^*e, e)_J = e + e^* + 2e^*e \in U$

and the square

(III) $(e + e^* + 2e^*e)^2 = e + 3e^*e + e^* + 2e^*e \in U \,.$

Thus, subtracting (III) from twice (II) we have that the symmetric idempotent $e + e^* - e^*e \in U$.

Subcase 2. $T = (\theta)$ or $e^*ee^*A = e^*A$. Thus, there exists $z \in e^*A$ such that $e^*ez = e^*e$ and so that for all $y \in e^*A$, $zy = y$. In particular, $ze^* = e^*$ or, applying the involution, $e = ez^*$. It now follows that $e^*e(z - zz^*) = \theta$. Thus, $z - zz^* \in T$, and hence z is a symmetric idempotent. As $e^*A = e^*eA$ and $e^*e \in B$ we have $z \in B$. But, then $z = (z, (1/2)z)_J \in U$. The desired conclusion.

Case 2. $e^*eA = (\theta)$ which means that $e^*e = \theta$. Now, as noted $e^* \notin \mathscr{L}(B)$ so $e^*B = e^*A$. Hence, there exists a $b \in B$ such that $e^* = e^*b$. Now, $(e + e^*)b \in B$ and as $eb \in B$ we conclude $e^* \in B$. We are now in the same position to argue on e^* as we did previously on e, and we conclude that the theorem is proved unless it is also the case that $ee^* = \theta$. We immediately see that under these conditions $e + e^* \in B$ and is symmetric idempotent. To complete the argument we note that

$$e + e^* = \left(e + e^*, \frac{1}{2}(e + e^*)\right)_J \in U \,.$$

COROLLARY 22. *Let A be a semiprime ring, let $\zeta^a = (\theta)$, then every nonzero Jordan ideal of S contains an idempotent.*

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