Pacific Journal of Mathematics

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Vol. 31, No. 3 BadMonth 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

ENTIRE FUNCTIONS OF SEVERAL VARIABLES WITH ALGEBRAIC DERIVATIVES AT CERTAIN ALGEBRAIC POINTS[1]

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The purpose of this paper is to extend certain theorems on the arithmetic properties of analytic functions due to Straus to functions of several variables.

Numerous papers have been written on the arithmetic properties of analytic functions (e.g., Straus [7], Buck [1], Kakeya [3], Selberg [5]). The author is not aware of any analogous studies for analytic functions of several variables. Since the generalization from two to several variables involves no new difficulties that are not already encountered in the generalization from one to two variables, we shall for the sake of simplicity, restrict our discussion to functions of two variables.

2. Preliminaries.

We begin with a generalization of order and type.

DEFINITION 1. Let $f(z_1, z_2)$ be an entire function of the two variables. Let $M(r_1, r_2) = M(r)$ denote the maximum value of $|f|$ on the surface given by $|z_i| = r_i (i = 1, 2)$. (ρ_1, ρ_2) is said to be an order point of f, if for any $\varepsilon > 0$, as $r_1 + r_2$ approaches infinity

$$M(r)/\exp\,(r_1^{\rho_1+\varepsilon} + r_2^{\rho_2+\varepsilon})$$

is bounded, while

$$M(r)/\exp\,(r_1^{\rho_1} + r_2^{\rho_2-\varepsilon})$$

and

$$M(r)/\exp\,(r_1^{\rho_1-\varepsilon} + r_2^{\rho_2})$$

are both unbounded. The set, ρ, of all such points (ρ_1, ρ_2) is called the order of f.

DEFINITION 2. Let $f(z_1, z_2)$ be as above and let (ρ_1, ρ_2) be one of its order points. (σ_1, σ_2) is said to be a type point of f at (ρ_1, ρ_2) if for any $\varepsilon > 0$, as $r_1 + r_2$ approaches infinity

[1] In a dissertation written by the author under the direction of Professor E. G. Straus and submitted to U.C.L.A. in July 1962, variations of the results in this paper were proved by a generalization of an argument used by Straus in [7]. The arguments presented here are somewhat briefer.

$$M(r)/\exp\left((\sigma_1+\varepsilon)r_1^{\rho_1}+(\sigma_2+\varepsilon)r_2^{\rho_2}\right)$$

is bounded, while

$$M(r)/\exp\left(\sigma_1 r_1^{\rho_1}+(\sigma_2-\varepsilon)r_2^{\rho_2}\right)$$

and

$$M(r)/\exp\left((\sigma_1-\varepsilon)r_1^{\rho_1}+\sigma_2 r_2^{\rho_2}\right)$$

are both unbounded. The set of points, $\sigma_{\rho_1\rho_2}$, of all such points (σ_1, σ_2) is called the type of f at (ρ_1, ρ_2).

For the sake of simplicity, we add the following.

DEFINITION 3. An entire function $f(z_1, z_2)$ will be said to have $\{(\rho_1, \sigma_1), (\rho_2, \sigma_2)\}$ as an order-type point if (ρ_1, ρ_2) is an order point of f and (σ_1, σ_2) is a type point of f at (ρ_1, ρ_2). We shall say that $(\rho_i, \sigma_i) < (x, y)$ if either $\rho_i < x$ or $\rho_i = x$ and $\sigma_i < y$ $(i = 1, 2)$.

We state some lemmas whose proofs are contained in [2].

LEMMA 1. (*Generalized Taylor series.*) *Let* $f(z_1, z_2)$ *be entire and let* z_{ij} $(i = 1, 2;\ j = 1, 2, \cdots)$ *be two infinite sequences of complex numbers whose terms are bounded. Then one may write*

$$f(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1 n_2} \prod_{i=1}^{2} \prod_{j=1}^{n_i} (z_i - z_{ij}) \ , \tag{1}$$

with

$$a_{n_1 n_2} = \frac{1}{(2\pi i)^2} \int_{|z|=r_1} \int_{|z|=r_2} \frac{f(z_1, z_2)\,dz_1 dz_2}{\prod_{i=1}^{2} \prod_{j=1}^{n_i+1} (z_i - z_{ij})} \tag{2}$$

where

$$r_i > \max_j |z_{ij}| \qquad (i = 1, 2; j = 1, 2, \cdots) \ .$$

Proof. Same as Lemma 2.1 in [2].

When z_{ij} is a finite set of integers, α_j $(j = 0, 1, \cdots; k_1 - 1)$ and z_{2j} is a finite set of integers, β_j $(j = 0, 1, \cdots; k_2 - 1)$ then (1) may be written as

$$\begin{aligned} f(z_1, z_2) = \sum_{s=0}^{\infty} \sum_{l=0}^{k_1-1} \sum_{t=0}^{\infty} \sum_{h=0}^{k_2-1} a_{(sk_1+l)(tk_2+h)}(a_1 - \alpha_0)^{s+1} \cdots \\ (z_1 - \alpha_{l-1})^{s+1}(z_1 - \alpha_l)^s \cdots (z_1 - \alpha_{k_1-1})^s (z_2 - \beta_0)^{t+1} \cdots \\ (z_2 - \beta_{k_2-1})^t \ . \end{aligned} \tag{1a}$$

By means of the residue theorem and (2), one obtains

LEMMA 2. *If*

$$\partial^{n_1+n_2}f(z_1, z_2)/\partial z_1^{n_1}\partial z_2^{n_2}$$

is integral for $(z_1, z_2) = (\alpha_i, \beta_j)(i = 0, 1, \cdots, k_1 - 1; j = 0, 1, \cdots, k_2 - 1)$ *and for all nonnegative integers* n_1 *and* n_2, *then the coefficients* $a_{(sk_1+l)(tk_2+l)}$ *on the right side of* (1a) *are rational numbers whose denominators divide the least common multiple of the quantities*

$$(3)\qquad \begin{gathered}(s-x)!\prod_{j=0;\, j\neq m}^{l}(\alpha_m-\alpha_j)^{s+1+x_j}\prod_{j=l+1;\, j\neq m}^{k_1-1}(\alpha_m-\alpha_j)^{s+x_j}(t-y)!\\ \prod_{i=0;\, i\neq n}^{h}(\beta_n-\beta_i)^{t+1+y_i}\prod_{i=h+1;\, i\neq n}^{k_2-1}(\beta_n-\beta_i)^{t+y_i}\,,\end{gathered}$$

$m = 0, 1, \cdots, k_1 - 1$; $n = 0, 1, \cdots, k_2 - 1$; $\{x_0 + x_1 + \cdots + x_{k_1-1} = x, y_0 + y_1 + \cdots + y_{k_2-1} = y\}$; $x = 0, 1, \cdots, s$ *and* $y = 0, 1, \cdots t$.

Proof. See proof of Theorem 3.6 pages 134 and 135 in [2].

An argument almost identical to this gives the following.

LEMMA 2A. *If the* α'*s*, β'*s and partial derivatives in Lemma* 2 *are algebraic integers, then each of the coefficients is a ratio of two algebraic integers whose denominator is the least common multiple of the expressions* (3).

LEMMA 3. *Let* f *and* $a_{n_1n_2}$ *be as in Lemma* 1 *and suppose that* $\{(\rho_1, \sigma_1), (\rho_2, \sigma_2\}$ *is an order-type point of* f. *Then the inequality*

$$M(r) < \exp(r_1^{\rho_1+\varepsilon} + r_2^{\rho_2+\varepsilon})$$

holds for $\varepsilon > 0$ *and all sufficiently large (depending on* ε) $r_1 + r_2$ *if, and only if, the inequality*

$$(4)\qquad |a_{n_1n_2}| < \prod_{i=1}^{2} n_i^{-n_i/(\rho_i+\varepsilon)}$$

holds for $\varepsilon > 0$ *and all sufficiently large (depending on* ε) $n_1 + n_2$. *Furthermore, the inequality*

$$M(r) < \exp((\sigma_1 + \varepsilon)r_1^{\rho} + (\sigma_2 + \varepsilon)r_2^{\rho_2})$$

holds for $\varepsilon < 0$ *and all sufficiently large (depending on* ε) $r_1 + r_2$ *if, and only if, the inequality*

$$(5)\qquad |a_{n_1n_2}| < \prod_{i=1}^{2}((e\rho_i\sigma_i + \varepsilon)/n_i)^{n_i/\rho_i}$$

holds for $\varepsilon > 0$ *and all sufficiently large (depending on* ε) $r_1 + r_2$.

Proof. The proof of this lemma is entirely analogous to the one variable case (see e.g. [6]).

3. Main result. We first consider the case where assumptions are made about the value of the function and its partial derivatives at a single point.

THEOREM 1. *Let* $f(z_1, z_2)$ *be an entire function such that*

$$\partial^{n_1+n_2} f(0, 0)/\partial z_1^{n_1} \partial z_2^{n_2} = \alpha_{n_1 n_2} ,$$

where $\alpha_{n_1 n_2}$ *is an algebraic number of degree* $\leqq d$ *for* $n_1, n_2 = 0, 1, \cdots$. *Let* $q_{n_1 n_2}$ *be a positive rational integer such that* $q_{n_1 n_2}\alpha_{n_1 n_2}$ *is an algebraic integer. Assume that for some positive numbers* A, B, s_i *and* t_i $(i = 1, 2)$ *and any positive* ε

$$\overline{|\alpha_{n_1 n_2}|} = 0((A + \varepsilon)^{n_1+n_2} n_1^{s_1 n_1} n_2^{s_2 n_2}$$

and

$$q_{n_1 n_2} = 0((B + \varepsilon)^{n_1+n_2} n_1^{t_1 n_1} n_2^{t_2 n_2}) . \tag{6}$$

Let

$$\begin{aligned} \rho_{i0} &= ((s_i + t_i)(d - 1) + t_i + 1)^{-1} \\ \sigma_{i0} &= (e\rho_{i0})^{-1}(eA^{-(d-1)}B^{-d})^{\rho_{i0}} \qquad (i = 1, 2) . \end{aligned}$$

If for some order-type point, $\{(\rho_1, \sigma_1), (\rho_2, \sigma_2)\}$, *of* f, *there holds*

$$(\rho_i, \sigma_i) < (\rho_{i0}, \sigma_{i0}) \qquad (i = 1, 2) ,$$

then f *is a polynomial.*

Proof. We may write

$$f(z_i, z_2) = \sum a_{n_1 n_2} z_1^{n_1} z_2^{n_2} ,$$

where

$$a_{n_1 n_2} = \alpha_{n_1 n_2}/n_1! n_2! .$$

Furthermore, it follows from the hypotheses of the theorem that

$$\overline{|q_{n_1 n_2}\alpha_{n_1 n_2}|} = 0((AB + \varepsilon)^{n_1+n_2} n_1^{(s_1+t_1)n_1} n_2^{(s_2+t_2)n_2}) . \tag{7}$$

Assume that f is not a polynomial. Since $q_{n_1 n_2}\alpha_{n_1 n_2}$ is an algebraic integer, it follows that for an infinite sequence of pairs (n_1, n_2)

$$|\operatorname{Norm} q_{n_1 n_2}\alpha_{n_1 n_2}| \geqq 1 . \tag{8}$$

Consequently, for these n_1 and n_2

$$|q_{n_1 n_2}\alpha_{n_1 n_2}| \geqq |\operatorname{Norm} q_{n_1 n_2}\alpha_{n_1 n_2}| \, \overline{|q_{n_1 n_2}\alpha_{n_1 n_2}|}^{-(d-1)} . \tag{9}$$

Thus, from (6), (7) and (9) we obtain

$$\begin{aligned}|\operatorname{Norm} q_{n_1n_2}\alpha_{n_1n_2}| \leqq \frac{|\alpha_{n_1n_2}|}{n_1!n_2!}\Big[0\Big(\prod_{i=1}^{2}((AB+\varepsilon)^{(d-1)n_i} \\ \times (B+\varepsilon)^{n_i}e^{-n_i}n_i^{[(s_i+t_i)(d-1)+t_i+1]n_i})\Big)\Big]\,. \end{aligned} \tag{10}$$

On the other hand, it follows from (4) of Lemma 3 that

$$\frac{|\alpha_{n_1n_2}|}{n_1!n_2!} < |a_{n_1n_2}| < \prod_{i=1}^{2} n_i^{-n_i/(\rho_i+\varepsilon)}\,. \tag{11}$$

If for $i = 1, 2$, $\rho_i < \rho_{i0}$, then for some positive ε satisfying $\rho_i + \varepsilon < \rho_{i0} - \varepsilon$ and some positive ε_0

$$n_i^{-n_i/(\rho_i+\varepsilon)} < n_i^{-n_i[(s_i+t_i)(d-1)+t_i+1]-\varepsilon_0 n_i} \qquad (i = 1, 2)\,. \tag{12}$$

From (10), (11) and (12), one easily concludes that for sufficiently large $n_1 + n_2$

$$|\operatorname{Norm} q_{n_1n_2}\alpha_{n_1n_2}| < 1\,. \tag{13}$$

Thus, in this case, we get a contradiction between (8) and (13).

If $\rho_i = \rho_{i0}$ and $\alpha_i < \alpha_{i0}$ for either $i = 1$ or $i = 2$ or both, then one can similarly use (5) of Lemma 3 (instead of (4)) together with (10) to again arrive at the contradiction between (8) and (13). This completes the proof of the theorem.

We now proceed to the case where something is known about the value of the function and its partial derivatives at several points.

THEOREM 2. *Let $f(z_1, z_2)$ be entire and suppose that for all non-negative integers n_1 and n_2*

$$\partial^{n_1+n_2}f(z_1, z_2)/\partial z_1^{n_1}\partial z_2^{n_2}$$

is integral for $(z_1, z_2) = (a_i, b_j)(i = 1, 2, \cdots, k_1, j = 1, 2, \cdots, k_2)$ with $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$, where a_i and b_j are integers. If f has an order type point satisfying

$$(\rho_1, \sigma_1) < (k_1, |V(a_j)^{-2}|)$$
$$(\rho_2, \sigma_2) < (k_2, |V(b_i)^{-2}|)\,,$$

where $V(a_j)$ and $V(b_j)$ are the Vandermondes of the a_j's and b_j's respectively, then f is a polynomial.

Proof. By Lemma 1, we may write

$$f(z_1, z_2) = \sum \alpha_{n_1n_2}(z_1 - a_1)(z_1 - a_2)\cdots(z_1 - a_{n_1})(z_2 - b_1)\cdots(z_2 - b_{n_2})\,.$$

where $a_{k_1+n} = a_n$ and $b_{k_2+n} = b_n$ $(n = 1, 2, \cdots)$. Using Lemma 2 with $s = [n_1/k_1]$ and $t = [n_2/k_2]$ ($[r]$ = greatest integer less than r), one easily concludes by looking at the expressions (3) that $\alpha_{n_1n_2}$ is a rational number expressible as $c_{n_1n_2}/d_{n_1n_2}$, $c_{n_1n_2}$ integers and

$$d_{n_1n_2} = [n_1/k_1]!\,[n_2/k_2]!\,V(a_i)^{2[n_1/k_1]}V(b_j)^{2[n_2/k_2]}\,.$$

If $\rho_i < k_i$ $(i = 1, 2)$, then using (4) of Lemma 3, we obtain

$$|c_{n_1n_2}| = |\alpha_{n_1n_2}|\,|d_{n_1n_2}| < \prod_{i=1}^{2}(n_i^{-n_i/k_i}[n_i/k_i]!\,|V_i|^{2[n_i/k_i]})\,, \tag{14}$$

where V_1 and V_2 are $V(a_j)$ and $V(b_j)$ respectively.

For sufficiently large $n_1 + n_2$, the right side of (14) is less than 1. Thus, $c_{n_1n_2}$ and consequently $\alpha_{n_1n_2}$ must be zero, so that in this case, f must be a polynomial. If $\rho_i = k_i$ and $\sigma_i < V_i^{-2}$ for one of the values i, then by virtue of (5) Lemma 3

$$\begin{aligned}|c_{n_1n_2}| &< ((ek_i\,|V_i|^{-2} + \varepsilon)/n_i)^{n_i/k_i}\\ &[n_i/k_i]!\;V_i^{2[n_i/k_i]})\,|\,(\text{second factor})\,|\,.\end{aligned} \tag{15}$$

It is easy to see that the first factor on the right side of (15) is less than 1 for sufficiently large n_i. The second factor is either of the same form as the first or has the form of the right factors appearing in (14). Thus, in any case the right side of (15) is less than 1 for sufficiently large $n_1 + n_2$ and the theorem follows.

Instead of considering functions with integral values and partial derivatives at the integers one can consider more generally functions whose values and derivatives evaluated at a certain set, F, of algebraic numbers are themselves numbers in F.

THEOREM 3. *Let $f(z_1, z_2)$ be an entire function such that*

$$\frac{\partial^{n_1+n_2}f(z_1, z_2)}{\partial z_1^{n_1}\partial z_2^{n_2}}$$

has the values $\alpha_{n_1n_2ij}$ at the points $(z_1, z_2) = (\alpha_i, \beta_j)$; $i = 0, \cdots, k_1 - 1$, $j = 0, \cdots, k_2 - 1$, $\alpha_0 = \beta_0 = 0$; $\alpha_{i_1} \neq \alpha_{i_2}$, $\beta_{i_1} \neq \beta_{i_2}$ when $i_1 \neq i_2$. Assume that $\alpha_{n_1n_2ij}$, α_i and β_j belong to an algebraic number field K of degree d for $n_1 = 0, 1, \cdots$; $n_2 = 0, 1, \cdots$; $i = 0, 1, \cdots, k_1 - 1$ and $j = 0, 1, \cdots, k_2 - 1$. Let

$$M_1 = 2\max_i \overline{|\alpha_i|}\,, \tag{16}$$

$$M_2 = 2\max_j \overline{|\beta_j|} \tag{17}$$

and let c be a positive rational integer such that $c\alpha_i^{(\nu)}$, $c\beta_j^{(\nu)}$ are algebraic integers for $i = 1, \cdots; k_1 - 1$ and $j = 1, \cdots, k_2 - 1$, where $\alpha_i^{(\nu)}$, $\beta_j^{(\nu)}(\nu = 1, \cdots, d)$ are the conjugates of α_i and β_i respectively. Let $q_{n_1n_2}$ be a positive rational integer such that $q_{n_1n_2}\alpha_{n_1n_2ij}$ is an algebraic integer and assume that for some positive reals A_1, s_1, s_2, B, t_1, t_2

$$\overline{|\alpha_{n_1n_2ij}|} = 0((A + \varepsilon)^{n_1+n_2}n_1^{s_1n_1}n_2^{s_2n_2}) \tag{18}$$

and

$$q_{n_1n_2} = 0((B + \varepsilon)^{n_1+n_2}n_1^{t_1n}n_2^{t_2n_2}) \tag{19}$$

for

$$i = 0, 1, \cdots, k_1 - 1; j = 0, 1, \cdots, k_2 - 1; n_1 = 0, 1, \cdots$$

and $n_2 = 0, 1, \cdots$. Let $\lambda_i = 2k_i(k_i - 1)$,

$$\rho_{i0} = k_i[(dt_i + (d - 1)s_i)k_i + d]^{-1}$$

and

$$\sigma_{i0} = (e\rho_{i0})^{-1}((k_ie)^dM_i^{\lambda_i(d-1)})^{\rho_i/k_i}$$
$$(A^{(d-1)}B^d\,|\,V_i\,|^{2/k_i}c^{d\lambda_i/k_i})^{-\rho_{i0}}$$

for $i = 1, 2$.

If f has an order-type point satisfying

$$(\rho_i, \sigma_i) < (\rho_{i0}, \sigma_{i0}) \qquad (i = 1, 2)\ ,$$

then f is a polynomial.

Proof. Let $f(z_1, z_2)$ be given by (1). If $\alpha_{n_1n_2ij}$, α_i and β_j were algebraic integers, then applying Lemma 2A one would be able to express the coefficients of the series as a ratio of two algebraic integers $c_{n_1n_2}/d_{n_1n_2}$ and one would get an upper bound for $|c_{n_1n_2}|$ as in the proof of the previous theorem. From the hypotheses of the theorem one can also get an upper bound for $|c_{n_1n_2}|$ and subsequently arrive at the conclusion that $|\mathrm{Norm}\, c_{n_1n_2}| < 1$ for sufficiently large $n_1 + n_2$. Though in our case $\alpha_{n_1n_2ij}$, α_i, β_j are not algebraic integers, multiplication by the appropriate rational integers effectively reduces it to the simpler case just mentioned.

For the sake of convenience let us also express f in the equivalent form (1a) with $s = [n_1/k_1]$ and $t = [n_2/k_2]$. From the second equation on page 135 of [2] one can easily verify that one may write

$$\begin{aligned}\frac{c_{n_1n_2}}{d_{n_1n_2}} &= q_{n_1n_2}a_{n_1n_2}c^{(\lambda_1[n_1/k_1]+\lambda_2[n_2/k_2])} \\ &= q_{(sk_1+l)(tk_2+h)}c^{(\lambda_1s+\lambda_2t)}a_{(sk_1+l)(tk_2+h)}\end{aligned} \tag{20}$$

with

$$(21) \qquad |d_{n_1 n_2}| < \prod_{i=1}^{2} |V_i|^{2[n_i/k_i]}\left[\frac{n_i}{k_i}\right]!$$

and $c_{n_1 n_2}$ an algebraic integer of the form

$$(22) \qquad \sum I_l \gamma_l \eta_l ,$$

where I_l is a positive rational integer satisfying for all l

$$I_l < 4 \prod_{i=1}^{2}\left(\left[\frac{n_i}{k_i}\right]^{k_i+2}\left[\frac{n_i}{k_i}\right]!\right) c^{\lambda_1[n_1/k_1]+\lambda_2[n_2/k_2]} O((B+\varepsilon)^{n_1+n_2} n_1^{t_1 n_1} n_2^{t_2 n_2}) ;$$

the γ_l are products of at most $\lambda_1/2$ terms of the form $(\alpha_i - \alpha_j)^{u_{ij}}$ and at most $\lambda_2/2$ terms of the form $(\beta_i - \beta_j)^{v_{ij}}$ with

$$u_{ij} < 2\left[\frac{n_1}{k_1}\right] v_{ij} < 2\left[\frac{n_2}{k_2}\right] ;$$

and the η_l are one of the numbers $\alpha_{m_1 m_2 ij}$ with $m_i < [n_i/k_i]$ $(i = 1, 2)$.

Using (16), (17) and (18) one can easily show that for each summand in (22)

$$(23) \qquad \overline{|I_l \gamma_l \eta_l|} < \prod_{i=1}^{2}\left(\left[\frac{n_i}{k_i}\right]^{k_i+2}\left[\frac{n_i}{k_i}\right]! \, M_i^{\lambda_i[n_i/k_i]} c^{\lambda_i[n_i/k_i]} \right. \\ \left. O((A+\varepsilon)^{n_i} n_i^{s_i n_i}) O((B+\varepsilon)^{n_i} n_i^{t_i n_i})\right) .$$

The number of summands in (22) does not exceed $\prod_{i=1}^{2} k_i[n_i/k_i]$ and hence (23) implies

$$(24) \qquad \overline{|c_{n_1 n_2}|} < \prod_{i=1}^{2}\left(\left[\frac{n_i}{k_i}\right]^{k_i+3} c^{\lambda_i[n_i/k_i]}\left[\frac{n_i}{k_i}\right]! \, M_i^{\lambda_i[n_i/k_i]} \right. \\ \left. O((A+\varepsilon)^{n_i} n_i^{s_i n_i}) O((B+\varepsilon)^{n_i} n_i^{t_i n_i})\right) .$$

Using (4), (19), (20), (21) and (24) we obtain for any $\varepsilon > 0$.

$$(25) \qquad \begin{aligned} |\mathrm{Norm}\, c_{n_1 n_2}| &\leqq |c_{n_1 n_2}| \, \overline{|c_{n_1 n_2}|}^{(d-1)} \\ &\leqq \prod_{i=1}^{2}\left(|V_i|^{2[n_i/k_i]}\left[\frac{n_i}{k_i}\right]! \, n_i^{-n_i/(\rho_i+\varepsilon)} O((B+\varepsilon)^{n_i} n_i^{t_i n_i} c^{\lambda_i[n_i/k_i]d}\right) \\ &\quad \cdot \left(\left[\frac{n_i}{k_i}\right]^{k_i+3}\left[\frac{n_i}{k_i}\right]! \, M_i^{\lambda_i[n_i/k_i]}(O((A+\varepsilon)^{n_i} n_i^{s_i n_i}) \right. \\ &\quad \left. \times O((B+\varepsilon)^{n_i} n_i^{t_i n_i}))^{(d-1)}\right) . \end{aligned}$$

If $\rho_i < \rho_{i0}$, then a simple calculation shows that the right side of (25) is less than 1 for sufficiently large $n_1 + n_2$ and the desired conclusion follows in this case. Using (5), (19), (20), (21) and (24) one obtains similarly

$$
\begin{aligned}
&|\operatorname{Norm} c_{n_1 n_2}| \leqq \prod_{i=1}^{2} \Bigg(| V_i |^{2[n_i/k_i]} \left[\frac{n_i}{k_i}\right]! \, (e\rho_i(\sigma_i + \varepsilon)/n_i)^{n_i/\rho_i} \\
(26) \qquad &O((B + \varepsilon)^{n_i} n_i^{t_i n_i}) c^{\lambda_i d[n_i/k_i]} \Bigg(\left[\frac{n_i}{k_i}\right]^{k_i+3} \left[\frac{n_i}{k_i}\right]! M_i^{\lambda_i[n_i/k_i]} \\
&O((A + \varepsilon)^{n_i} n_i^{s_i n_i}) O((B + \varepsilon)^{n_i} n_i^{t_i n_i}) \Bigg)^{(d-1)} \Bigg) .
\end{aligned}
$$

If $\rho_i \leqq \rho_{i0}$, $i = 1, 2$ and $\rho_i = \rho_{i0}$, $\sigma_i < \sigma_{i0}$ for at least one of the i, then again a simple calculation shows that the right side of (26) is less than 1 for $n_1 + n_2$ sufficiently large and the theorem follows.

The question of generalizing the results of one variable to functions which are not entire, such as meromorphic functions, has already been suggested by Straus [7]. More generally it would be interesting to consider meromorphic functions of several complex variables. Though it is difficult to see how the methods of this paper can be applied to this more general case, even with the aid of Nevanlinna theory, it is quite possible that other methods, such as for example the one used in the proof of Theorem 2 in [4], might yield interesting analogues of our results in the meromorphic case.

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Received November 24, 1965, in revised form June, 1969.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics
Vol. 31, No. 3 BadMonth, 1969