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**ZERO SQUARE RINGS**

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A ring  $R$  for which  $x^2 = 0$  for all  $x \in R$  is called a *zero-square ring*. Zero-square rings are easily seen to be locally nilpotent. This leads to two problems: (1) constructing finitely generated zero-square rings with large index of nilpotence, and (2) investigating the structure of finitely generated zero-square rings with given index of nilpotence. For the first problem we construct a class of zero-square rings, called *free* zero-square rings, whose index of nilpotence can be arbitrarily large. We show that every zero-square ring whose generators have (additive) orders dividing the orders of the generators of some free zero-square ring is a homomorphic image of the free ring. For the second problem, we assume  $R^n \neq 0$  and obtain conditions on the additive group  $R_+$  of  $R$  (and thus also on the order of  $R$ ). When  $n = 2$ , we completely characterize  $R_+$ . When  $n > 3$  we obtain the smallest possible number of generators of  $R_+$ , and the smallest number of generators of order 2 in a minimal set of generators. We also determine the possible orders of  $R$ .

Trivially every null ring (that is,  $R^2 = 0$ ) is a zero-square ring. From every nonnull commutative ring  $S$  we can make  $S \times S \times S$  into a nonnull zero square ring  $R$  by defining addition componentwise and multiplication by

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (0, 0, x_1y_2 - x_2y_1).$$

In this example we always have  $R^3 = 0$ . If  $S$  is a field, then  $R$  is an algebra over  $S$ . Zero-square algebras over a field have been investigated in [1].

**2. Preliminaries.** Every zero-square ring is anti-commutative, for  $0 = (x + y)^2 = x^2 + xy + yx + y^2 = xy + yx$ . From anti-commutativity we get  $2R^3 = 0$ , for  $yzx = y(-xz) = -(yx)z = xyz$  and  $(yz)x = -x(yz)$ , so  $2xyz = 0$  for all  $x, y, z \in R$ . It follows that a zero-square ring  $R$  is commutative if and only if  $2R^3 = 0$ .

If  $R$  is a zero-square ring with  $n$  generators, then any product of  $n + 1$  generators must contain two factors the same. By applying anti-commutativity we get a square factor in the product; hence  $R^{n+1} = 0$ . In particular, every zero-square ring is locally nilpotent.

If  $G$  is a finitely generated abelian group, then by the fundamental theorem on abelian groups we have

$$(1) \quad G = C_{a_1} \oplus \cdots \oplus C_{a_n}, \quad a_i \mid a_{i+1} \text{ for } 1 \leq i \leq k-1, \\ a_{k+1} = \cdots = a_n = \infty,$$

where  $C_{a_i}$  is a cyclic group of order  $a_i$ . If  $X = \{x_1, \dots, x_n\}$  generates  $G$  and if there is some decomposition (1) for which  $x_i$  generates  $C_{a_i}$ ,  $1 \leq i \leq n$ , then we call  $X$  a *standard set of (group) generators* for  $G$ . Now let  $R$  be any finitely generated ring with a minimal set of ring generators  $X' = \{x_1, \dots, x_n\}$ . Let  $\langle X' \rangle$  denote the additive group generated by  $X'$  (whose elements are considered now as group, not ring, generators), and let  $X$  be a standard set of generators for  $\langle X' \rangle$ . Then  $X$  generates  $R$  as a ring since it generates  $\langle X' \rangle$  as a group. Such a set  $X$  will be called a *standard set of ring generators* for  $R$ , and it follows that every finitely generated ring has a standard set of ring generators.

**3. Free zero-square rings.** For every positive integer  $n$  and every  $n$ -tuple  $(a_1, \dots, a_n)$ , where  $a_i \mid a_{i+1}$  for  $i = 1, \dots, k-1$ , and  $a_{k+1} = \dots = a_n = \infty$ , we define the *free zero-square ring*  $R_F(a_1, \dots, a_n)$  and derive its basic properties. Free zero-square rings are constructed from combinations of indeterminates called special monomials.

**DEFINITION 3.1.** Let  $a_1, \dots, a_n$  be integers  $\geq 2$  or  $\infty$ , such that for some  $k \leq n$ ,  $a_i \mid a_{i+1}$  for  $i = 1, \dots, k-1$ , while  $a_{k+1} = \dots = a_n = \infty$ ; and let  $x_1, \dots, x_n$  be indeterminates. We say that  $x_{i_1}x_{i_2} \dots x_{i_q}$  is a *special monomial* if  $1 \leq i_1 < i_2 < \dots < i_q \leq n$ , and if  $a_{i_1}$  is even or  $\infty$  whenever  $q > 2$ .

Thus the special monomials consist of

$$\begin{aligned} x_i, & \quad 1 \leq i \leq n \\ x_i x_j, & \quad 1 \leq i < j \leq n \\ x_{i_1} x_{i_2} \dots x_{i_q}, & \quad q \geq 3 \text{ and } a_{i_1} \text{ even or } \infty. \end{aligned}$$

Now let  $y_1, y_2, \dots, y_r$  denote the  $r$  distinct special monomials (in some order) corresponding to  $a_1, a_2, \dots, a_n$ . If  $y_j = x_{i_1}x_{i_2} \dots x_{i_q}$  is a special monomial, we define

$$b_j = b(y_j) = \begin{cases} a_{i_1}, & \text{if } q = 1 \text{ or } 2 \\ 2, & \text{if } q \geq 3. \end{cases}$$

Let  $R_F(a_1, \dots, a_n)$  denote the set of formal sums

$$\begin{aligned} R_F(a_1, \dots, a_n) = \left\{ \sum_{i=1}^r c_i y_i \mid 0 \leq c_j < b_j \text{ if } b_j \neq \infty, \right. \\ \left. -\infty < c_j < \infty \text{ if } b_j = \infty \right\}. \end{aligned}$$

We define addition and multiplication on  $R_F$  as follows:

*Addition.* Define

$$\sum_{i=1}^r c_i y_i + \sum_{i=1}^r d_i y_i = \sum_{i=1}^r e_i y_i ,$$

where  $e_i \equiv c_i + d_i \pmod{b_i}$ ,  $0 \leq e_i < b_i$  if  $b_i \neq \infty$ ,  $e_i = c_i + d_i$  if  $b_i = \infty$ . We are adding the  $i^{\text{th}}$  components mod  $b_i$ .

*Multiplication.* We first define multiplication of special monomials. If  $y_i$  and  $y_j$  have a factor  $x_s$  in common, define  $y_j y_i = y_i y_j = 0$ . In particular,  $x_s^2 = 0$ . If  $y_i = x_s$ ,  $y_j = x_t$  with  $s < t$ , define  $(ay_i)(by_j) = \overline{ab} x_s x_t$ ; where if  $b_i \neq \infty$ , then  $\overline{ab}$  is defined by  $\overline{ab} \equiv ab \pmod{b_i}$ ,  $0 \leq \overline{ab} < b_i$ , while if  $b_i = \infty$ , then  $\overline{ab} = ab$ . If we think of  $a$  and  $b$  as representatives of the congruence classes mod  $b_i$  and  $b_j$ , then since  $b_i | b_j$  the product  $ab$  always represents the same element mod  $b_i$  regardless of the choice of  $a$  and  $b$ . Similarly define  $(by_j)(ay_i) = -\overline{ab}(x_s x_t)$ . If  $y_i$  and  $y_j$  do not have a factor  $x_s$  in common, and if at least one of  $y_i, y_j$  contains at least two distinct factors  $x_s$  and  $x_t$ , then define  $(ay_i)(by_j) = cy_l$ , where  $y_l$  is obtained by rearranging the factors  $x_h$  of  $y_i$  and  $y_j$  in ascending subscript order and defining

$$c = \begin{cases} 0, & \text{if } a_q \text{ is odd} \\ 0, & \text{if } a_q \text{ is even or } \infty \text{ and } ab \text{ is even} \\ 1, & \text{if } a_q \text{ is even or } \infty \text{ and } ab \text{ is odd,} \end{cases}$$

where  $a_q$  is the order of the indeterminate  $x_q$  with least subscript appearing in  $y_l$ .

We now define in general

$$\left( \sum_i c_i y_i \right) \left( \sum_j d_j y_j \right) = \sum_{i,j} (c_i y_i)(d_j y_j) ,$$

where this sum is to be rearranged according to the previously defined rules of special monomial multiplication and of addition.

We call this set  $R_F(a_1, \dots, a_n)$ , together with the operations of addition and multiplication just defined, the *free zero square ring*  $R_F(a_1, \dots, a_n)$ .

**THEOREM 3.2.**  $R_F(a_1, \dots, a_n)$  is a zero-square ring.

*Proof.* All the desired properties follow from the definitions except associativity of multiplication and the zero-square property.

It follows from the definition of multiplication that we need only to verify associativity for monomials  $c_h y_h$ , where  $c_h$  is a constant between 0 and  $b_h - 1$  for  $b_h \neq \infty$ , while  $-\infty < c_h < \infty$  for  $b_h = \infty$ ,

and  $y_h$  is a special monomial. But if either of  $y_h, y_i, y_j$  contain an indeterminate  $x_s$  of odd order, then  $(c_h y_h)(c_i y_i)(c_j y_j) = 0$  upon any association, while if all orders are even or  $\infty$ , then

$$(c_h y_h)(c_i y_i)(c_j y_j) = \begin{cases} 0, & \text{if two of } y_h, y_i, y_j \text{ contain} \\ & \text{a common factor } x_s \\ 0, & \text{if any of } c_h, c_i, c_j \text{ is even} \\ y_h y_i y_j, & \text{otherwise} \end{cases}$$

upon any association.

It remains only to show  $(\sum c_i y_i)^2 = 0$ . Now

$$(\sum c_i y_i)^2 = \sum_{i < j} c_i c_j (y_i y_j + y_j y_i) + \sum c_i^2 y_i^2.$$

The latter sum is 0 by definition of special monomial multiplication. If  $y_i y_j$  is the product of more than two indeterminates, then

$$c_i c_j (y_i y_j + y_j y_i) = 2c_i c_j y_i y_j = 0,$$

since either  $y_i y_j = 0$  or  $2c_i c_j$  is taken mod 2. This completes the proof.

**THEOREM 3.3.** *If  $a_n \neq \infty$  and  $i$  is the least integer for which  $a_i$  is even (except that if  $a_n$  is odd, put  $i = n$ ), then  $R_F(a_1, \dots, a_n)$  has order*

$$a_1^n a_2^{n-1} \dots a_n^1 2^{2^{n-i}+1-1} 2^{-(n-i+1)(n-i+2)/2}.$$

*Proof.* In general there are  $\binom{n-j}{k-1}$  distinct special monomials with  $k$  factors such that  $j$  is the least subscript appearing among the factors. Such a monomial has order  $a_j$  if  $k \leq 2$ , while if  $k > 2$  the monomial has order 2 when  $a_j$  is even and vanishes when  $a_j$  is odd. Thus the order of  $R_F$  is given by

$$\begin{aligned} (a_1 a_2 \dots a_n) & (a_1^{n-1} a_2^{n-2} \dots a_{n-1}) \left[ 2^{\binom{n-i}{2} + \binom{n-i-1}{2} + \dots + \binom{2}{2}} \right] \\ & \cdot \left[ 2^{\binom{n-i}{3} + \binom{n-i-1}{3} + \dots + \binom{3}{3}} \right] \dots \left[ 2^{\binom{n-i}{n-i}} \right] \\ & = a_1^n a_2^{n-1} \dots a_n^1 2^{2^{n-i}+1-1} 2^{-(n-i+1)(n-i+2)/2}, \end{aligned}$$

as asserted.

The next theorem elucidates the “free” nature of  $R_F$ .

**THEOREM 3.4.** *If  $R$  is a zero-square ring with a standard set of ring generators  $x'_1, \dots, x'_n$  of orders  $a'_1, \dots, a'_n$ , and if  $R_F(a_1, \dots, a_n)$  is a free zero-square ring with  $a'_i | a_i$  for  $1 \leq i \leq n$  (with the convention that every integer and  $\infty$  are divisors of  $\infty$ ), then  $R$  is a homomorphic image of  $R_F(a_1, \dots, a_n)$ .*

*Proof.* Let  $x_1, \dots, x_n$  be the indeterminates (generators) of  $R_F$ . Let  $y_1, \dots, y_r$  be the special monomials of  $R_F$  and  $y'_1, \dots, y'_r$  the corresponding monomials of  $R$ , so that if  $y_i = x_{i_1} \cdots x_{i_g}$ , then  $y'_i = x'_{i_1} \cdots x'_{i_g}$ . (Of course for some  $i$  we may have  $y'_i = 0$ .) We then claim that the mapping  $\varphi: \sum c_i y_i \rightarrow \sum c_i y'_i$  is the desired homomorphism.

Since  $a'_i | a_i$ , the ring of integers mod  $a'_i$  is a homomorphic image of the ring of integers mod  $a_i$ . It follows from its definition that  $\varphi$  preserves sums and products. It remains only to verify that  $\varphi$  is onto  $R$ , i.e., that every element of  $R$  occurs among  $\sum c_i y'_i$ ,  $0 \leq c_i < b_i$  if  $b_i \neq \infty$ ,  $-\infty < c_i < \infty$  if  $b_i = \infty$ . This, however, is an immediate consequence of the fact that  $R$  is anti-commutative and satisfies  $R^{n+1} = 0$  and  $2R^3 = 0$ , and that the order of an anti-commutative product cannot exceed the g.c.d. of the orders of its factors. This completes the proof.

In general, a subring (or ideal) of  $R_F(a_1, \dots, a_n)$  need not be free. For instance, if  $n > 2$  and each  $a_i$  is even, then  $R_F^{[(n/2)+1]}$  is a null ideal with more than one generator.

If  $R_F = R_F(a_1, \dots, a_n)$  is a free zero-square ring such that  $i$  is the least integer for which  $a_i$  is even or  $\infty$ , and if  $n - i \geq 1$ , then it is easily verified that  $R_F$  has index of nilpotence  $n - i + 2$ . Thus free zero-square rings provide examples of zero-square rings with arbitrarily large index of nilpotence.

**4. Nonnull finite zero-square rings.** In this section we characterize the additive groups of nonnull finite zero-square rings and as a corollary characterize the orders of such rings. For this purpose we introduce a function  $f(G)$  of a finitely generated abelian group  $G$ .

**DEFINITION 4.1.** If  $G$  is a finitely generated abelian group, define  $f(G) = \max \{n: R \text{ is a zero-square ring, } R^n \neq 0, G \text{ is isomorphic to the additive group } R_+ \text{ of } R\}$ .

It follows from the local nilpotence of zero-square rings that  $f(G)$  is finite. In this section and the next we assume  $G$  is finite to avoid looking at a large number of cases. The results can easily be extended to arbitrary finitely generated  $G$ .

**THEOREM 4.2.** *Let  $G$  be a finite abelian group. Then  $f(G) \geq 2$  if and only if either of the following hold:*

- (i) *The dimension of  $G$  is greater than two; or*
- (ii)  *$G = C_{a_1} \oplus C_{a_2}$ , where  $a_1 | a_2$  and either  $(a_2/a_1, a_1) \neq 1$  or  $a_1$  is divisible by a square  $> 1$ . (This condition on  $a_1$  and  $a_2$  is equivalent to  $a_1 | a_2$  and the existence of an integer  $b$ ,  $0 < b < a_2$ , such that  $a_2 | b(a_1, b)$ .)*

*Proof.* We first prove sufficiency of (i) and of (ii). Assume that  $G = C_{a_1} \oplus C_{a_2} \oplus \cdots \oplus C_{a_n}$ , with  $a_i \mid a_{i+1}$  and  $n \geq 3$ . Let  $Z$  be the null ring with additive group  $C_{a_4} \oplus C_{a_5} \oplus \cdots \oplus C_{a_n}$ . Let  $x_1, x_2$  be generators for the free ring  $R_F(a_2, a_3)$ , and let  $J$  be the ideal of  $R_F$  generated by  $a_1 x_1 x_2$ . Then it is easily seen that the ring  $(R_F/J) \oplus Z$  is a nonnull zero-square ring with additive group isomorphic to  $G$ . This proves the sufficiency of (i).

The equivalency of the two conditions in (ii) can be verified straightforwardly. To prove the sufficiency of (ii), assume that  $G = C_{a_1} \oplus C_{a_2}$  where  $a_1$  and  $a_2$  satisfy the conditions of (ii). In view of Theorem 3.4 we need to prove that if  $R_F(a_1, a_2)$  is generated by  $x_1, x_2$ , then the ideal  $J$  generated by  $x_1 x_2 - b x_2$  does not contain  $x_1 x_2$ , where  $b$  is defined in (ii). Assume to the contrary that  $x_1 x_2 \in J$ . Then for some  $y \in R_F$  and some integer  $c$ ,

$$x_1 x_2 = c(x_1 x_2 - b x_2) + y(x_1 x_2 - b x_2).$$

Since  $y(x_1 x_2 - b x_2)$  contains no term in  $x_2$ , we must have  $c b x_2 = 0$ . This means  $a_2 \mid b c$ . The remaining way an  $x_1 x_2$  term can appear is for  $y = d x_1$ . Thus we get

$$x_1 x_2 = (c - b d) x_1 x_2.$$

We therefore have  $(a_1, c - b d) = 1$ , since the order of  $x_1 x_2$  in  $R_F(a_1, a_2)$  is  $a_1$ . This implies  $(a_1 b, b c - b^2 d) = b$ . We have just proved  $a_2 \mid b c$ , and from  $a_2 \mid b(a_1, b)$  we get  $a_2 \mid a_1 b$  and  $a_2 \mid b^2$ . Thus  $a_2 \mid (a_1 b, b c - b^2 d)$ , or  $a_2 \mid b$ , contradicting  $0 < b < a_2$ . This proves the sufficiency of (ii).

If  $G$  has one generator, then  $R$  is clearly null. Hence to prove necessity, we need to show that if  $R$  is generated by  $x_1, x_2$  of orders  $a_1, a_2$  with  $a_1 \mid a_2$  and  $R^2 \neq 0$ , then  $a_1$  and  $a_2$  satisfy the conditions in (ii). Let

$$x_1 x_2 = b_1 x_1 + b_2 x_2$$

in  $R$ . Without loss of generality it may be assumed that  $0 \leq b_1 < a_1$ ,  $0 \leq b_2 < a_2$ .

Assume first that  $b_2 = 0$ . Then  $x_1 x_2 = b_1 x_1$ , so  $0 = x_1 x_2^2 = b_1 x_1 x_2 = b_1^2 x_1$ ; hence  $a_1 \mid b_1^2$ . If  $a_1$  is divisible by a square  $> 1$ , we have satisfied one of the conditions. Otherwise  $b_1 = 0$  since  $b_1 < a_1$ . In this case  $R$  is null, a contradiction.

Now consider the remaining case  $x_1 x_2 = b_1 x_1 + b_2 x_2$ ,  $b_2 \neq 0$ . Let  $c$  be the order of  $x_1 x_2$ . Then from  $0 = c x_1 x_2 = c b_1 x_1 + c b_2 x_2$ , we get  $0 = c b_1 x_1 = c b_2 x_2$ , so  $a_2 \mid c b_2$ . Moreover,  $0 = x_1^2 x_2 = b_2 x_1 x_2$  gives  $c \mid b_2$ . Thus  $a_2 \mid b_2^2$ . But  $a_1 x_1 x_2 = a_1 b_2 x_2$  gives  $a_2 \mid a_1 b_2$ . Then from  $a_2 \mid b_2^2$  and  $a_2 \mid a_1 b_2$  we deduce  $a_2 \mid b_2(a_1, b_2)$ . Since  $b_2 \neq 0$ , we can take  $b = b_2$ . This completes the proof.

**COROLLARY 4.3.** *There exists a nonnull finite zero-square ring of order  $r$  if and only if  $r$  is divisible by a cube.*

**COROLLARY 4.4.** *The smallest nonnull zero-square ring has order 8.*

A simple direct proof of Corollary 4.4 is given in [2], (see also Th. 5.7.) It can be shown that there are exactly two nonisomorphic nonnull zero-square rings of order 8. One of these is  $R_F(2, 2)$ .

**5. Additive group structure of finite zero-square rings.** In this section we extend Theorem 4.2 by considering conditions on  $G$  which make  $f(G) \geq n$  for  $n > 2$ . Theorem 5.5 gives some necessary conditions, while Theorem 5.6 provides a converse.

$R$  will denote a finite zero-square ring and  $R_+$  its additive group, while  $G$  denotes a finite abelian group and  $G_2$  its Sylow 2-subgroup. Let  $x_1, x_2, \dots, x_n$  be a fixed standard set of ring generators of  $R$ . Let  $x$  denote the element  $x_1 x_2 \cdots x_n$  and  $\bar{x}_i$  the element  $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$ . More generally, if  $y = x_{i_1} x_{i_2} \cdots x_{i_m}$ ,  $i_1 < i_2 < \cdots < i_m$ , then  $\bar{y}$  denotes the element  $x_{j_1} x_{j_2} \cdots x_{j_{n-m}}$ ,  $j_1 < j_2 < \cdots < j_{n-m}$ , such that the  $i$ 's and  $j$ 's include all the integers  $1, 2, \dots, n$ . When  $n > 2$  note that  $y\bar{y} = x$ . If  $x \neq 0$ , we call  $m$  the *length* of  $y$ , denoted by  $|y|$ . Note  $|y| + |\bar{y}| = n$ . If  $z \in R$ , then  $c(z)$  denotes the additive order of  $z$ .

**LEMMA 5.1.** *Every symmetric matrix of odd order over  $GF(2)$  with 0's on the main diagonal is singular.*

The proof is a straightforward application of the definition of the determinant and will be omitted.

**LEMMA 5.2.** *If a matrix  $E$  has the form*

$$E = \begin{pmatrix} E_1 & & & 0 \\ & E_2 & & \\ & & \ddots & \\ * & & & E_t \end{pmatrix},$$

where the  $E_k$  are square matrices and some  $E_j$  is singular, then  $E$  is singular.

*Proof.* This is a special case of the well-known result  $\det E = (\det E_1) \cdots (\det E_t)$ .



The next theorem reduces the problem of evaluating  $f(G)$  to the case where  $G$  is a 2-group.

**THEOREM 5.3.** *If  $G$  is a finite abelian group and  $f(G) \geq 3$ , then  $f(G) = f(G_2)$ .*

*Proof.* Let  $R$  be a finite zero-square ring with  $R^n \neq 0$ ,  $n \geq 3$ . By anti-commutativity the elements  $R'$  of  $R$  whose additive order is a power of two form a subring. If  $z_i \in R$ ,  $1 \leq i \leq n$ , such that  $z_1 z_2 \cdots z_n \neq 0$ , where  $c(z_i) = a_i 2^{b_i}$ ,  $a_i$  odd, then  $a_i z_i \in R'$  and

$$(a_1 z_1)(a_2 z_2) \cdots (a_n z_n) \neq 0$$

since  $2z_1 z_2 \cdots z_n = 0$ . Hence  $(R')^n \neq 0$ , so  $f(G_2) \geq f(G)$ .

Conversely, assume  $R^n \neq 0$  and  $R_+$  is a 2-group. If  $G$  is a finite abelian group with  $G_2 \cong R_+$ , write  $G = G_2 \oplus H$ , and let  $S$  be the null ring with  $S_+ \cong H$ . Then  $(R \oplus S)_+ \cong G$  and  $(R \oplus S)^n \neq 0$ , so that  $f(G_2) \leq f(G)$ . Thus  $f(G) = f(G_2)$  and the theorem is proved.

We can now assume in what follows that the additive group  $R_+$  of  $R$  is a 2-group.

**LEMMA 5.4.** *Let  $R$  be a finite zero-square ring (with  $R_+$  a 2-group) with  $n \geq 3$  elements  $x_1, \dots, x_n$  satisfying  $x = x_1 \cdots x_n \neq 0$ .*

- (i) *There exists a standard set of group generators for  $R_+$  containing every special monomial  $y_j$  in the  $x_i$  of length  $3 \leq |y_j| \leq n - 2$ .*
- (ii) *The group generated by those  $y_j$  satisfying  $1 \leq |y_j| \leq n - 2$  is generated irredundantly by them (though not necessarily standardly).*
- (iii) *If we assume  $x_1, \dots, x_n$  is a standard set of ring generators for the ring  $R'$  they generate, then there exists a standard set of group generators for  $R'_+$  containing every special monomial  $y_j$  in the  $x_i$  satisfying  $|y_j| = 1$  or  $3 \leq |y_j| \leq n - 2$ .*

*Proof.* (i) If  $G$  is a finite abelian  $p$ -group and  $t_1, \dots, t_s \in G$ , then  $t_1, \dots, t_s$  extend to a standard set of group generators for  $G$  if and only if the following two condition are satisfied:

- (1) For any integers  $a_1, \dots, a_s$ ,

$$\sum_{i=1}^s a_i t_i = pz \Rightarrow p \sum_{i=1}^s b_i t_i = pz,$$

for some integers  $b_1, \dots, b_s$ .

- (2) For any integers  $a_1, \dots, a_s$ ,

$$\sum_{i=1}^s a_i t_i = 0 \Rightarrow a_i t_i = 0, \text{ all } i.$$

To prove (1) in our case, assume

$$(3) \quad \sum_{3 \leq |y_i| \leq n-2} a_i y_i = 2z.$$

Since  $2y_i = 0$  when  $|y_i| \geq 3$ , we can take  $a_i = 0$  or 1. Let  $y_j$  be a special monomial of minimal length satisfying  $a_j = 1$ . Then from (3) we get

$$x = \overline{y_j} \sum_{3 \leq |y_i| \leq n-2} a_i y_i = 2(\overline{y_j} z) = 0,$$

since  $\overline{y_j} z \in R^3$  when  $|y_j| \leq n-2$ . This contradicts  $x \neq 0$  and proves (1).

To prove (2), assume

$$(4) \quad \sum_{3 \leq |y_i| \leq n-2} a_i y_i = 0,$$

where at least one  $a_i y_i \neq 0$ . As in (1), let  $y_j$  be of minimal length such that  $a_j y_j \neq 0$ . Multiplying (4) by  $\overline{y_j}$  gives the contradiction  $x = 0$ , and completes the proof of (i).

(ii) We need to prove that

$$(5) \quad \sum_{1 \leq |y_i| \leq n-2} a_i y_i = 0 \Rightarrow a_i = 2b_i \text{ all } i.$$

Letting  $y_j$  be of minimal length such that  $a_i \neq 2b_i$  for any integer  $b_i$ , an argument similar to those used in (i) leads to a contradiction.

(iii) We must show (1) and (2) hold, where the  $t_i$ 's are the  $y_j$ 's satisfying  $|y_j| = 1$  or  $3 \leq |y_j| \leq n-2$ , and  $p=2$ . The proof of (1) is similar to the proof of (5). To prove (2), assume that

$$\sum_{i=1}^n a_i x_i + \sum_{3 \leq |y_j| \leq n-2} b_j y_j = 0.$$

By (1), each  $b_j y_j = 0$ . It follows that each  $a_i x_i = 0$  since  $x, \dots, x_n$  is a standard set of ring generators. This completes the proof of the lemma.

We can now give necessary conditions for  $f(G) \geq n \geq 4$ .

**THEOREM 5.5.** *Let  $G$  be a finite abelian 2-group.*

(i) *If  $f(G) \geq n \geq 4$ , then the dimension of  $G$  is at least  $2^n - 2[(n+2)/2]$ , i.e., every generating set of  $G$  has at least  $2^n - 2[(n+2)/2]$  elements. (Brackets denote the integer part.)*

(ii) *If  $f(G) \geq n \geq 4$ , then at least  $2^n - n(n+1)/2 - 2[(n+2)/2]$  generators in a standard set of generators for  $G$  have order 2.*

*Proof.* (i) Suppose  $R$  is a zero-square ring with  $R^n \neq 0$  and  $R_+ \cong G$ , and that  $x_1 x_2 \cdots x_n \neq 0$  in  $R(n \geq 4)$ . Let  $R'$  be the subring

of  $R$  generated by  $x_1, x_2, \dots, x_n$ . Since  $\dim R'_+ \leq \dim R_+$ , it suffices to show  $\dim R'_+ \geq 2^n - 2[(n+2)/2]$ . By Lemma 5.4 (ii),  $\dim R'_+$  is equal to at least the number of special monomials  $y_i$  in the indeterminates  $x_1, \dots, x_n$  satisfying  $1 \leq |y_i| \leq n-2$ . The number of such special monomials is  $\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-2} = 2^n - n - 2$ . Hence to complete the proof of (i) we need only to prove that when  $n$  is odd, we cannot have  $\dim R'_+ = 2^n - n - 2$ .

Assume that  $n$  is odd and  $R'_+$  has  $2^n - n - 2$  generators,  $R'^n \neq 0$ . By Lemma 5.4 there is a standard set of group generators for  $R'_+$  containing (1)  $x_1, \dots, x_n$ , (2) all special monomials  $y_j$  in the  $x_i$  satisfying  $3 \leq |y_j| \leq n-2$ , and (3) a standard set of generators  $y'_1, \dots, y'_m$  ( $m = \binom{n}{2}$ ) for the group generated by all  $y_j$  of length 2. Since this accounts for  $2^n - n - 2$  generators, these in fact generate all of  $R'_+$ . In particular the special monomials  $\bar{x}_1, \dots, \bar{x}_n$  of length  $n-1$  can be written as

$$(6) \quad \bar{x}_j = \sum_{1 \leq |y_i| \leq n-2} b_{ij} y_i, \quad j = 1, \dots, n,$$

where  $b_{ij}$  is an integer. (This representation may not be unique since the  $y_i$ 's of length 2 need not be standard generators.)

We show that  $b_{ij}$  is even. Let  $y_k$  be a term appearing on the right side of (6) whose coefficient  $b_{kj}$  is odd, such that no  $y_i$  of smaller length has an odd coefficient. Then we get  $0 = \bar{x}_j \bar{y}_k = b_{kj} y_k \bar{y}_k = x_j$ , a contradiction, so every  $b_{ij}$  is even. In particular, the terms  $b_{ij} y_i$  with  $|y_i| \geq 3$  vanish since  $2R'^3 = 0$ . If we re-express  $\bar{x}_j$  as a linear combination of the standard generators given above, then the terms  $b_{ij} y_i$  with  $|y_i| = 1$  remain the same. Since  $2\bar{x}_j = 0$  when  $n > 3$ , we have  $b_{ij} = 1/2 c(y_i)$  or  $b_{ij} = 0$  whenever  $|y_i| = 1$ . (This is where the argument fails for  $n = 3$ .) Hence we can rewrite (6) as

$$(7) \quad \bar{x}_j = 2z_j + \sum_{i=1}^n h_{ij} \left( \frac{1}{2} c(x_i) \right) x_i,$$

where

$$2z_j = \sum_{|y_i|=2} b_{ij} y_i,$$

and where  $h_{ij} = 0$  or  $1$ .

We claim that the matrix  $H = (h_{ij})$  over  $GF(2)$  is nonsingular. If  $H$  were singular, then if we regard  $1/2 c(x_j) x_j$  as indeterminates over  $GF(2)$ , we can eliminate them from (7) and get a relation of the form

$$\sum_{i=1}^n r_i (\bar{x}_i + 2z_i) = 0,$$

where some  $r_j = 1$ . But then

$$x = x_j \sum_{i=1}^n r_i (\overline{x_i} + 2z_i) = x_j \cdot 0 = 0 ,$$

a contradiction. Hence  $H$  is nonsingular.

Therefore we can solve (7) for the  $n$  unknowns  $1/2 c(x_j)x_j$  over  $GF(2)$  to get

$$(8) \quad \frac{1}{2} c(x_j)x_j = 2 \sum_{i=1}^n e_{ij}z_i + \sum_{i=1}^n e_{ij}\overline{x_i}, \quad j = 1, \dots, n ,$$

where each  $e_{ij} = 0$  or  $1$ . If  $E$  denotes the matrix  $(e_{ij})$  over  $GF(2)$ , then  $E = H^{-1}$ , so  $E$  is nonsingular.

We will now reach a contradiction by showing that  $E$  is singular. We first show  $e_{jj} = 0$ . We have

$$0 = \frac{1}{2} c(x_j)x_j^2 = 2 \sum_{i=1}^n e_{ij}z_ix_j + \sum_{i=1}^n e_{ij}\overline{x_i}x_j = e_{jj}x .$$

Since  $x \neq 0$ ,  $e_{jj} = 0$ .

Define  $s_1, s_2, \dots, s_t$  by

$$\begin{aligned} c(x_1) &= c(x_2) = \dots = c(x_{s_1}) \\ &< c(x_{s_1+1}) = c(x_{s_1+2}) = \dots = c(x_{s_2}) \\ &< \dots < c(x_{s_{t-1}+1}) = \dots = c(x_{s_t}) , \end{aligned}$$

where  $s_t = n$ . Let  $E_k$  be the square submatrix of  $E$  defined by  $E_k = (e_{ij})$ ,  $s_{k-1} + 1 \leq i, j \leq s_k$ , for  $k = 1, 2, \dots, t$ . (Here  $s_0$  is taken to be  $0$ .) We show that each  $E_k$  is symmetric. Assume  $e_{ij} = 1$  for some  $s_{k-1} + 1 \leq i, j \leq s_k$ . Then from (8) we get  $1/2 c(x_j)x_ix_j = x$ , so  $1/2 c(x_j)x_ix_j \neq 0$ . But  $1/2 c(x_j) = 1/2 c(x_i)$ , as  $s_{k-1} + 1 \leq i, j \leq s_k$ . Hence  $1/2 c(x_i)x_ix_j \neq 0$ . From (8) we again get  $0 \neq 1/2 c(x_i)x_ix_j = e_{ji}x$ , so  $e_{ji} = 1$ . This proves that  $E_k$  is symmetric. Moreover,  $E_k$  has  $0$ 's on the main diagonal since each  $e_{jj} = 0$ .

We now show that if for some  $k$  we have  $i \leq s_k, j > s_k$ , then  $e_{ij} = 0$ . As in the previous paragraph we have

$$(9) \quad \frac{1}{2} c(x_j)x_ix_j = e_{ij}x .$$

Since  $i \leq s_k, j > s_k$  we have  $c(x_i) < c(x_j)$ . Therefore  $1/2 c(x_i)x_ix_j \neq 0$ . But from (4),  $1/2 c(x_i)x_ix_j = e_{ji}x$ , so  $c(x_i)x_ix_j = 2e_{ji}x = 0$  (since  $2x = 0$ ). But  $c(x_i) < c(x_j)$  implies  $c(x_i) \leq 1/2 c(x_j)$ , so  $1/2 c(x_j)x_ix_j = 0$ . Comparing with (9) shows  $e_{ij} = 0$ , as asserted.

This shows that  $E$  has the form given in Lemma 5.2. Since the

sum of the orders of the  $E_j$  must be the order of  $E$ , some  $E_k$  has odd order. Then by Lemma 5.1  $E_k$  is singular, so by Lemma 5.2  $E$  is singular, a contradiction. This completes the proof of (i).

(ii) Using the notation of part (i), it follows from  $2R'^3 = 0$  that every special monomial  $y_i$  satisfying  $3 \leq |y_i| \leq n-2$  has order 2. There are  $\binom{n}{3} + \binom{n}{4} + \cdots + \binom{n}{n-2} = 2^n - n(n+1)/2 - (n+2)$  such  $y_i$ , and by Lemma 5.4 they extend to a standard set of group generators for  $R_+$ . Moreover, we have just shown that when  $n$  is odd, there is at least one  $y_j$  with  $|y_j| = n-1$  which cannot be expressed in the form  $y_j = \sum_{i \leq |y_j| \leq n-2} s_i y_i$ . Exactly as in the proof of Lemma 5.4 it follows that the set of all  $y_i$  satisfying  $3 \leq |y_i| \leq n-2$ , along with  $y_j$ , extend to a standard set of group generators for  $R_+$ . Thus we have found  $2^n - n(n+1)/2 - 2[(n+2)/2]$  generators of order 2, proving (ii) and completing the proof of the theorem.

The following theorem shows that the results of the previous theorem are best possible.

**THEOREM 5.6.** *Let  $n \geq 4$  be an integer.*

(i) *Given any integer  $N \geq 2^n - 2[(n+2)/2]$ , there exists a finite abelian 2-group  $G$  of dimension  $N$ , such that  $f(G) = n$ .*

(ii) *Given any integer  $M \geq 2^n - n(n+1)/2 - 2[(n+2)/2]$ , there exists a finite abelian 2-group  $G$  with precisely  $M$  generators of order 2 (in a standard set of generators), such that  $f(G) = n$ .*

*Proof.* Clearly to prove both (i) and (ii) it suffices to construct a finite zero-square ring  $R$  with  $R^n \neq 0$  ( $n \geq 4$ ), such that  $R_+$  has precisely  $N = 2^n - 2[(n+2)/2]$  standard group generators, with precisely  $M = 2^n - n(n+1)/2 - 2[(n+2)/2]$ , of these generators of order 2. Let  $m = [n/2]$  and let  $R_F(\alpha_1 = 8, \alpha_2 = 8, \dots, \alpha_n = 8)$  be a free ring with generators  $x_1, \dots, x_n$  (as defined in §3). If  $n$  is even let  $J$  be the ideal generated by  $\{\bar{x}_1 - 4x_2, \bar{x}_2 - 4x_1, \bar{x}_3 - 4x_4, \bar{x}_4 - 4x_3, \dots, \bar{x}_{n-1} - 4x_n, \bar{x}_n - 4x_{n-1}\}$ , while if  $n$  is odd let  $J$  be generated by  $\{\bar{x}_1 - 4x_2, \bar{x}_2 - 4x_1, \dots, \bar{x}_{n-2} - 4x_{n-1}, \bar{x}_{n-1} - 4x_{n-2}\}$ . Let  $R = R_F/J$ . Then  $R$  is generated by the images of all  $y_i$  satisfying  $1 \leq |y_i| \leq n-2$  when  $n$  is even; with the additional generator  $\bar{x}_n$  when  $n$  is odd. This gives a total of  $2^n - 2[(n+2)/2]$  generators, as desired. Moreover, when  $n$  is even, a standard set of group generators for  $R_+$  has  $n+1$  elements of order 8,  $2m^2 - m - 1$  elements of order 4, and exactly  $M$  elements of order 2. When  $n$  is odd, there are  $n+1$  elements of order 8,  $2m^2 + m - 1$  elements of order 4, and exactly  $M$  elements of order 2. Hence it remains to prove that the image of  $x$  in  $R_F/J$  is not 0, i.e., that  $x \notin J$ . We treat only the case when  $n$  is even; the case  $n$  odd is

almost exactly the same.

Assume  $x \in J$ . Then

$$\begin{aligned} x = z_1(\bar{x}_1 - 4x_2) + z_2(\bar{x}_2 - 4x_1) + \cdots + z_n(\bar{x}_n - 4x_{n-1}) \\ + b_1(\bar{x}_1 - 4x_2) + b_2(\bar{x}_2 - 4x_1) + \cdots + b_n(\bar{x}_n - 4x_{n-1}), \end{aligned}$$

where  $z_i \in R_F$ ,  $b_i = 0$  or  $1$ . Hence we need at least one  $z_j = x_j$ , say  $z_1 = x_1$ . We then also get the term  $-4x_1x_2$ , which can only be cancelled by  $z_2 = x_2$ , giving another  $-4x_1x_2$ . But this also gives another  $x$ , and  $x + x = 0$ . Hence  $x \notin J$ , and the theorem is proved.

REMARK. The proofs of Theorems 5.5 and 5.6 are not valid for  $n = 3$ , basically because from  $|y_i| = n - 1$  we cannot deduce  $2y_i = 0$ . If Theorem 5.5 (i) were false for  $n = 3$ , then there would be a 2-group  $G$  with three generators such that  $f(G) = 3$ . Although this seems highly unlikely, the question remains open. Clearly  $G$  cannot have less than three generators. Note that Theorem 5.5 (ii) is trivially satisfied for  $n = 3$ . Finally, Theorem 5.6 is easy to verify for  $n = 3$  (though in part (ii) we of course must have  $M \geq 0$ ).

It is considerably simpler to get results on the order of zero-square rings satisfying  $R^n \neq 0$ .

THEOREM 5.7. *Assume  $n > 2$ . Then there exists a zero-square ring of order  $r$  satisfying  $R^n \neq 0$  if and only if  $2^{2^{n-1}} | r$ .*

*Proof.* Assume  $R^n \neq 0$ . We know from the proof of Theorem 5.3 that there are elements  $x_1, \dots, x_n$  in the Sylow 2-subgroup  $R_2$  of  $R_+$  such that  $x_1 \cdots x_n \neq 0$ . Let  $y_1, y_2, \dots, y_{2^{n-1}}$  be the special monomials in the  $x_i$ . Claim that the  $2^{2^{n-1}}$  elements of the form  $\sum_{i=1}^{2^{n-1}} b_i y_i$ ,  $b_i = 0$  or  $1$ , are all distinct, otherwise we would have a relation of the form

$$\sum b_i y_i = 0,$$

with at least one  $b_i = 1$ . Let  $y_j$  be a special monomial of shortest length such that  $b_j = 1$ . Then multiplying (6) by  $\bar{y}_j$  gives  $x = 0$ , a contradiction. Hence  $R_2$  has order at least  $2^{2^{n-1}}$ , so that  $2^{2^{n-1}} || R_2|$ . Hence  $2^{2^{n-1}} | r$ .

Conversely, if  $2^{2^{n-1}} \cdot s = r$ , then we take  $R_+$  to be  $C_2^{2^{n-1}} \oplus C_s$ . If we impose the free ring  $R_F(2, 2, \dots, 2)$  on  $C_2^{2^{n-1}}$  and the null ring on  $C_s$ , then  $R^n \neq 0$ .

Finally we have the result of [3].

COROLLARY 5.8. *The smallest zero-square ring  $R$  satisfying  $R^n \neq 0$ ,  $n > 1$ , has order  $2^{2^{n-1}}$ .*

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