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COEFFICIENT MULTIPLIERS OF H^p AND B^p SPACES

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COEFFICIENT MULTIPLIERS OF H^p AND B^p SPACES

P. L. DUREN AND A. L. SHIELDS

This paper describes the coefficient multipliers of $H^p(0 into <math>\mathcal{Z}^q(p \leq q \leq \infty)$ and into $H^q(1 \leq q \leq \infty)$. These multipliers are found to coincide with those of the larger space B^p into $\mathcal{Z}^q(1 \leq q \leq \infty)$ and into $H^q(1 \leq q \leq \infty)$. The multipliers of H^p and B^v into $B^q(0 are also characterized.$

A function f analytic in the unit disk is said to be of class $H^p(0 if$

$$M_{p}(r,f) = \left\{ rac{1}{2\pi} \int_{0}^{2\pi} |\, f(re^{i heta})\,|^{p} \; d heta
ight\}^{1/p}$$

remains bounded as $r \to 1$. H^{∞} is the space of all bounded analytic functions. It was recently found ([2], [4]) that if p < 1, various properties of H^p extend to the larger space B^p consisting of all analytic functions f such that

$$\int_{0}^{1} (1-r)^{1/p-2} \, M_{1}(r,f) dr < \infty$$
 .

Hardy and Littlewood [8] showed that $H^p \subset B^p$.

A complex sequence $\{\lambda_n\}$ is called a *multiplier* of a sequence space A into a sequence space B if $\{\lambda_n a_n\} \in B$ whenever $\{a_n\} \in A$. Aspace of analytic functions can be regarded as a sequence space by identifying each function with its sequence of Taylor coefficients. In [4] we identified the multipliers of H^p and $B^p(0 into <math>\mathcal{L}^1$. We have also shown ([2], Th. 5) that the sequence $\{n^{1/q-1/p}\}$ multiplies B^p into B^{q} . We now extend these results by describing the multipliers of $H^p(0 into <math>\ell^q(p \leq q \leq \infty)$, of B^p into $\ell^q(1 \leq q \leq \infty)$, and of both H^p and B^p into $B^q(0 < q < 1)$. We also extend a theorem of Hardy and Littlewood (whose proof was never published) by characterizing the multipliers of H^p and B^p into $H^q(0 .$ In almost every case considered, the multipliers of B^{p} into a given space are the same as those of H^{p} .

2. Multipliers into \swarrow^q . We begin by describing the multipliers of H^p and B^p into \swarrow^{∞} , the space of bounded complex sequences.

THEOREM 1. For $0 , a sequence <math>\{\lambda_n\}$ is a multiplier of H^p into \swarrow^{∞} if and only if

$$\lambda_n = O(n^{1-1/p}) \cdot$$

For p < 1, the condition (1) also characterizes the multipliers of B^p into \swarrow^{∞} .

Proof. If
$$f(z) = \sum a_n z^n$$
 is in B^p , then by Theorem 4 of [2],

(2)
$$a_n = o(n^{1/p-1})$$
.

If $f \in H^1$, then $a_n \to 0$ by the Riemann-Lebesgue lemma. This proves the sufficiency of (1). Conversely, suppose $\{\lambda_n\}$ is a multiplier of H^p into \checkmark^{∞} . Then the closed linear operator

$$\Lambda : f \longrightarrow \{\lambda_n a_n\}$$

maps H^p into \swarrow^{∞} . Thus Λ is bounded, by the closed graph theorem (which applies since H^p is a complete metric space with translation invariant metric; see [1], Chapter 2). In other words,

(3)
$$\sup |\lambda_n a_n| = ||A(f)|| \leq K ||f||.$$

Now let

$$g(z) = (1 - z)^{-1 - 1/p} = \sum b_n z^n$$
 ,

where $b^n \sim Bn^{1/p}$; and choose f(z) = g(rz) for fixed r < 1. Then by (3)

 $|\lambda_n| \ n^{1/p} r^n \leq C (1-r)^{-1}$.

The choice r = 1 - 1/n now gives (1). Note that $\{\lambda_n\}$ multiplies H^p or B^p into \swarrow^{∞} if and only if it multiplies into c_0 (the sequences tending to zero).

As a corollary we may show that the estimate (2) is best possible in a rather strong sense. For functions of class H^p , this estimate is due to Hardy and Littlewood [8]. Evgrafov [6] later showed that if $\{\delta_n\}$ tends monotonically to zero, then there is an $f \in H^p$ for which $a_n \neq O(\delta_n n^{1/p-1})$. A simpler proof was given in [5]. The result may be reformulated: if $a_n = O(d_n)$ for all $f \in H^p$, then $d_n n^{1-1/p}$ cannot tend monotonically to zero. We can now sharpen this statement as follows.

COROLLARY. If $\{d_n\}$ is any sequence of positive numbers such that $a_n = O(d_n)$ for every function $\sum a_n z^n$ in H^p , then there is an $\varepsilon > 0$ such that

$$d_{n}n^{{\scriptscriptstyle 1}-{\scriptscriptstyle 1}/p} \geqq arepsilon > 0 \;, \qquad n=1,\,2,\,\cdots \,.$$

Proof. If $a_n = O(d_n)$ for every $f \in H^p$, then $\{1/d_n\}$ multiplies H^p into \mathscr{L}^{∞} . Thus $1/d_n = O(n^{1-1/p})$, as claimed.

We now turn to the multipliers of H^p and B^p into $\swarrow^q(q < \infty)$, the space of sequences $\{c_n\}$ with $\sum |c_n|^q < \infty$. The following theorem generalizes a previously known result [4] for \checkmark^1 .

Theorem 2. Suppose 0 .

(i) A complex sequence $\{\lambda_n\}$ is a multiplier of H^p into $\mathcal{E}^q(p \leq q < \infty)$ if and only if

(4)
$$\sum_{n=1}^{N} n^{q/p} |\lambda_n|^q = O(N^q)$$
.

(ii) If $1 \leq q < \infty$, $\{\lambda_n\}$ is a multiplier of B^p into \mathcal{E}^q if and only if (4) holds.

(iii) If q < p, the condition (4) does not imply that $\{\lambda_n\}$ multiplies H^p into \swarrow^q ; nor does it imply that $\{\lambda_n\}$ multiplies B^p into \swarrow^q if q < 1.

Proof. (i) A summation by parts (see [4]) shows that (4) is equivalent to the condition

(5)
$$\sum_{n=N}^{\infty} |\lambda_n|^q = O(N^{q(1-1/p)})$$
.

Assume without loss of generality that $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n^q = 1$. Let $s_1 = 0$ and

$$s_n = 1 - \left\{\sum\limits_{k=n}^\infty \lambda_k^q
ight\}^{1/eta}, \qquad n=2,\,3,\,\cdots\,,$$

where $\beta = q(1/p - 1)$. Note that s_n increases to 1 as $n \to \infty$. By a theorem of Hardy and Littlewood ([8], p. 412), $f \in H^p(0 implies$

$$(\ 6\) \qquad \qquad \int_{_0}^{^1}(1-r)^{eta-1}\,M_{_1}^q(r,f)dr < \infty \ , \qquad p \leq q < \infty \ .$$

Thus if $f(z) = \sum a_n z^n$ is in H^p and $\{\lambda_n\}$ satisfies (4) with $p \leq q < \infty$, it follows that

$$egin{aligned} &\sim &> \sum_{n=1}^{\infty} \int_{s_n}^{s_{n+1}} (1-r)^{eta-1} M_1^q(r,f) dr \ &&\geq \sum_{n=1}^{\infty} |\, a_n \,|^q \int_{s_n}^{s_{n+1}} (1-r)^{eta-1} r^{nq} dr \ &&\geq \sum_{n=1}^{\infty} |\, a_n \,|^q \, (s_n)^{nq} \int_{s_n}^{s_{n+1}} (1-r)^{eta-1} dr \ &&= rac{1}{eta} \sum_{n=1}^{\infty} |\, a_n \,|^q \, (s_n)^{nq} \{ (1-s_n)^eta - (1-s_{n+1})^eta \} \ &&= rac{1}{eta} \sum_{n=1}^{\infty} |\, a_n \,|^q \, (s_n)^{nq} \lambda_n^q \ , \end{aligned}$$

by the definition of s_n . But by (5),

$$\left\{\sum_{k=n}^{\infty} \lambda_k^q\right\}^{1/\beta} \leq rac{C}{n}$$
 ,

which shows, by the definition of s_n , that

$$(s_n)^{nq} \ge (1 - C/n)^{nq} \longrightarrow e^{-Cq} > 0$$
.

Since these factors $(s_n)^{nq}$ are eventually bounded away from zero, the preceding estimates show that $\sum |a_n|^q \lambda_n^q < \infty$. In other words, $\{\lambda_n\}$ is a multiplier of H^p into \swarrow^q if it satisfies the condition (4).

(ii) The above proof shows that $\{\lambda_n\}$ multiplies B^p into \checkmark^1 under the condition (4) with q = 1. (This was also shown in [4].) The more general statement (ii) now follows by showing that if $\{\lambda_n\}$ satisfies (4), then the sequence $\{\mu_n\}$ defined by

$$\mu_n = |\lambda_n|^q \, n^{(1/p-1)(q-1)}$$

satisfies (4) with q = 1. Hence $\{\mu_n\}$ is a multiplier of B^p into \swarrow^1 , and in view of (2), $\{\lambda_n\}$ is a multiplier of B^p into \swarrow^q . Alternatively, it can be observed that $f \in B^p$ implies (6) for $1 \leq q < \infty$, so that the foregoing proof applies directly. Indeed, if $f \in B^p$, then (as shown in [2], proof of Theorem 3)

$$M_1(r, f) = O((1 - r)^{1 - 1/p});$$

hence, if $1 \leq q < \infty$,

$$\int_{_{0}}^{_{1}}(1-r)^{_{q(1/p-1)-1}}M_{_{1}}^{q}(r,f)dr \leq C\!\!\int_{_{0}}^{^{1}}(1-r)^{_{1/p-2}}M_{_{1}}(r,f)dr < \infty$$
 .

(iii) That (4) does not imply $\{\lambda_n\}$ multiplies H^p into $\swarrow^q (q < p)$ or B^p into $\swarrow^q (q < 1)$, follows from the fact [4] that the series

$$\sum_{n=1}^{\infty} n^{q(1-1/p)-1} |a_n|^q$$

may diverge if $f \in H^p$ and q < p, or if $f \in B^p$ and q < 1.

To show the necessity of (4), we again appeal to the closed graph theorem. If $\{\lambda_n\}$ multiplies H^p into $\swarrow^q (0 , then$

$$\Lambda: f \longrightarrow \{\lambda_n a_n\}$$

is a bounded operator:

$$\left\{\sum_{n=0}^{\infty}|\lambda_na_n|^q
ight\}^{1/q}\leq C\mid\mid f\mid\mid$$
 , $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H^p$.

Choosing f(z) = g(rz) as in the proof of Theorem 1, we now find

$$\left\{\sum_{n=1}^{\infty} n^{q/p} \, |\, \lambda_n|^q \, r^{nq}
ight\}^{1/q} \leq C (1-r)^{-1} \; ;$$

and (4) follows after terminating this series at n = N and setting r = 1 - 1/N. Note that the argument shows (4) is necessary even if $p \ge 1$ or q < p.

COROLLARY 1. If $\{n_k\}$ is a lacunary sequence of positive integers $(n_{k+1}/n_k \ge Q > 1)$, and if $f(z) = \sum a_n z^n$ is in $H^p(0 , then$

$$\sum\limits_{k=1}^\infty n_k^{q(1-1/p)} \, | \, a_{n_k} |^q < \infty$$
 , $p \leq q < \infty$.

COROLLARY 2. If $f(z) = \sum a_n z^n$ is in $H^p(0 , then <math>\sum n^{p-2} |a_n|^p < \infty$.

The first corollary extends a theorem of Paley [13] that $f \in H^1$ implies $\{a_{n_k}\} \in \mathbb{Z}^2$. The second is a theorem of Hardy and Littlewood [7]. It is interesting to ask whether the converse to Corollary 1 (with q = p) is valid. That is, if $\{c_k\}$ is a given sequence for which

$$\sum_{k=1}^{\infty} n_k^{p-1} \, | \, c_k \, |^p < \, \infty$$
 ,

then is there a function $f(z) = \sum a_n z^n$ in H^p with $a_{n_k} = c_k$? We do not know the answer.

Hardy and Littlewood [9] also proved that $\{\lambda_n\}$ multiplies H^1 into H^2 (alias \swarrow^2) if (and only if)

$$\sum\limits_{n=1}^{N} n^2 \, |\, \lambda_n \,|^2 = \, O(N^2) \; .$$

From this it is easy to conclude that (4) characterizes the multipliers of H^1 into \mathbb{Z}^q , $2 \leq q < \infty$. Indeed, let $\{\lambda_n\}$ satisfy (4) and let $\mu_n = |\lambda_n|^{q/2}$. Then, by the Hardy-Littlewood theorem, $\{\mu_n\}$ multiplies H^1 into \mathbb{Z}^2 (see [3], p. 253). Hence $\{\lambda_n\}$ multiplies H^1 into \mathbb{Z}^q . (See also Hedlund [12].)

On the other hand, the condition (4) is not sufficient if p = 1 and q < 2. This may be seen by choosing a lacunary series

$$f(z)=\sum\limits_{k=1}^{\infty}c_kz^{n_k}$$
 , $n_{k+1}\!/n_k\geqq Q>1$,

with $\sum |c_k|^2 < \infty$ but $\sum |c_k|^q = \infty$ for all q < 2. The sequence $\{\lambda_n\}$ with $\lambda_n = 1$ if $n = n_k$ and $\lambda_n = 0$ otherwise then satisfies (4) but does not multiply H^1 into ℓ^q , q < 2.

3. Multipliers into B^q . The following theorem may be regarded

as a generalization of our previous result ([2], Th. 5) that if $f \in B^p$, then its fractional integral of order (1/p - 1/q) is in B^q . (A fractional integral of negative order is understood to be a fractional derivative.)

THEOREM 3. Suppose 0 and <math>0 < q < 1. Let ν be the positive integer such that $(\nu + 1)^{-1} \leq p < \nu^{-1}$. Then $\{\lambda_n\}$ is a multiplier of H^p or B^p into B^q if and only if $g(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ has the property

(7)
$$M_1(r, g^{(\nu)}) = O((1 - r)^{1/p - 1/q - \nu}) .$$

Proof. Let $\{\lambda_n\}$ satisfy (7), let $f(z) = \sum a_n z^n$ be in B^p , and let $h(z) = \sum \lambda_n a_n z^n$. Then

$$h(
ho z) = rac{1}{2\pi} \int_{0}^{2\pi} f(
ho e^{it}) g(z e^{-it}) dt \;, \qquad 0 <
ho < 1 \;.$$

Differentiation with respect to z gives

(8)
$$\rho^{\nu}h^{(\nu)}(\rho z) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\rho e^{it}) g^{(\nu)}(z e^{-it}) e^{-i\nu t} dt$$
.

Hence

$$egin{aligned} &
ho^
u M_{ ext{\tiny 1}}(r
ho,\,h^{(
u)}) &\leq M_{ ext{\tiny 1}}(r,\,g^{(
u)}) M_{ ext{\tiny 1}}(
ho,\,f) \ &\leq C(1\,-\,r)^{1/p-1/q-
u} M_{ ext{\tiny 1}}(
ho,\,f) \ , \end{aligned}$$

where r = |z|. Taking $r = \rho$, we now see that $f \in B^p$ implies $h^{(\nu)} \in B^s$, $1/s = 1/q + \nu$. Thus $h \in B^q$, by Theorem 5 of [2].

Conversely, let $\{\lambda_n\}$ multiply H^p into B^q . Then by the closed graph theorem,

 $\Lambda: \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n$

is a bounded operator from $H^{\,p}$ to $B^{\,q}$. If $(
u+1)^{-1} \leq p <
u^{-1}$, let

$$f(z) =
u! \, z^
u(1-z)^{-
u-1} = \sum_{n=
u}^{\infty} a_n z^n$$
 ,

where $a_n = n!/(n - \nu)!$, and observe that

(9)
$$h(z) = \sum_{n=\nu}^{\infty} \lambda_n \alpha_n z^n = z^{\nu} g^{(\nu)}(z) .$$

Let $f_r(z) = f(rz)$ and $h_r(z) = h(rz)$. Since Λ is bounded, there is a constant C independent of r such that

$$||h_r||_{B^q} = ||A(f_r)|| \leq C ||f_r||_{H^p}$$
 .

In other words,

$$\int_{0}^{1} (1-t)^{1/q-2} M_{1}(tr, h) dt \leq C M_{p}(r, f)$$
$$= O((1-r)^{1/p-\nu-1}) .$$

It follows that

$$M_1(r^2, h) \int_r^1 (1-t)^{1/q-2} dt = O((1-r)^{1/p-\nu-1}) ,$$

or

$$M_{\scriptscriptstyle 1}(r^2,\,h) = O((1\,-\,r)^{{\scriptscriptstyle 1/p}-{\scriptscriptstyle 1/q}-
u})$$
 .

But in view of (9), this proves (7).

COROLLARY. The sequence $\{\lambda_n\}$ multiplies B^p into B^p if and only if

(10)
$$M_1(r, g') = O\left(\frac{1}{1-r}\right).$$

Proof. If p = q, the condition (10) is equivalent to (7). (see [8], p. 435.) This corollary is essentially the same as a result of Zygmund ([14], Th. 1), who found the multipliers of the Lipschitz space Λ_{α} or λ_{α} into itself. Because of the duality between these spaces and B^{p} (see [2], §§ 3, 4), the multipliers from Λ_{α} to Λ_{α} and from λ_{α} to λ_{α} ($0 < \alpha < 1$) are the same as those from B^{p} to B^{p} . Similar remarks apply to the spaces Λ_{*} and λ_{*} , also considered in [14].

4. Multipliers into H^q . By combining Theorem 3 with the simple fact that $f' \in B^{1/2}$ implies $f \in H^1$, it is possible to obtain a sufficient condition for $\{\lambda_n\}$ to multiply H^p into H^q , 0 . However, this method leads to a sharp result only in the case <math>q = 1. The following theorem provides the complete answer.

THEOREM 4. Suppose $0 , and let <math>(\nu + 1)^{-1} \leq p < \nu^{-1}$, $\nu = 1, 2, \cdots$. Then $\{\lambda_n\}$ is a multiplier of H^p or B^p into H^q if and only if $g(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ has the property

(11)
$$M_q(r, g^{(\nu+1)}) = O((1-r)^{1/p-\nu-2}).$$

Hardy and Littlewood ([9], [10]) stated in different terminology that (11) implies $\{\lambda_n\}$ is a multiplier of H^p into $H^q(0 , but they never published the proof. Our proof will make use of the following lemma.$

LEMMA. Let f be analytic in the unit disk, and suppose

$$\int_{\scriptscriptstyle 0}^{\scriptscriptstyle 1}(1-r)^{\scriptscriptstyle lpha}M_{\scriptscriptstyle q}(r,f')dr < \infty$$
 ,

where $\alpha > 0$ and $1 \leq q \leq \infty$. Then

$$\int_{\scriptscriptstyle 0}^{\scriptscriptstyle 1}(1-r)^{lpha-1}M_{\scriptscriptstyle q}(r,f)dr<\infty$$
 .

Proof of Lemma. Without loss of generality, assume f(0) = 0, so that

$$f(re^{i heta}) = \int_{_0}^r f'(se^{i heta})e^{i heta}ds \; .$$

The continuous form of Minkowski's inequality now gives

(12)
$$M_q(r,f) \leq \int_0^r M_q(s,f') ds$$

Hence an interchange of the order of integration shows that

$$\int_{0}^{1}(1-r)^{lpha-1}M_{q}(r,f)dr \leq rac{1}{lpha} \int_{0}^{1}(1-s)^{lpha}M_{q}(s,f')ds \; ,$$

which proves the lemma.

Proof of Theorem 4. Suppose first that $\{\lambda_n\}$ satisfies (11). Given $f(z) = \sum a_n z^n$ in B^p , we are to show that $h(z) = \sum \lambda_n a_n z^n$ belongs to H^q . By (8), with ν replaced by $(\nu + 1)$, we have

$$|
ho^{
u+1} \, | \, h^{(
u+1)}(
ho z) \, | \, \leq rac{1}{2\pi} \int_0^{2\pi} | \, f(
ho e^{it}) \, | \, | \, g^{(
u+1)}(z e^{-it}) \, | \, dt \; .$$

Since $q \ge 1$, it follows from Jensen's inequality ([11], §6.14) that

$$egin{aligned} &
ho^{
u+1}\,M_q(r
ho,\,h^{(
u+1)}) &\leq M_1(
ho,\,f)M_q(r,\,g^{(
u+1)}) \ &\leq C(1-r)^{1/p-
u-2}M_1(
ho,\,f) \ , \end{aligned}$$

where r = |z| and (11) has been used. Now set $r = \rho$ and use the hypothesis $f \in B^p$ to conclude that

$$\int_{0}^{1}(1-r)^{
u}M_{q}(r,\,h^{(
u+1)})dr < \infty$$
 .

But by successive applications of the lemma, this implies

$$\int_{\scriptscriptstyle 0}^{\scriptscriptstyle 1} M_{\scriptscriptstyle q}(r,h') dr < \infty$$
 .

Thus, in view of the inequality (12), it follows that $h \in H^{q}$, which was to be shown.

Conversely, suppose $\{\lambda_n\}$ is a multiplier of H^p into H^q for arbitrary $q(0 < q \leq \infty)$. Then by the closed graph theorem,

$$\Lambda: \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n$$

is a bounded operator from H^p to H^q . An argument similar to that used in the proof of Theorem 3 now leads to the estimate (11).

COROLLARY. If $0 and <math>f \in B^p$, then its fractional integral $f_{\alpha} \in H^q$, where $\alpha = 1/p - 1/q$. This is false if q < 1.

This corollary can also be proved directly. Indeed, since ([2], Th. 5) the fractional integral of order (1/p - 1/s) of a B^p function is in B^s (0 < s < 1), and since ([8], p. 415) the fractional integral of order (1 - 1/q) of an H^1 function is in $H^q(1 \le q \le \infty)$, it suffices to show that $f' \in B^{1/2}$ implies $f \in H^1$. But this is easy; it follows from (12) with q = 1. That the corollary is false for q < 1 is a consequence of the fact ([2], Th. 5) that the fractional derivative of order (1/p - 1/q) of every B^q function is in B^p .

The converse is also false. That is, if $f \in H^q$, its fractional derivative of order (1/p - 1/q) need not be in $B^p(0 .$ $As before, this reduces to showing that <math>f \in H^1$ does not imply $f' \in B^{1/2}$. To see this, let $f(z) = \sum c_k z^{n_k}$, where $\{n_k\}$ is lacunary, $\{c_k\} \in \mathbb{Z}^2$, and $\{c_k\} \notin \mathbb{Z}^1$. Then $f \in H^2 \subset H^1$, but $f' \notin B^{1/2}$, since it was shown in [4] (Th. 3, Corollary 2) that

$$\sum\limits_{k=1}^{\infty}n_k^{1-1/p}\left|a_{n_k}
ight|<\infty$$

whenever $\sum a_n z^n \in B^p$ and $\{n_k\}$ is a lacunary sequence.

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