G-SPACES, *H*-SPACES AND *W*-SPACES

JERROLD NORMAN SIEGEL

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The notions of G-space, W-space, H-space, and higher order Whitehead product are differentiated through example.

In [3], [4] and [5] D. H. Gottlieb introduces certain subgroups, $G_n(X, x_0)$, of the homotopy groups of a space. These groups are related to the problem of sectioning fibrations with fibre X. Related to the groups $G_n(X, x_0)$ is the notion of a G-space. A G-space is a space with $G_n(X, x_0) = \pi_n(X, x_0)$ for all n. It is a simple matter to show that every H-space is a G-space (see below). However, till recently the status of the converse remained undecided. Recently, Gottlieb produced an example of a two-stage Postnikov system that is a G-space but not an H-space (unpublished). The purpose of this note is to clarify the situation further. We produce a 3-dimensional manifold that is a G-space but not an H-space. Incidently, the theory of G-spaces tells us that our example is also a W-space, that is, a space whose Whitehead products all vanish.

Finally we would like to resolve a question of G. Porter [6]. Namely, our example is also an example of a space whose higher order Whitehead products all vanish but, again, is not an H-space.

We would like to acknowledge the priority of D. H. Gottlieb's example mentioned above and thank him for his help in the preparation of this paper.

1. **Preliminaries.** In this section we review the elementary theory of G-spaces presented in [4] and [5].

NOTATION 1.1. We assume all our spaces X are path connected C. W. complexes with base point x_0. We let X^X be the space of maps X to X. We let $M(X)$ be the component of the identity map $1: X \to X$ in X^X. Consider the evaluation map $e: M(X) \to X$ given by $e(f) = f(x_0)$. This map gives a fibration with fibre $M(X)_0$, the space of maps in $M(X)$ with $f(x_0) = x_0$.

DEFINITION 1.2. We define

$$G_n(X, x_0) = e_*(\pi_n(M(X), 1)) \subseteq \pi_n(X, x_0) .$$

THEOREM 1.3. *The groups $G_n(X, x_0)$ are invariant with respect to base point and homotopy type but not natural with respect to maps.*

Proof. [5].

THEOREM. 1.4.

$$G_n(X, x_0) = \{[f] \mid \cdot \exists \cdot F : X \times S^n \longrightarrow X \ with \ F/X \vee S^n = 1 \vee f\} \ .$$

Proof. [5].

THEOREM 1.5.

$$G_n(X, x_0) = \{[f] \mid \cdot \exists \ a \ fibration \ X \subseteq E \xrightarrow{p} S^{n+1} \ with \ [f] = \partial_*[1]\} \ ,$$

where $1 : S^{n+1} \to S^{n+1}$ is the identity map.

Proof. [4].

DEFINITION 1.6. $P_n(X, x_0)$ is the subgroup of elements $[f]$ in $\pi_n(X, x_0)$ with $[[f], [g]] = 0$ (Whitehead product) for all m and all $[g] \in \pi_m(X, x_0)$.

THEOREM 1.7. $G_n(X, x_0) \subseteq P_n(X, x_0)$.

Proof. [5] (see 1.4 above).

REMARK. Ganea [1] has shown that in general $G_n(X, x_0) \neq P_n(X, x_0)$. (see 3.4 below).

DEFINITION 1.8. (a) A *G-space* is a space X with $G_n(X, x_0) = \pi_n(X, x_0)$, all n.
(b) A *W-space* is a space X with $P_n(X, x_0) = \pi_n(X, x_0)$ for all n.

THEOREM 1.9. (a) *Every H-space is a G-space.*
(b) *Every G-space is a W-space.*

Proof. [5]. (a) Follows from 1.4.
(b) Follows from 1.7.

2. **A G-space that is not an H-space.** As mentioned in 1.3 the groups $G_n(X, x_0)$ are not natural with respect to maps. However, we can prove the following.

LEMMA 2.1. *Suppose we are given a map* $F : Y \times X \to Y$ *with* $F/Y \vee X = 1 \vee f$ *then* $f_* : \pi_n(X, x_0) \to G_n(Y, y_0)$.

Proof. For $g : S^n \to X$ consider the composition

$$Y \times S^n \xrightarrow{1 \times g} Y \times X \xrightarrow{F} Y \ .$$

Now apply 1.4.

EXAMPLE 2.2. Let H be a closed subgroup of a Lie group G. Let $(H \backslash G)$ be the left coset space. Let $p \colon G \to (H \backslash G)$ be the projection. We have the usual pairing $(H/G) \times G \xrightarrow{F} (H/G)$ with $F/(H/G) \vee G = 1 \vee p$.

THEOREM 2.3. *In the situation of 2.2 assume $i_* \colon \pi_n(H, e) \to \pi_n(G, e)$ is an inclusion for all n, then $(H \backslash G)$ is a G-space hence a W-space.*

Proof. Since i_* is an inclusion $p_* \colon \pi_n(G, e) \to \pi_n(H \backslash G, [e])$ is an epimorphism. On the other hand, by 2.1 $p_* \pi_n(G, e) \subseteqq G_n(H \backslash G, [e])$ hence $G_n(H \backslash G, [e]) = \pi_n(H \backslash G, [e])$ or $H \backslash G$ is a G-space.

We are now prepared to produce our example. We represent S^1 by the complex numbers $e^{i\theta}$ $0 \leqq \theta \leqq 2\pi$. In $SO(3)$ we let the symbol (θ) denote the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

2.4. *Example of a G-space that is not an H-space.* Embed $S^1 \subseteqq SO(3) \times S^1$ as a subgroup by the following map $i(e^{i\theta}) = (2\theta) \times e^{i3\theta}$. We let

$$T \overset{\mathrm{def}}{\equiv} i(S^1) \backslash SO(3) \times S^1 \ .$$

LEMMA 2.5. *T is a G-space, hence a W-space.*

Proof. By 2.3 we need only check

$$i_* \colon \pi_1(S^1) \longrightarrow \pi_1(SO(3) \times S^1)$$
$$\| \qquad\qquad\qquad \| $$
$$Z \qquad\qquad\qquad Z_2 \oplus Z$$

is an inclusion, but it is easy to check $i_*(1) = 0 \oplus 3$. Note this implies $\pi_1(T) = Z_2 \oplus Z_3$.

LEMMA 2.6. *T is not an H-space.*

Proof. (a) T is a 3-dimensional manifold hence $H^n(T, Z_3) = 0$,

$n > 3$.

(b) $H'(T, Z_3) = Z_3$, generated say by α. This is by remarks at the end of 2.5.

(c) From the universal coefficient theorem we know there is $\beta \neq 0$ in $H^2(T, Z_3)$ β indecomposable $(\alpha^2 = 0)$.

(a), (b) and (c) implies that $H^*(T, Z_3)$ does not support a Hopf algebra structure, hence, T is not an H space. In particular if $T \times T \xrightarrow{\ h\ } T$ is a Hopf map.

$$0 = h^*(\beta^2) = (1 \otimes \beta + \beta \otimes 1 + r(\alpha \otimes \alpha))^2 = 2\beta \otimes \beta + \cdots \neq 0 \ .$$

We could also note that $T = Z_3 \backslash SO(3)$ where Z_3 is the group (0), $(2/3\pi)$, $(4/3\pi)$. Then, using the spectral sequence of a covering we have

$$H^n(T, Z_3) = \begin{cases} Z_3 & n = 0, 1, 2, 3 \\ 0 & n > 3 \ . \end{cases}$$

This does not support a Hopf algebra structure.

3. Higher order Whitehead products. The purpose of this section is to point out that our example also answers a question of G. Porter [6].

DEFINITION 3.1. A space X is said to have *trivial higher order Whitehead products*. If given any set of homotopy elements

$$[f_i] \in \pi_{p_i}(X, x_0) \ 1 \leq i \leq n \ .$$

The map $\bigvee_{i=1}^{n} f_i \colon \bigvee S^{p_i} \to X$ extends to some $f \colon \bigtimes_{i=1}^{n} S^{p_i} \to X$. (see [6]).

THEOREM 3.2. *Any G-space has trivial spherical Whitehead products.*

Proof.

LEMMA. *Given any $n - 1$ elements $[f_i] \in \pi_{p_i}(X, x_0)$ $1 \leq i \leq n - 1$ we can find a map*

$$h \colon \left(\bigtimes_{i=1}^{n-1} S^{p_i} \right) \times X \longrightarrow X \qquad \qquad with$$

$$h \Big/ \left(\bigvee_{i=1}^{n-1} S^{p_i} \right) \vee X = \left(\bigvee_{i=1}^{n-1} f_i \right) \vee 1 \ .$$

This is proved by induction. For $n = 2$ this is 1.4. Suppose we

have a map

$$\bar{h}: \left(\overset{n-2}{\underset{i=1}{\times}} S^{p_i}\right) \times X \longrightarrow X$$

with the required property.

Consider $\tilde{h}: S^{p_{n-1}} \times X \to X$ an extension of $f_{n-1} \vee 1$ (1.4). Finally, consider the composition

$$S^{p_{n-1}} \times \left(\left(\overset{n-2}{\underset{i=1}{\times}} S^{p_i}\right) \times X\right) \xrightarrow{1 \times \bar{h}} S^{p_{n-1}} \times X \xrightarrow{\tilde{h}} X \;.$$

Set $h = \tilde{h}(1 \times \bar{h})$.

We now finish the proof by noting that the composition

$$\left(\overset{n-1}{\underset{i=1}{\times}} S^{p_i}\right) \times S^{p_n} \xrightarrow{1 \times f_n} \left(\overset{n-1}{\underset{i=1}{\times}} S^{p_i}\right) \times X \xrightarrow{h} X$$

is the required extension of $\bigvee_{i=1}^{n} f_i$.

THEOREM 3.3. *There exists finite dimensional spaces with trivial higher order Whitehead products that are not H-spaces.*

Proof. The space T of 2.4 is such an example.

FINAL REMARKS 3.4. Ganea [2] has constructed an infinite dimensional example of a W-space that is not a G-space. G. Lang (unpublished) points out that using recent results of Gottlieb [5] one can show that $CP(3)$ is a finite dimensional example of such a space. In [1] it is shown that $CP(3)$ is a W-space, but in [5] it is shown that every finite dimensional G-space has Euler-Poinare characteristic 0 hence $CP(3)$ is not a G-space.

Porter [7] shows that $CP(3)$ has nontrivial higher order Whitehead products. It would be interesting to have examples of spaces with vanishing higher order Whitehead products that are not G-spaces.

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