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WREATH PRODUCTS OF ORDERED PERMUTATION GROUPS

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In this paper we define the ordered wreath product of a chain of ordered permutation groups and characterize it by certain completeness properties. We then use the construction to prove a nonabelian analogue of Hahn's theorem for totally ordered abelian groups.

One of the fundamental tools in the theory of totally ordered groups is Hahn's theorem [4; a detailed discussion may be found in Fuchs [3]], which asserts, roughly, that every abelian totally ordered group can be embedded in a lexico-graphically ordered (unrestricted) direct sum of copies of the ordered group of real numbers. Almost any general question regarding the structure of abelian totally ordered groups can be answered by reference to Hahn's theorem. Although Conrad [2] has used the Schreier theory to describe the structure of certain classes of nonabelian ordered groups, no tool like Hahn's theorem has been available for the general case.

The natural nonabelian analogue of the direct product of two groups (at least for embedding problems) is the wreath product. Lloyd [9] and Reilly [12] have shown that it is possible to lattice-order the wreath product of two lattice-ordered groups. A definition has recently been given [8] for the wreath product of an arbitrary set of groups, and in this paper we shall investigate the consequences of that construction for certain partially ordered groups. Although we shall occasionally refer to theorems proved in [8], familiarity with the material in [8] is *not* assumed.

One of our results will be an analogue of Hahn's theorem: every totally ordered group can be embedded in a lexicographically ordered wreath product of copies of the ordered group of real numbers. This theorem suffers by comparison with Hahn's theorem in one critical aspect; namely, the wreath product of totally ordered groups is not totally ordered, but only lattice-ordered. Thus, we shall broaden our interest to consider partially ordered groups which are transitive subgroups of the partially ordered group $A(S)$ of all order-preserving permutations of a chain S (ordered by $f \leqq g$ if and only if $sf \leqq sg$ for all $s \in S$). This class includes every totally ordered group (permuting itself in the Cayley representation), and a large class of lattice-ordered groups [6]. We shall define the wreath product of a collection of such ordered permutation groups and show that the wreath product is itself an ordered permutation group. If the factors are lattice-

ordered, then so is the wreath product.

A transitive group G of order-preserving permutations of a chain S determines a set of "o-primitive components" which are themselves ordered permutation groups, and G can be embedded in the ordered wreath product of these components. One corollary of this is the analogue of Hahn's theorem mentioned before. We also characterize ordered wreath products by certain completeness properties.

2. The ordered wreath product of two ordered permutation groups. The set $A(S)$ of all order-preserving permutations (o-permutations) of a chain S becomes a lattice-ordered group when we define $f \leqq g$ if and only if $sf \leqq sg$ for all $s \in S$. The lattice operations are $s(g \vee h) = sg \vee sh$, and dually. If G is a subgroup of $A(S)$ with the inherited partial order, we shall say that (G, S) is an *ordered permutation group* (*o-permutation group*).

We want to motivate an investigation of generalized ordered wreath products by a brief discussion of the ordered wreath product of two ordered permutation groups. Most of these definitions and results can be found in [9] and [12]. We shall omit proofs because we shall prove much more general theorems later.

Let (H, T) and (K, V) be o-permutation groups. The (unrestricted) wreath product $(H, T)Wr(K, V) = (W, R)$ is defined (see, e.g., M. Hall [5]) to be the group of all permutations of $R = T \times V$ of the form $(t, v) \to (th_v, vk)$, where $k \in K$ and $h_v \in H$. If we order $T \times V$ lexicographically by setting $(t_1, v_1) \leqq (t_2, v_2)$ if and only if $v_1 < v_2$ or $v_1 = v_2$ and $t_1 \leqq t_2$, then $T \times V$ becomes a chain $\overleftarrow{T \times V}$ and each element of the wreath product becomes an o-permutation of that chain. Thus W is a subgroup of $A(R)$. Moreover, if H and K are both lattice-ordered (closed under the lattice operation defined above), so is W.

Now we describe a certain universal property possessed by the ordered wreath product. If F is any set of permutations of a chain S, then a *convex F-congruence* on S is an equivalence relation Q on S such that each Q-class is convex ($x \leqq y \leqq z$ and xQz implies xQy) and such that if xQz and $f \in F$, then $(xf)Q(zf)$. Now let (G, S) be a transitive o-permutation group. If $g \in G$ and $s \in S$, then $g \mid sQ$ is an order isomorphism (o-isomorphism) of the chain sQ upon the chain $(sg)Q$, so all Q-classes are o-isomorphic. Let $T = sQ$ be any fixed Q-class and let $G_T = \{h \in G \mid Th = T\}$. There is a natural o-homomorphism of G_T into $A(T)$, whereby each $g \in G_T$ corresponds to its restriction to T. The image of G_T under this homomorphism is denoted by $G \mid T$. Thus if $H = G \mid T$, (H, T) is an o-permutation group.

The set $V = S/Q$ of all Q-classes bears a natural total order in which $xQ < yQ$ if and only if $x < y$ and $xQ \neq yQ$. Moreover, each

$g \in G$ induces a natural o-permutation $\bar{g} \in A(V)$, given by $(xQ)\bar{g} = (xg)Q$. The map $g \to \bar{g}$ is an o-homomorphism of G into $A(V)$. If the image of G in $A(V)$ is denoted by $K = G/Q$, then (K, V) is an o-permutation group. Since G is transitive on S, so is H transitive on T and K transitive on V; and all of the permutation groups (H, T) for various Q-classes T are o-isomorphic. We shall call (H, T) and (K, V) the local and global Q-components, respectively, of (G, S).

There is a natural o-isomorphism between the chains S and $\overleftarrow{T \times V}$ which can be realized as follows. Fix some point $O \in S$, let $T = OQ$, and for each Q-class xQ, choose $g_{xQ} \in G$ such that $Og_{xQ} \in xQ$ (by transitivity of G). Then map S onto $\overleftarrow{T \times V}$ by $s \to (t, v)$, where $v = sQ$ and $t = sg_{sQ}^{-1}$. Identifying S with $\overleftarrow{T \times V}$ in this way, each $g \in G$ is an element of the ordered wreath product (W, R). Thus (G, S) is embedded in the ordered wreath product of its local and global components. Under the identification of S and $R = \overleftarrow{T \times V}$, the convex G-congruence Q is described by $(t_1, v_1)Q(t_2, v_2)$ if and only if $v_1 = v_2$. Thus Q is also a convex W-congruence. Moreover, the local and global components of (W, R) are just (H, T) and (K, V), respectively. Roughly speaking, the wreath product of the Q-components of G is gotten by adding to G as many o-permutations of S as possible without changing the Q-components.

In the following sections, we shall develop a theory which is more general than the one discussed here in that instead of a single G-congruence Q on S, we shall have a more or less arbitrary collection of convex G-congruences on S, producing components which are neither local nor global, but somewhere in between.

3. Construction of the general ordered wreath product. What we define here is a special case of the general wreath product defined for (nonordered) permutation groups in [8]. Let Γ be a chain, and for each $\gamma \in \Gamma$, let S_γ be a chain and (G_γ, S_γ) a transitive o-permutation group. Let $\bar{R} = \prod_{\gamma \in \Gamma} S_\gamma$ be the cartesian product, and choose any point $O \in \bar{R}$. Thinking of the elements of $\bar{R}$ as functions on Γ, we define the *support* of $v \in \bar{R}$ to be $\{\gamma \in \Gamma \mid r(\gamma) \neq 0(\gamma)\}$. Now let $R = \{r \in \bar{R} \mid r$ has inversely well ordered support$\}$, so that for $r, s \in R$, $\{\gamma \in \Gamma \mid r(\gamma) \neq s(\gamma)\}$ is inversely well ordered. For $r, s \in R$, $r \neq s$, we define $r < s$ if and only if $r(\alpha) < s(\alpha)$ in S_α, where α is the largest element of the set $\{\gamma \in \Gamma \mid r(\gamma) \neq s(\gamma)\}$. This makes R a chain.

For each $\gamma \in \Gamma$, we define an equivalence relation K_γ on R by setting $rK_\gamma s$ if and only if $r(\alpha) = s(\alpha)$ for all $\alpha \geqq \gamma$. Let $K^\gamma = \bigcap_{\beta > \gamma} K_\beta$ (and $K^\gamma = R \times R$ if there is no $\beta > \gamma$). Equivalently, $rK^\gamma s$ if and only if $r(\alpha) = s(\alpha)$ for all $\alpha > \gamma$. All K_γ-classes and all K^γ-classes are convex.

Let $P(R)$ denote the set of all (not necessarily order-preserving) permutations of R. Let $W' = \{f \in P(R) \mid K_\gamma$ is a (convex) $\{f, f^{-1}\}$-congruence for each $\gamma \in \Gamma\}$, a subgroup of $P(R)$. Then for every $f \in W'$ and $\gamma \in \Gamma$, K^γ is also a (convex) $\{f\}$-congruence. Clearly K^γ contains K_γ and the chain $(OK^\gamma)/K_\gamma$ (ordered in the natural way) is o-isomorphic to the chain S_γ.

Let $g \in W'$, $\gamma \in \Gamma$, and $x \in R$. g induces a permutation $g_{\gamma,x}$ of S_γ, defined in the following way. If $a \in S_\gamma$, there exists $x' \in R$ such that $x'K^\gamma x$ and $x'(\gamma) = a$. (We need only alter the γ^{th} component of x to obtain x'.) Now define $ag_{\gamma,x} = (x'g)(\gamma)$. $g_{\gamma,x}$ is well defined, for if $x''K^\gamma x$ and $x''(\gamma) = a$, then $x''K_\gamma x'$, whence $(x''g)K_\gamma(x'g)$, and in particular, $(x''g)(\gamma) = (x'g)(\gamma)$. Thus we have the rules

(1) $(xg)(\gamma) = (x(\gamma))g_{\gamma,x}$, and

(2) $g_{\gamma,x} = g_{\gamma,y}$ if $xK^\gamma y$.

Of course (2) states that $g_{\gamma,x}$ really depends only on γ and the congruence class xK^γ.

It is easy to see [8] that $g_{\gamma,x}$ is a permutation of S_γ and that $g \in W'$ is determined by the matrix $\{g_{\gamma,x} \mid \gamma \in \Gamma, x \in R\}$ of its components.

THEOREM 1. *Let $g \in W'$. Then g preserves order if and only if all $g_{\gamma,x}$'s preserve order.*

Proof. First, suppose that $g \in A(R)$. Let $a, b \in S_\gamma$ with $a < b$ and let $x'K^\gamma xK^\gamma x''$ with $x'(\gamma) = a$ and $x''(\gamma) = b$. It follows that $x' < x''$, so that $x'g < x''g$. But as $(x'g)K^\gamma(x''g)$, we must have $(x'g)(\gamma) < (x''g)(\gamma)$, i.e., $ag_{\gamma,x} < bg_{\gamma,x}$. Therefore $g_{\gamma,x} \in A(S_\gamma)$.

Conversely, suppose that g is a permutation in W' such that for each $\gamma \in \Gamma$, $x \in R$, $g_{\gamma,x} \in A(S_\gamma)$. If $x, y \in R$ with $x < y$, let γ be the largest element of Γ such that $x(\gamma) \neq y(\gamma)$. Then $x(\gamma) < y(\gamma)$; and $xK^\gamma y$, so $(xg)K^\gamma(yg)$. Now $(xg)(\gamma) = (x(\gamma))g_{\gamma,x} < (y(\gamma))g_{\gamma,x} = (y(\gamma))g_{\gamma,y} = (yg)(\gamma)$. Hence $xg < yg$.

If $\{(G_\gamma, S_\gamma)\}$ is a collection of o-permutation groups indexed by a chain Γ, we define the *ordered wreath product* ${}^*\prod_{\gamma\in\Gamma}(G_\gamma, S_\gamma)$ to be (W, R), where R is as specified above and W is the set of all permutations $g \in W'$ such that $g_{\gamma,x} \in G_\gamma$ for all $\gamma \in \Gamma$, $x \in R$.

The ordered wreath product we have defined here is the same as the general wreath product defined in [8]. In [8] it is shown that the general wreath product is a subgroup of $P(R)$; that it is transitive; and that, to within o-isomorphism, it is independent of which element 0 is chosen from R'. By Theorem 1, each $g \in W$ preserves order. Thus we have

THEOREM 2. *The ordered wreath product* $(W, R) = {}^*\prod_{\gamma\in\Gamma}(G_\gamma, S_\gamma)$

of transitive ordered permutation groups (G_γ, S_γ), Γ *a chain, is a transitive ordered permutation group.*

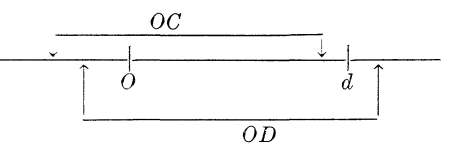
In case Γ is a two element chain, say $0 < 1$, then $R = \bar{R} = S_0 \times S_1$, and the ordered wreath product we have constructed is the same as the one developed by Lloyd and Reilly that was described in §2.

4. The o-primitive components of an ordered permutation group. Let $\gamma = (Q_\gamma, Q^\gamma)$ be a pair of convex G-congruences of a transitive o-permutation group (G, S), with Q_γ properly contained in Q^γ. Let O be any point in S. Let S_γ be the chain $(OQ^\gamma)/Q_\gamma$ and let $G_{OQ^\gamma} = \{g \in G \mid (OQ^\gamma)g = OQ^\gamma\}$. Let G_γ be the image in $A(S_\gamma)$ of G_{OQ^γ} under the natural o-homomorphism $g \to \bar{g}$, where $(xQ_\gamma)\bar{g} = (xg)Q_\gamma$. Then (G_γ, S_γ) is called the γ-*component* of (G, S). It is an easy consequence of transitivity that, to within o-isomorphism, (G_γ, S_γ) is independent of the choice of $O \in S$.

The following argument from [11] generalizes a theorem from [7].

THEOREM 3. *If* (G, S) *is a transitive ordered permutation group, the set of all convex* G*-congruences is a chain under inclusion.*

Proof. Since G is transitive, it suffices to pick any $O \in S$ and to show that for any convex G-congruences C and D, either $OC \subseteqq OD$ or $OD \subseteqq OC$. Suppose that neither of these two *convex* sets is contained in the other. By transitivity, we may pick $g \in G$ such that $Og = d \in OD \backslash OC$. Since Og lies in the G-congruence class OD, $(OD)g = OD$. But $OC \cap OD$ is either a left ray of OD(i.e., $s \in OC \cap OD$, $t \in OD$, and $t \leqq s$ implies $t \in OC \cap OD$) or a right ray of OD. Since g preserves order, $(OC)g$ meets OC and thus $(OC)g = OC$, contradicting the assumption that $Og = d \notin OC$.



We denote the chain of convex G-congruences by $\Delta = \Delta(G, S)$. An ordered pair (s, t) of elements of a chain T is a *covering pair* if $s < t$ and if there is no $u \in T$ such that $s < u < t$; and we order the set of covering pairs by $(s, t) < (u, v)$ if $t \leqq u$. We denote by $\Gamma = \Gamma(G, S)$ the chain of covering pairs of Δ. We shall find in §8 that every chain occurs as $\Gamma(G, S)$ for some o-permutation group (G, S). If we denote the γ^{th} covering pair by $\gamma = (C_\gamma, C^\gamma)$, then $C^\gamma = C_\eta$ if and only if (γ, η) is a covering pair of elements of Γ. Also, considering a G-congruence as a subset of $S \times S$, we see that Δ is closed under arbitrary intersection and union. Thus Δ is obtained from Γ by replacing by a pair of points any point $\gamma \in \Gamma$ which is not the

first half of a covering pair of elements of Γ, and then forming the completion by Dedekind cuts (with end points). The chains which occur as $\Delta(G, S)$ for some (G, S) are those obtained in this way from an arbitrary chain Γ.

A transitive o-permutation group (G, S) is said to be *o-primitive* if the only convex G-congruences are the two trivial ones. O-primitivity is a quite strong condition, and a great deal of information about o-primitive permutation groups is contained in [10] and [11]. Our aim in the rest of this paper is to show how an arbitrary transitive o-permutation group can be built up from o-primitive ones.

If C and K are convex G-congruences with $C \subset K$, it is easy to check that K covers C if and only if the (C, K)-component (G_γ, S_γ) of (G, S) is o-primitive. Thus the *o-primitive components* (G_γ, S_γ) of (G, S) are precisely those components arising from covering pairs $\gamma = (C_\gamma, C^\gamma)$ of convex G-congruences.

THEOREM 4. *Let (G, S) be an ordered permutation group with o-primitive components (G_γ, S_γ) arising from covering pairs (C_γ, C^γ) of convex G-congruences. Then every convex G-congruence D is equal to $\cap \{C_\gamma \mid D \subseteqq C_\gamma\}$.*

Proof. Since the chain Δ of all convex G-congruences is the Dedekind completion of the subchain consisting of those convex G-congruences which are (either first or second) members of covering pairs of Δ, we are finished unless $C_\gamma \subset D \subset C^\gamma$ for some γ; and this is impossible because (C_γ, C^γ) is a covering pair.

An o-permutation group (G_γ, S_γ) is *trivial* if S_γ contains only one point. We now show that if the (G_γ, S_γ)'s are all o-primitive and nontrivial, then Γ (W, R) and the o-primitive components of (W, R) are what they ought to be.

THEOREM 5. *Let $(W, R) = {}^*\prod_{\gamma \in \Gamma} (G_\gamma, S_\gamma)$ be the ordered wreath product of nontrivial o-primitive ordered permutation groups (G_γ, S_γ), Γ a chain. Then the chain Γ (W, R) of all covering pairs of convex W-congruences is just $\{(K_\gamma, K^\gamma) \mid \gamma \in \Gamma\}$, and the correspondence $(K_\gamma, K^\gamma) \leftrightarrow \gamma$ is an o-isomorphism between Γ (W, R) and Γ. The (K_γ, K^γ)-component of (W, R) is just (G_γ, S_γ).*

Proof. Always $K_\gamma \subseteqq K^\gamma$. Since (G_γ, S_γ) is nontrivial, $K_\gamma \subset K^\gamma$. Since (G_γ, S_γ) is o-primitive, (K_γ, K^γ) is a covering pair of $\Delta(W, R)$. Now let (C, D) be any covering pair of $\Delta(W, R)$. Pick $x, y \in R$ such that $(x, y) \in D \backslash C$ and let η be the greatest element of $\{\gamma \in \Gamma \mid x(\gamma) \neq y(\gamma)\}$. Then $(x, y) \in K^\eta \backslash K_\eta$. Since (C, D) and (K_η, K^η) are both covering pairs, $(C, D) = (K_\eta, K^\eta)$. The rest is clear.

COROLLARY 6. *Under the hypotheses of Theorem* 5, *every convex W-congruence D is equal to* $\cap \{K_\gamma \mid D \subseteqq K_\gamma\}$.

5. **Completeness of the ordered wreath product.** The ordered wreath product (W, R) has two completeness properties which we may describe as "completeness of the set R" and "completeness of the group W". Although the intersection of a collection of convex congruences is always a convex congruence, it may happen in an o-permutation group (G, S) that there is a tower of convex G-congruence classes (from different congruences) whose intersection is empty. An example is given in §8. The tower will be said to be *complete* if its intersection is *not empty*. If all towers are complete, then S (actually S together with the set of all G-congruences) is termed *complete*. By Theorem 4, it suffices to consider towers $\{x_\lambda C_\lambda : \lambda \in \Lambda\}$ in which every congruence is the first half of a covering pair and in which Λ is a right ray of Γ. Henceforth we shall consider only towers of this sort. We note that if Γ is well ordered, S is automatically complete.

THEOREM 7. *Let* $(W, R) = {}^*\prod_{\gamma \in \Gamma} (G_\gamma, S_\gamma)$ *be the ordered wreath product of o-primitive permutation groups* (G_γ, S_γ). *Then R is complete.*

Proof. Let $\{x_\lambda K_\lambda : \lambda \in \Lambda\}$ be a tower of the prescribed kind. (We may use K_γ's because by Theorem 5, the (K_γ, K^γ)'s constitute *all* covering pairs in $\Delta(W, R)$). Define y by setting $y(\gamma) = x_\gamma(\gamma)$ if $\gamma \in \Lambda$; and $y(\gamma) = 0(\gamma)$ otherwise. If $\lambda \in \Lambda$, then $y(\eta) = x_\lambda(\eta)$ for all $\eta \geqq \lambda$ because $y(\eta) = x_\eta(\eta)$ and $x_\lambda \in x_\eta K_\eta$. It follows that y has inversely well ordered support, for if Σ is contained in the support of y and $\sigma \in \Sigma$, then $x_\sigma(\tau) = y(\tau)$ for all $\tau \in \Sigma$ such that $\tau \geqq \sigma$, so Σ has a greatest element. Since $y(\eta) = x_\lambda(\eta)$ for $\eta \geqq \lambda$, $y \in x_\lambda K_\lambda$ for all $\lambda \in \Lambda$.

Regarding the completeness of the group, it can easily happen that an o-permutation group (G, S) is a proper subgroup of an o-permutation group (H, S) on the same set S, that each convex G-congruence is also an H-congruence, and yet that the o-primitive components of H are no larger than those of G. (It is clear that the γ-component (G_γ, S_γ) is a subgroup of the γ-component (H_γ, S_γ)). If an o-permutation group (G, S) is not properly contained in this manner in any (H, S), we shall say that the group G is *complete*. In view of Theorem 5, the definition of ordered wreath product yields.

THEOREM 8. *Let* $(W, R) = {}^*\prod_{\gamma \in \Gamma} (G_\gamma, S_\gamma)$ *be the ordered wreath product of o-primitive permutation groups* (G_γ, S_γ). *Then W is complete.*

Let (G, S) and (H, T) be transitive o-permutation groups on different sets. We wish to make precise the idea of (G, S) being contained in (H, T) and having the same o-primitive components as (H, T). Accordingly, we shall say that (H, T) is a *c-extension* of (G, S) if the following hold

(i) S is a subchain of T.

(ii) G is the faithful restriction to S of some subgroup $\bar{G}$ of H such that $S\bar{g} = S$ for each $\bar{g} \in \bar{G}$.

(iii) Restriction to S is an isomorphism (and thus an o-isomorphism) from $\Gamma(H, T)$ onto $\Gamma(G, S)$. We note that (iii) holds if and only if restriction to S is an isomorphism from $\Delta(H, T)$ onto $\Delta(G, S)$, which holds if and only if every convex G-congruence on S is the restriction to S of a unique convex H-congruence on T.

(iv) If $(D_\gamma, D^\gamma) \in \Gamma(H, T)$, and if $s \in S$, then every D_γ-class contained in sD^γ meets S. (iv) provides a natural o-isomorphism from S_γ onto T_γ.

(v) If $(D_\gamma, D^\gamma) \in \Gamma(H, T)$, and if $s \in S$, then for every $h \in H$ such that $(sD^\gamma)h = sD^\gamma$, there exists $\bar{g} \in \bar{G}$ such that $E\bar{g} = Eh$ for every D_γ-class E contained in sD^γ. (v) provides a natural o-isomorphism from G_γ onto H_γ [8].

The concept of *immediate extension* in [8] specializes for *ordered* wreath products to the present concept of c-extension. It is easy to check that a c-extension of a c-extension of (G, S) is itself a c-extension of (G, S). We shall call (G, S) *c-closed* if it has no proper c-extensions.

THEOREM 9. *Let (G, S) be a transitive ordered permutation group such that S and G are both complete. Then (G, S) is c-closed.*

Proof. Suppose that (H, T) is a c-extension of (G, S). Suppose there exists $t \in T \backslash S$. Let M be the intersection of all H-congruence classes tB, $B \in \Delta(H, T)$, such that tB meets S. Then $M \cap S = \cap (tB \cap S)$ is not empty because S is complete. Let D be the convex H-congruence of which M is a congruence class. Pick $s \in M \cap S$. Let C be the union of all convex H-congruences E such that $sE \neq tE$. Then C is a convex H-congruence, and D covers C. By (iv), every C-class contained in the D-class M must meet S. In particular, tC must meet S, contradicting the fact that $C \subset D$. Therefore $S = T$. Since G is complete, $G = H$.

In the next section, we shall prove the converse of Theorem 9.

COROLLARY 10. *Let $(W, R) = {}^*\prod_{\gamma \in \Gamma} (G_\gamma, S_\gamma)$ be the ordered wreath product of o-primitive permutation groups (G_γ, S_γ). Then (W, R) is c-closed.*

6. **The embedding theorem.** It follows from results in [8] that if (G, S) is a transitive o-permutation group with o-primitive components (G_γ, S_γ), $\gamma \in \Gamma$, then (G, S) can be embedded as a permutation group in the general wreath product of the groups (G_γ, S_γ). Here we show that the embedding is into the ordered wreath product and preserves order. To do so requires that we carefully describe the embedding.

Pick any $x_0 \in S$. Our first task is to embed the chain S in R, the set of those elements of $\prod_{\gamma\in\Gamma} S_\gamma$ with inversely well ordered support. The embedding will send x_0 onto 0.

LEMMA 11 [8]. *If H is any group, there exists a set $\{T(A) \mid A$ a subgroup of $H\}$ such that $T(A)$ is a set consisting of exactly one element from each right coset Ah of A in H, $T(A) \cap A$ is the identity, and if $A \subseteqq B$, then $T(A) \supseteqq T(B)$.*

Now pick any $\gamma \in \Gamma$ and let (G_γ, S_γ) be the o-primitive component of (G, S) determined by $\gamma = (C_\gamma, C^\gamma)$. Let $G_{x_0C^\gamma} = \{g \in G \mid (x_0C^\gamma)g = x_0C^\gamma\}$. The function $\varphi: S \to \prod_{\gamma\in\Gamma} S_\gamma$ is defined in the following way. Let $s \in S$. There exists exactly one $g_s \in T(G_{x_0C^\gamma})$ such that $(x_0g_s)C^\gamma s$. Let $(s\varphi)(\gamma) = sg_s^{-1}C_\gamma \in S_\gamma = (x_0C^\gamma)/C_\gamma$. In [8] it is shown that φ is a one-to-one map of S into R and that $(C_\gamma, C^\gamma)\varphi = (K_\gamma, K^\gamma) \mid S\varphi$, where (K_γ, K^γ) is the γ^{th} covering pair of convex W-congruences (Theorem 5).

We show here that φ preserves order (and in the process prove again that φ is one-to-one). If $x < y$ are elements of S, let B be the intersection of all convex G-congruences E such that xEy, and let A be the union of all convex G-congruences F such that $xF \neq yF$. Then (A, B) is a covering pair (C_γ, C^γ) of convex G-congruences. Since $xC^\gamma y$, $g_x = g_y$. Thus $(x\varphi)(\gamma) = xg_x^{-1}C_\gamma < yg_x^{-1}C_\gamma = (y\varphi)(\gamma)$. However, for all $\beta > \gamma$, $C^\gamma \subseteqq C_\beta$, so that $xC_\beta y$, and hence $(x\varphi)(\beta) = (y\varphi)(\beta)$. Therefore $x\varphi < y\varphi$. Thus the chain S is o-embedded in the chain R.

Next we define an embedding (also denoted by φ) of G in W. Let $g \in G$, $\gamma \in \Gamma$, and $x \in R$. g induces an o-permutation of S_γ, which is formally denoted by $(g\varphi)_{\gamma,x}$, as follows. If xK^γ meets $S\varphi$, then for each $a \in S_\gamma$, there exists $s \in S$ such that $s\varphi \in xK^\gamma$ and $(s\varphi)(\gamma) = a$. Now define $a(g\varphi)_{\gamma,x} = ((sg)\varphi)(\gamma)$. If xK^γ does not meet $S\varphi$, define $(g\varphi)_{\gamma,x}$ to be the identity 1_γ on S_γ. In [8] it is shown that $(g\varphi)_{\gamma,x}$ is well defined and is a permutation of S_γ. Here we show it preserves order. Suppose we have $a, b \in S_\gamma$ with $a < b$, and $s, t \in S$ with $(s\varphi)K^\gamma xK^\gamma(t\varphi)$ and $(s\varphi)(\gamma) = a$, $(t\varphi)(\gamma) = b$. Then $s\varphi < t\varphi$, so $s < t$, so $sg < tg$, so $(sg)\varphi < (tg)\varphi$. Since $(s\varphi)K^\gamma(t\varphi)$, we have $sC^\gamma t$, so $(sg)C^\gamma(tg)$, and thus $((sg)\varphi)K^\gamma((tg)\varphi)$. Since also $((sg)\varphi)K_\gamma \neq ((tg)\varphi)K_\gamma$, we have $a(g\varphi)_{\gamma,x} = ((sg)\varphi)(\gamma) < ((tg)\varphi)(\gamma) = b(g\varphi)_{\gamma,x}$. Thus $(g\varphi)_{\gamma,x}$ pre-

serves order.

Now $g\varphi$ is defined as a function on R by setting $(r(g\varphi))(\gamma) = (r(\gamma))(g\varphi)_{\gamma,r}$. It is proved in [8] that $g\varphi$ is a permutation of R, and moreover is contained in the general wreath product; and that its component $(g\varphi)_{\gamma,x}$ defined in §3 coincides with the permutation which is formally denoted above by $(g\varphi)_{\gamma,x}$. By Theorem 1, $g\varphi$ preserves order on R. It was also shown in [8] that φ is an embedding of the group G in W satisfying $(sg)\varphi = (s\varphi)(g\varphi)$ for all $s \in S$, $g \in G$. Thus g is a positive o-permutation of S if and only if $g\varphi$ is a positive o-permutation of $S\varphi$.

To show that $\varphi: G \to W$ is an o-embedding, it remains to show that if $g\varphi$ is positive on $S\varphi$, it is also positive on R. Thus suppose that $g\varphi$ is positive on $S\varphi$ and let $r \in R$. Let γ be the largest element of Γ such that $r(\gamma) \neq (r(g\varphi))(\gamma)$. We may assume that rK^γ meets $S\varphi$ (otherwise $(r(g\varphi))(\gamma) = r(\gamma)(g\varphi)_{\gamma,r} = r(\gamma)1_\gamma = r(\gamma)$) and thus that there exists $s\varphi \in rK_\gamma$. Since $g\varphi$ is positive on $S\varphi$ and since $((s\varphi)K^\gamma)(g\varphi) = (s\varphi)K^\gamma$, we have $r(\gamma) = (s\varphi)(\gamma) < ((s\varphi)(g\varphi))(\gamma) = (r(g\varphi))(\gamma)$. Therefore $g\varphi$ is positive on R, and thus $\varphi: G \to W$ is an o-embedding. It follows easily (and is essentially included in [8]) that (W, R) is a c-extension of $(G\varphi, S\varphi)$. We recapitulate the foregoing discussion in the following theorem.

THEOREM 12. *Let (G, S) be a transitive ordered permutation group with o-primitive components (G_γ, S_γ), $\gamma \in \Gamma$. Let (W, R) be the ordered wreath product ${}^*\prod_{\gamma\in\Gamma}(G_\gamma, S_\gamma)$. Then there is an embedding $\varphi: (G, S) \to (W, R)$ of the following sort. First, $\varphi: S \to R$ is an o-embedding, and $(C_\gamma, C^\gamma)\varphi = (K_\gamma, K^\gamma) \mid S\varphi$. Next, $\varphi: G \to W$ is an o-embedding of the partially ordered group G in the partially ordered group W. Finally, $(s\varphi)(g\varphi) = (sg)\varphi$ for all $s \in S$, $g \in C$; and (W, R) is a (c-closed) c-extension of $(G\varphi, S\varphi)$.*

COROLLARY 13. *Let (G, S) be a transitive ordered permutation group. Then the following are equivalent:*

(1) *(G, S) is o-isomorphic to the ordered wreath product (W, R) of its o-primitive components.*

(2) *(G, S) is c-closed.*

(3) *S and G are both complete.*

Proof. (1) implies (3) by Theorems 7 and 8. (3) implies (2) by Theorem 9. (2) implies (1) by Theorem 12.

COROLLARY 14. *Let (G, S) be a transitive ordered permutation group and $(W, R) = {}^*\prod_{\gamma\in\Gamma}(G_\gamma, S_\gamma)$ the ordered wreath product of its*

o-primitive components. Then any other c-closed c-extension (H, T) *of* (G, S) *is o-isomorphic to* (W, R).

Proof. Since (H, T) is a c-extension of (G, S), the ordered wreath product of *its* o-primitive components is also (W, R). Since (H, T) is c-closed, (H, T) is o-isomorphic to (W, R).

We now identify (G, S) with $(G\varphi, S\varphi)$ and describe the way in which S has been filled out to obtain R. Let $\{x_\lambda C_\lambda \mid \lambda \in \Lambda\}$ be an *incomplete* tower of convex G-congruence classes in S. By Theorem 7, $M = \cap \{x_\lambda K_\lambda \mid \lambda \in \Lambda\} \subseteq R \backslash S$ is not empty. Thus M is a convex congruence class of the W-congruence $\cap \{K_\lambda \mid \lambda \in \Lambda\}$; and consists of more than one point if and only if $\Lambda \neq \Gamma$. M has been inserted in the Dedekind cut of S which corresponds to the left ray $\{s \in S \mid$ there exists $\lambda = \lambda_s$ with $s < y$ for all $y \in x_\lambda C_\lambda\}$. Conversely, every $r \in R \backslash S$ arises in this way from the incomplete tower $\{x_\lambda C_\lambda \mid \lambda \in \Lambda\}$ defined by $x_\lambda \in rK_\lambda \cap S$ if rK_λ meets S; and x_λ undefined otherwise. Clearly $x_\lambda C_\lambda$ depends only on λ; and $\cap \{x_\lambda C_\lambda \mid \lambda \in \Lambda\}$ is empty, for if $s \in \cap \{x_\lambda C_\lambda\}$, then rK_γ meets S for some $\gamma \notin \Lambda$.

In particular, $S = R$ if and only if S is complete (as we had essentially shown already); and S is order dense in R if and only if for every incomplete tower $\{x_\lambda C_\lambda : \lambda \in \Lambda\}$, $\Lambda = \Gamma$. Also, it is not hard to show that for $x \in S$, xK^γ is the convex hull of xC^γ, i.e., for any $y \in xK^\gamma$, there exist $u, v \in xC^\gamma$ such that $u \leqq y \leqq v$.

7. **Lattice-ordered permutation groups.** If (G_γ, S_γ) is an ordered permutation group, so that G_γ is a subgroup of $A(S_\gamma)$, then if G_γ is closed under the lattice operations, G_γ is a lattice-ordered group (l-group). In this case we shall call (G_γ, S_γ) a *lattice-ordered permutation group.*

THEOREM 15. *If* (G, S) *is a transitive lattice-ordered permutation group and* $\gamma = (Q_\gamma, Q^\gamma)$ *is a pair of convex G-congruences with* $Q_\gamma \subset Q^\gamma$, *then the component* (G_γ, S_γ) *is also a lattice-ordered permutation group.*

Proof. G_{oQ^γ} is an l-subgroup of G; and the map $g \to \bar{g}$ from G_{oQ^γ} into $A(S_\gamma)$ is an l-homomorphism since

$$(xQ_\gamma)\overline{(g \vee 1)} = (x(g \vee 1))Q_\gamma = (xg \vee x)Q_\gamma = (xg)Q_\gamma \vee xQ_\gamma = (xQ_\gamma)(\bar{g} \vee \bar{1}) .$$

Thus the image G_γ is an l-subgroup of $A(S_\gamma)$.

THEOREM 16 [10]. *If each* (G_γ, S_γ) *is a transitive lattice-ordered permutation group, then the ordered wreath product* $(W, R) = {}^*\prod_{\gamma \in \Gamma} (G_\gamma, S_\gamma)$ *is also a (transitive) lattice-ordered permutation group, and conversely.*

Proof. The converse follows from the previous theorem. Suppose now that each (G_γ, S_γ) is lattice-ordered. Let $w \in W$ have as components $\{w_{\gamma,x}\}$. Define z by setting $z_{\gamma,x}$ equal to

(i) $w_{\gamma,x}$ if $(xw)K^\gamma > xK^\gamma$,

(ii) 1_γ if $(xw)K^\gamma < xK^\gamma$,

(iii) $w_{\gamma,x} \vee 1_\gamma$ if $(xw)K^\gamma = xK^\gamma$ (i.e., if $(xw)K^\gamma x$). $z_{\gamma,x}$ depends only on xK^γ, for if also $y \in xK^\gamma$, then $xK^\gamma y$, so $(xw)K^\gamma(yw)$, and thus the same one of the three cases applies. If δ is the greatest element of Γ at which sw differs from s, then $sz = sw$ if $(sw)(\delta) > s(\delta)$, while $sz = s$ if $(sw)(\delta) < s(\delta)$, so that $z = w \vee 1$.

THEOREM 17. *Let (G, S) be a transitive lattice-ordered permutation group. Then the embedding $\varphi: G \to W$ described in Theorem 12 is an l-isomorphism.*

Proof. Since $\varphi: G \to G\varphi$ is an o-isomorphism, it is an l-isomorphism, and thus for all $s \in S$, $(s\varphi)((g \vee 1)\varphi) = (s\varphi)(g\varphi) \vee s\varphi$. Now let $r \in R$, $\gamma \in \Gamma$. If there exists $s \in S$ such that $(s\varphi)K_\gamma r$, then

$$[r((g \vee 1)\varphi)](\gamma) = [s\varphi((g \vee 1)\varphi)](\gamma) = [(s\varphi)(g\varphi) \vee s\varphi](\gamma)$$
$$= [r(g\varphi) \vee r](\gamma) = [r(g\varphi \vee 1)](\gamma) .$$

If, on the other hand, there is no such s, then $[r((g \vee 1)\varphi)](\gamma) = r(\gamma) = [r(g\varphi \vee 1)](\gamma)$. Therfore $r((g \vee 1)\varphi) = r(g\varphi \vee 1)$, so that φ is an l-isomorphism.

If $(G, S) = (A(S), S)$, the situation is even nicer. The o-primitive component (G_γ, S_γ) is $(A(S_\gamma), S_\gamma)$. It is shown in [9] that if $(A(T), T)$ is o-primitive, then it is either o-2-transitive (i.e., for $t_1 < t_2$ and $u_1 < u_2$, there exists $g \in A(T)$ such that $t_1 g = u_1$ and $t_2 g = u_2$) or the regular representation of a rather special subgroup of the reals. Also, if S is complete, the embedding of Theorem 12 maps $A(S)$ onto W (as well as mapping S onto R).

8. **Totally ordered groups.** If G is a totally ordered group, then the right regular representation (G, G) is an o-permutation group. A convex G-congruence amounts to a partition of G by the right cosets Lg of some convex subgroup L of G; and the ordering of the convex G-congruences corresponds to the ordering by inclusion of the convex subgroups of G.

Thus an o-primitive component of (G, G) arises from a covering pair (C_γ, C^γ) of convex subgroups of G. Hence in the o-primitive component $(G_\gamma S_\gamma)$, S_γ is just the chain C^γ/C_γ. It is well kown [3] that C_γ is normal in C^γ and that the quotient group C^γ/C_γ is o-isomorphic to a subgroup of the real numbers. The kernel of the natural

map $g \to \bar{g}$ from the group C^γ into $A(S_\gamma)$ is $\{g \in C^\gamma \mid (C_\gamma x)g = C_\gamma x$ for all $x \in C^\gamma\}$, which is just C_γ since C_γ is normal in C^γ. Hence G_γ is o-isomorphic to the o-group C^γ/C_γ, a subgroup of the reals. Thus (G_γ, S_γ) is just the regular representation of a subgroup of the reals. Using only the group embedding $\varphi: G \to W$ of Theorem 12, we have an analogue of Hahn's theorem.

THEOREM 18. *If G is a totally ordered group, then G can be o-embedded in an ordered wreath product of subgroups of the real numbers.*

As mentioned before, this theorem is somewhat unsatisfactory since the wreath product is not totally ordered, but by Theorem 16, it is at least lattice-ordered.

We mention without proof that if the transversals of Lemma 11 can be chosen so that for each $\gamma \in \Gamma$, G is the direct sum of $G_{x_0 C^\gamma}$ and $T(C_{x_0 C^\gamma})$, then the embedding φ can be chosen so that for each $g \in G$, $\gamma \in \Gamma$, $(g\varphi)_{\gamma,x} = (g\varphi)_{\gamma,y}$ for all $x, y \in S$, i.e., all of the γ-components of g are the same. In this case, φ embeds G in the (unrestricted) *direct* product $\prod_{\gamma \in \Gamma} G_\gamma$. According to the lemma of Banaschewski [1], this hypothesis is satisfied if G is a divisible totally ordered abelian group. Thus Hahn's theorem is a corollary of the theory developed here. However, a proof of Hahn's theorem along these lines is essentially the same as the proof in [1].

Hahn groups provide several examples which help to clarify the theory of ordered wreath products. Let Γ be an arbitrary chain and let G be a Hahn group on Γ. Then $\Gamma(G, G) = \Gamma$, so every chain Γ occurs as $\Gamma(G, S)$ for some o-permutation group (G, S). Let H be a *restricted* Hahn group on Γ (containing only elements having finite support). If Γ is the chain N of negative integers, (H, H) has incomplete towers, but at least every incomplete tower has $\Lambda = \Gamma$. If Γ is the ordinal sum $N + N$, (H, H) has imcomplete towers with $\Lambda \neq \Gamma$.

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