

# Pacific Journal of Mathematics

**MEAN VALUE ITERATION OF NONEXPANSIVE MAPPINGS IN  
A BANACH SPACE**

CURTIS L. OUTLAW

## MEAN VALUE ITERATION OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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**This paper applies a certain method of iteration, of the mean value type introduced by W. R. Mann, to obtain two theorems on the approximation of a fixed point of a mapping of a Banach space into itself which is nonexpansive (i.e., a mapping which satisfies  $\|Tx - Ty\| \leq \|x - y\|$  for each  $x$  and  $y$ ).**

**The first theorem obtains convergence of the iterates to a fixed point of a nonexpansive mapping which maps a compact convex subset of a rotund Banach space into itself.**

**The second theorem obtains convergence to a fixed point provided that the Banach space is uniformly convex and the iterating transformation is nonexpansive, maps a closed bounded convex subset of the space into itself, and satisfies a certain restriction on the distance between any point and its image.**

We note that a rotation  $T$  about zero of the closed unit disc in the complex plane satisfies the conditions of Theorems 1 and 2, but the usual sequence  $\{T^n x\}$  of iterates of  $x$  does not converge unless  $x$  is zero.

DEFINITIONS. If  $Y$  is a Banach space,  $T$  is a mapping from  $Y$  into itself, and  $x \in Y$ , then  $M(x, T)$  is the sequence  $\{v_n\}$  defined by  $v_1 = x$  and  $v_{n+1} = (1/2)(v_n + Tv_n)$ .

Following Wilansky [3, pp. 107-111], we say that a Banach space  $Y$  is *rotund* provided that if  $w \in Y$ ,  $y \in Y$ ,  $w \neq y$ , and  $\|w\| = \|y\| \leq 1$ , then  $(1/2)\|w + y\| < 1$ .

**THEOREM 1.** *Let  $Y$  be a rotund Banach space,  $E$  be a compact convex subset of  $Y$ , and  $T$  be a nonexpansive mapping which maps  $E$  into itself. If  $x \in E$  then  $M(x, T)$  converges to a fixed point of  $T$ .*

*Proof.* If, for some  $n$ ,  $v_n = Tv_n$ , then clearly  $M(x, T)$  converges to  $v_n$ .

Hence suppose that  $v_n \neq Tv_n$ , for each  $n$ . Let  $z$  be a fixed point of  $T$ . Then  $\{\|v_n - z\|\}$  is decreasing, for since  $Y$  is rotund and

$$\|Tv_n - z\| = \|Tv_n - Tz\| \leq \|v_n - z\|,$$

we have that

$$\|v_{n+1} - z\| = \left\| \frac{1}{2}(v_n + Tv_n) - z \right\| < \|v_n - z\|.$$

Suppose that  $\lim_n \|v_n - z\| = b > 0$ . Let  $y$  be a cluster value of  $\{v_n\}$ . Then clearly  $b = \|y - z\|$ .

Suppose first that  $y = Ty$ . Then for each  $n$ ,

$$\|Tv_n - y\| = \|Tv_n - Ty\| \leq \|v_n - y\|.$$

Since we have assumed that  $v_n \neq Tv_n$  for each  $n$ , we have by the rotundity of  $Y$  that

$$\|v_{n+1} - y\| = \left\| \frac{1}{2}(v_n + Tv_n) - y \right\| < \|v_n - y\|.$$

Thus  $\{\|v_n - y\|\}$  is decreasing, and since  $y$  is a cluster value of  $\{v_n\}$ ,  $M(x, T)$  converges to  $y$ .

Now suppose that  $y \neq Ty$ . Let  $d$  denote  $b - \|(1/2)(y + Ty) - z\|$ . Then  $d > 0$ , since  $Y$  is rotund, for

$$\|Ty - z\| = \|Ty - Tz\| \leq \|y - z\| = b.$$

Let  $n$  be such that  $\|y - v_n\| < d$ . Then since  $T$  is nonexpansive,

$$\begin{aligned} \left\| \frac{1}{2}(y + Ty) - v_{n+1} \right\| &= \left\| \frac{1}{2}(y + Ty) - \frac{1}{2}(v_n + Tv_n) \right\| \\ &\leq \frac{1}{2} \|y - v_n\| + \frac{1}{2} \|Ty - Tv_n\| \\ &\leq \|y - v_n\| < d. \end{aligned}$$

Hence

$$\begin{aligned} \|v_{n+1} - z\| &\leq \left\| v_{n+1} - \frac{1}{2}(y + Ty) \right\| + \left\| \frac{1}{2}(y + Ty) - z \right\| \\ &< d + (b - d) = b, \end{aligned}$$

a contradiction. Therefore  $b = \lim_n \|v_n - z\| = 0$ , so that  $M(x, T)$  converges to  $z$ .

F. E. Browder [1] has shown that each nonexpansive mapping which maps a closed bounded convex subset  $E$  of a uniformly convex Banach space into itself has a fixed point in  $E$ .

If such a mapping satisfies one additional requirement, we may approximate one of its fixed points using  $M(x, T)$ :

**THEOREM 2.** *Let  $Y$  be a uniformly convex Banach space,  $E$  be a closed bounded convex subset of  $Y$ , and let  $T$  be a nonexpansive mapping which maps  $E$  into itself. Let  $F$  denote the set of fixed point of  $T$  in  $E$ , and suppose that there is a number  $c$  in  $(0, 1)$  such that if  $x \in E$ , then*

$$\|x - Tx\| \geq cd(x, F),$$

where  $d(x, F)$  denotes  $\sup_{z \in F} \|x - z\|$ .

If  $x \in E$  then  $M(x, T)$  converges to a fixed point of  $T$ .

*Proof.* The theorem is trivial if  $x \in F$ . Suppose that  $x \in E - F$  and that  $M(x, T)$  does not converge to a member of  $F$ . Then  $v_n \notin F$  for each  $n$ . Since  $Y$  is uniformly convex, we have as in the proof of Theorem 1 that if  $z \in F$  then  $\{\|v_n - z\|\}$  is decreasing.

Suppose that  $b = \lim_n d(v_n, F) > 0$ . Since  $Y$  is uniformly convex, there is an  $r$  in  $(0, 2b)$  such that, for  $w, y$ , and  $z$  in  $Y$ , the relations

$$\|w - z\| \leq \|y - z\| \leq 2b \quad \text{and} \quad \|w - y\| \geq cb$$

imply that

$$\left\| \frac{1}{2}(w + y) - z \right\| \leq \|y - z\| - r.$$

There is a positive integer  $n$  and a member  $z$  of  $F$  such that

$$\|v_n - z\| < b + \frac{r}{2},$$

so that since

$$\|Tv_n - z\| = \|Tv_n - Tz\| \leq \|v_n - z\| < 2b$$

and

$$\|Tv_n - v_n\| \geq cd(v_n, F) \geq cb,$$

we have that

$$\begin{aligned} \|v_{n+1} - z\| &= \left\| \frac{1}{2}(v_n + Tv_n) - z \right\| \\ &\leq \|v_n - z\| - r < b + \frac{r}{2} - r < b, \end{aligned}$$

an contradiction. Hence  $\lim_n d(v_n, F) = 0$ .

We now need the following:

LEMMA. If  $s > 0, z \in F$ , and  $r > 0$  such that for some  $n, v_n$  is in the open sphere  $S(z, r)$  with center  $z$  and radius  $r$ , then there exist  $t$  in  $(0, s)$ ,  $w$  in  $F$ , and an  $m$  such that the closed sphere  $\bar{S}(w, t)$  lies in  $S(z, r)$ , and for each  $p, v_{m+p} \in S(w, t)$ .

*Proof.* Recall that  $\{\|v_p - z\|\}$  is decreasing and that we are supposing that  $\{v_p\}$  does not converge to  $z$ . Let  $a = \lim_p \|v_p - z\|$ .

Then  $0 < a < r$ . Let  $t = (1/3) \min \{r - a, s\}$ .

Since  $\lim_p \|v_p - z\| = a$ ,  $\lim_p d(v_p, F) = 0$ , and  $v_p \notin F$  for each  $p$ , there exist  $w$  in  $F$  and an  $m$  such that  $\|v_m - z\| < a + t$  and  $\|v_m - w\| < t$ .

Since  $w \in F$ ,  $\|v_{m+p} - w\|$  decreases as  $p$  increases, so that  $v_{m+p} \in S(w, t)$  for each  $p$ . Also, if  $y \in \bar{S}(w, t)$ , then  $y \in S(z, r)$ , for

$$\begin{aligned} \|y - z\| &\leq \|y - w\| + \|w - v_m\| + \|v_m - z\| \\ &< t + t + (a + t) \\ &\leq 3\left(\frac{r - a}{3}\right) + a = r. \end{aligned}$$

The lemma guarantees the existence of a sequence  $\{z_i\}$  in  $F$ , a sequence  $\{t_i\}$  of positive numbers with limit 0, and a subsequence  $\{v_{n_i}\}$  of  $\{v_n\}$  such that for each  $i$  and each  $p$ ,

$$\bar{S}(z_{i+1}, t_{i+1}) \text{ lies in } S(z_i, t_i)$$

and

$$v_{n_i+p} \in S(z_i, t_i).$$

By the Cantor Intersection Theorem,  $\bigcap_{i=1}^{\infty} S(z_i, t_i)$  contains exactly one point, say  $w$ . Clearly  $\{z_i\}$  converges to  $w$  and  $w \in F$ . Further,  $\{\|v_n - w\|\}$  is decreasing and  $\{v_{n_i}\}$  converges to  $w$ , so that  $\{v_n\}$  converges to  $w$ . Thus we have contradicted our assumption that  $M(x, T)$  does not converge to a member of  $F$ .

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