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A structure theorem is given for all locally compact rings such that x belongs to the closure of $\{x^n: n \geqq 2\}$, in particular, all such rings are commutative, a result which extends a well-known theorem of Jacobson. Similarly we show the commutativity of semisimple locally compact rings satisfying topological analogues of properties studied by Herstein.

Jacobson has shown that a ring is commutative if for every x there is some $n(x) \geqq 2$ such that $x^{n(x)} = x$ [5, Th. 1, p. 212]. Herstein has generalized this result, and certain of his and other generalizations are of interest here. A ring is commutative if (and only if) for all x and y there is some $n(x, y) \geqq 2$ such that $(x^{n(x,y)} - x)y = y(x^{n(x,y)} - x)$ [4, Th. 2]; a ring is commutative if (and only if) for all x and y there is some $n(x, y) \geqq 2$ such that $xy - yx = (xy - yx)^{n(x,y)}$ [3, Th. 6]; a semisimple ring is commutative if (and only if) for all x and y there is some $n(x, y) \geqq 1$ such that $x^{n(x,y)}y = yx^{n(x,y)}$ [4, Th. 1] or if for all x and y there are $n, m \geqq 1$ such that $x^n y^m = y^m x^n$ [1, Lemma 1]. The investigation of analogous conditions for topological rings is the major concern of this paper.

1. A topological analogue of Jacobson's condition.

If $x^n = x$ for some $n \geqq 2$, then an inductive argument shows that $x^{k(n-1)+1} = x$ for all $k \geqq 1$. A possible topological analogue of Jacobson's condition would thus be that for every x there is some $n(x) \geqq 2$ such that $\lim_k x^{k(n(x)-1)+1} = x$. But this implies that $x^{n(x)} = x$, since

$$x^{n(x)} = x^{n(x)-1}x = x^{n(x)-1} \lim_k x^{k(n(x)-1)+1} = \lim_k x^{(k+1)(n(x)-1)+1} = x .$$

Thus all topological rings having this property have Jacobson's property and hence are commutative.

A less trivial analogue of Jacobson's condition is that for every x in the topological ring A, x belongs to the closure of $\{x^n: n \geqq 2\}$. In our investigation of these rings, rings with no nonzero topological nilpotents play an important role. Recall that an element x of a topological ring is a *topological nilpotent* if $\lim_n x^n = 0$. We shall prove that a locally compact ring has no nonzero topological nilpotents if and only if it is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring B that is the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields. From this it is easy to derive a structure theorem for locally compact rings having the topological analogue of Jacobson's property mentioned above.

LEMMA 1. *If A is a locally compact ring with no nonzero topological nilpotents, then A is totally disconnected.*

Proof. The connected component C of zero in A is a closed ideal of A and so is itself a connected locally compact ring with no nonzero topological nilpotents. By hypothesis, C is not annihilated by any of its nonzero elements, for if $xC = (0)$, then $x^2 = 0$, so $x = 0$. Thus C is a finite-dimensional algebra over the real numbers (cf. [6, Th. III]). As the radical of a finite-dimensional algebra is nilpotent, C is a semisimple algebra. If $C \neq (0)$, then by Wedderburn's Theorem, C has an identity e, and clearly $(1/2)e$ would then be a nonzero topological nilpotent contrary to our hypothesis. Thus $C = (0)$, and so A is totally disconnected.

LEMMA 2. *A compact ring A has no nonzero topological nilpotents if and only if A is the Cartesian product of finite fields.*

Proof. Necessity: By Lemma 1, A is totally disconnected. Thus the radical $J(A)$ of A is topologically nilpotent [11, Th. 14], and hence is the zero ideal. Thus A is a compact semisimple ring, and so A is topologically isomorphic to the Cartesian product of a family of finite simple rings [11, Th. 16]. A finite simple ring is a matrix ring over a finite field, and unless the matrix ring is just the finite field itself, it will have nonzero nilpotent elements. Thus as A has no nonzero nilpotents, A is topologically isomorphic to the Cartesian product of a family of finite fields. Sufficiency: Clearly zero is the only topological nilpotent in the Cartesian product of a family of finite fields.

LEMMA 3. *If A is a ring with no nonzero nilpotents, then every idempotent is in the center of A.*

Proof. If e is an idempotent and if $a \in A$, an easy calculation shows that $(ae - eae)^2 = 0$, hence $ae - eae = 0$. Similarly, $ea = eae$ and thus $ae = ea$.

We recall that the local direct sum of a family $(A_\gamma)_{\gamma \in \Gamma}$ of topological rings with respect to open subrings $(B_\gamma)_{\gamma \in \Gamma}$ is the subring of the Cartesian product $\prod_\gamma A_\gamma$ consisting of all (a_γ) such that $a_\gamma \in B_\gamma$ for all but finitely many γ, topologized by declaring all neighborhoods of zero in the topological ring $\prod_\gamma B_\gamma$ to be a fundamental system of neighborhoods of zero in the local direct sum. It is easy to see that the local direct sum equipped with this topology is indeed a topological ring.

THEOREM 1. *A locally compact ring A has no nonzero topological nilpotents if and only if A is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring B (possibly the zero ring) that is topologically isomorphic to the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields.*

Proof. Necessity: As A is totally disconnected by Lemma 1, A contains a compact open subring F [7, Lemma 4]. By Lemma 2, F is topologically isomorphic to the product of finite fields. Consequently there exists in F a summable orthogonal family $(e_\gamma)_{\gamma\in\Gamma}$ of idempotents such that Fe_γ is a finite field and $\sum_{\gamma\in\Gamma} e_\gamma = e$, the identity of F.

By Lemma 3, e is in the center of A, so Ae and $A(1-e) = \{a - ae\colon a \in A\}$ are ideals. The continuous mappings $a \longrightarrow ae$ and $a \longrightarrow (a - ae)$ are the projections from A onto Ae and $A(1-e)$. Thus A is the topological direct sum of Ae and $A(1-e)$. As e is the identity of F, $F \cap A(1-e) = (0)$. Thus as F is open, $A(1-e)$ is discrete and hence has no nonzero nilpotents.

As F is open and as $Ae_\gamma \cap F = Fe_\gamma$, a finite field, Ae_γ is discrete and is an ideal as e_γ is in the center of A. Consequently Ae_γ has no nonzero nilpotents. It will therefore suffice to show that $B = Ae$ is topologically isomorphic to the local direct sum of the descrete rings Ae_γ, with respect to the finite subfields Fe_γ.

Let B' be the local direct sum of the Ae_γ's with respect to the Fe_γ's. Let $K\colon b \longrightarrow (be_\gamma) \in \prod_\gamma Ae_\gamma$. Clearly $b \longrightarrow be_\gamma$ is a continuous homomorphism for each γ, hence K is a continuous homomorphism from B into $\prod_\gamma Ae_\gamma$. If $b \in B$, then (be_γ) is summable and $\sum_\gamma be_\gamma = b(\sum_\gamma e_\gamma) = be = b$. Therefore as F is open in B, $be_\gamma \in F \cap Ae_\gamma = Fe_\gamma$ for all but finitely many $\gamma \in \Gamma$. Thus $K(B) \subseteqq B'$.

The mapping K is an isomorphism onto $K(B)$, since if $x \in B$ and if $xe_\gamma = 0$ for all $\gamma \in \Gamma$, then $x = xe = x(\sum_\gamma e_\gamma) = \sum_\gamma xe_\gamma = 0$. Let $y_\beta \in Fe_\beta$, and let $x_\gamma = 0$ for all $\gamma \neq \beta$, $x_\beta = y_\beta$; then $(x_\gamma) = K(y_\beta) \in K(F)$ since $(e_\gamma)\gamma$ is an orthogonal family. Thus $K(F)$ contains a dense subring of $\prod_\gamma Fe_\gamma$, and hence $K(F) = \prod_\gamma Fe_\gamma$ as $K(F)$ is compact. As the restriction of K to F is thus a continuous isomorphism from conpact F onto $\prod_\gamma Fe_\gamma$, F is topologically isomorphic to $\prod_\gamma Fe_\gamma$ under K.

Thus it sufficices to show that $K(B) \supseteqq B'$, for K is then, by the definition of the local direct sum, a topological isomorphism from B onto B'. If $(b_\gamma e_\gamma) \in B'$, then $b_\gamma e_\gamma \in Fe_\gamma$ for all but finitely many γ, say $\gamma_1, \cdots, \gamma_n$. Call this set Γ_1 and let $\Gamma - \Gamma_1 = \Gamma_2$. Thus $\sum_{\gamma\in\Gamma_1} b_\gamma e_\gamma \in B$ and $b_\gamma e_\gamma \in F$ for all $\gamma \in \Gamma_2$. Hence as F is topologically isomorphic to $\prod_\gamma Fe_\gamma$, $b' = \sum_{\gamma\in\Gamma_2} b_\gamma e_\gamma \in B$. Thus $b = b' + \sum_{\gamma\in\Gamma_1} b_\gamma e_\gamma \in B$, and $be_\gamma = b_\gamma e_\gamma$, so $K(b) = (b_\gamma e_\gamma)$. The sufficiency is clear.

We will call a ring A a *Jacobson ring* if given any $x \in A$ there is an $n(x) \geqq 2$ such that $x^{n(x)} = x$. All Jacobson rings are commutative [5, Th. 1, p. 212], and in extending this result to topological rings we give the following definition, noting that it reduces to Jacobson's condition in the discrete case.

DEFINITION. A topological ring A is a *J-ring* if for each $x \in A$, x belongs to the closure of $\{x^n : n \geqq 2\}$.

LEMMA 4. *If A is a J-ring, then A has no nonzero topological nilpotents.*

Proof. If $\lim_n x^n = 0$, then since x belongs to the closure of $\{x^n : n \geqq 2\}$, we conclude that $x = 0$.

THEOREM 2. *A locally compact ring A is a J-ring if and only if A is the topological direct sum of a discrete Jacobson ring and a ring B which is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields.*

Proof. Necessity: By Theorem 1 and Lemma 4, A is the topological direct sum of a discrete ring C and a ring B which is topologically isomorphic to the local direct sum of a family of discrete rings with respect to finite subfields. As each of these rings is an ideal of A, each is a discrete J-ring and so is a Jacobson ring.

Sufficiency: Let B be the local direct sum of a family of discrete Jacobson rings $B_\gamma, \gamma \in \Gamma$ with respect to finite subfields $F_\gamma, \gamma \in \Gamma$. Let $(x_\gamma) \in B$ and let U be a neighborhood of zero in B. Then we may assume that there is a finite subset Δ of Γ such that $x_\gamma \in F_\gamma$ for all $\gamma \notin \Delta$ and $U = \prod_\gamma G_\gamma$, where $G_\gamma = F_\gamma$ for all $\gamma \notin \Delta$. For each $\gamma \in \Delta$, let $n(\gamma) > 1$ be such that $x_\gamma^{n(\gamma)} = x_\gamma$. Let $n = 1 + \prod_{\gamma \in \Delta} (n(\gamma) - 1)$. An inductive argument shows that $x_\gamma^n = x_\gamma$ for all $\gamma \in \Delta$. Hence $(x_\gamma)^n - (x_\gamma) \in U$. Thus B is a J-ring, and consequently A is also a J-ring.

As all Jacobson rings are commutative we have the following analogue of Jacobson's Theorem:

COROLLARY. *A locally compact J-ring is commutative.*

THEOREM 3. *A locally compact ring A is a Jacobson ring if and only if there exists $N \geqq 2$ such that A is the topological direct sum of a discrete Jacobson ring and a ring B that is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields of order $\leqq N$.*

Proof. Necessity: Let $|B_\gamma|$ = the order of B_γ. By Theorem 2 it suffices to show that $\sup|B_\gamma| < +\infty$. If $\sup|B_\gamma| = +\infty$, then there exists $(x_\gamma) \in \prod_\gamma B_\gamma$ such that the orders of the x_γ's are unbounded. Consequently for no n does $x_\gamma^n = x_\gamma$ for all γ, i.e., for no n does $(x_\gamma)^n = (x_\gamma)$.

Sufficiency: Let $(A_\gamma)_{\gamma \in \Gamma}$ be a family of discrete Jacobson rings with finite subfields B_γ such that $|B_\gamma| \leqq N$ for all γ. Let (x_γ) be in the local direct sum of the A_γ's with respect to the B_γ's. There exists a finite subset Δ of Γ such that if $\gamma \notin \Delta$, $x_\gamma \in B_\gamma$. Since each A_γ is a Jacobson ring, for $\gamma \in \Delta$ there is $n(\gamma)$ such that $x_\gamma^{n(\gamma)} = x_\gamma$.

If $x_\gamma^{n(\gamma)} = x_\gamma$, an inductive argument shows that $x_\gamma^{k(n(\gamma)-1)+1} = x_\gamma$ for all k. If $x_\gamma \in B_\gamma$, then $|B_\gamma| \leqq N$, so since $|B_\gamma| - 1 < N$, $x_\gamma^{1+k(N!)} = x_\gamma$ for all k. Let $n = 1 + [(N!)\prod_{\gamma \in \Delta}(n(\gamma) - 1)]$. Then $x_\gamma^n = x_\gamma$ for all γ, i.e., $(x_\gamma)^n = (x_\gamma)$.

2. Analogues of four of Herstein's results.

An analogue for topological rings of the first of Herstein's conditions that are mentioned above is that for all x and y, $xy - yx$ is in the closure of $\{x^n y - yx^n: \geqq 2\}$, and we say such a topological ring is an H_1-*ring*. An analogue of the second of Herstein's conditions is that for all x and y, $xy - yx$ is in the closure of $\{(xy - yx)^n: n \geqq 2\}$, and we say such a topological ring is an H_2-*ring*. (If $(xy - yx)^{n(x,y)} = xy - yx$, then

$$(xy - yx)^{k[n(x,y)-1]+1} = xy - yx$$

for all $k \geqq 1$; hence another topological analogue is the assumption that for each $x, y \in A$, there exists $n(x, y) \geqq 2$ that $\lim_k (xy - yx)^{k[n(x,y)-1]+1} = xy - yx$; however by an argument similar to that of the first paragraph of § 1, this condition implies that $(xy - yx)^{n(x,y)} = xy - yx$.) Similarly an analogue of the third of Herstein's conditions is that for all x, y in A, $\lim_n x^n y - yx^n = 0$, and we say such topological rings are H_3-*rings*, just as we will call H_4-*rings* those topological rings in which for all x, y there is an $m(x, y) \geqq 1$ such that $\lim_n x^n y^{m(x,y)} - y^{m(x,y)} x^n = 0$. We shall prove that those H_i-*rings* which are semisimple and locally compact are commutative, $i = 1, 2, 3, 4$.

LEMMA 5. *All idempotents in an H_i-ring, $i = 1, 2, 3, 4$, commute.*

Proof. Let e and f be idempotents in such a ring A. Then $(efe - ef)^2 = 0$, so $\{(efe - ef)^n e - e(efe - ef)^n: n \geqq 2\} = \{0\}$. Therefore, if A is an H_1-ring, then $(efe - ef)e - e(efe - ef) = 0$, so

$$0 = (efe - ef)e = e(efe - ef) = efe - ef\,.$$

If A is an H_2-ring, then $(ef)e - e(ef) = efe - ef = 0$ since $efe - ef$ is in the closure of $\{[(ef)e - e(ef)]^n: n \geqq 2\} = \{0\}$. Similarly in either case

$efe = fe$, so $ef = fe$. As $0 = \lim_n e^n f - fe^n = \lim_n e^n f^m - f^m e^n = ef - fe$, the assention also holds for H_3 and H_4-rings.

Since it is clear that all subrings and quotient rings determined by closed ideals of H_i-rings are H_i-rings, $i = 1, 2, 3, 4$, and since all idempotents in such rings commute, we see that the following is applicable.

LEMMA 6. *Let P be a property of Hausdorff topological rings such that:*

(1) *if A is a Hausdorff topological ring with property P, then every subring of A has property P and A/B has property P where B is any closed ideal of A,*

(2) *if A has property P, then all idempotents in A commute. If A is a locally compact primitive ring with property P, then A is a division ring.*

Proof. Since A is a semisimple ring, A is the topological direct sum of a connected ring B and a totally disconnected ring C, where B is a semisimple algebra over R of finite dimension [7, Th. 2]. As A is primitive, either $A = B$ or $A = C$. In the former case A is a matrix ring since it is primitive, and so has idempotents which do not commute unless it is a division ring.

It suffices, therefore, to consider the case in which A is totally disconnected. We shall first prove the assertion under the additional assumption that A is a Q-ring (i.e., the set of quasi-invertible elements is a neighborhood of zero). We may consider A to be a dense ring of linear operators on a vector space E over a division ring D. If E is not one-dimensional, then E has a two-dimensional subspace M with basis $\{z_1, z_2\}$. Let $B = \{a \in A: a(M) \subseteqq M\}$, and let

$$N = \{a \in A: a(M) = (0)\} = K_1 \cap K_2$$

where $K_i = \{a \in A: a(z_i) = 0\}$, $i = 1, 2$.

There exists $u \in A$ such that $u(z_1) = z_1$, and hence $x - xu \in K_1$, for all $x \in A$. If $v \notin K_1$, then there exists $w \in A$ such that $wv(z_1) = z_1$, so as $u = wv + (u - wv)$ and $u - wv \in K_1$, $A = Au + K_1 = Av + K_1$. Therefore K_1, and similarly K_2, is a regular maximal left ideal, an observation of the referee that simplifies the proof. Hence K_1 and K_2 are closed (cf. [11, Th. 2]), so N is a closed ideal of B. By hypothesis B/N is therefore a Hausdorff topological ring having property P. Thus all idempotents in B/N commute; but B/N is isomorphic to the ring of all linear operators on M, a ring containing idempotents which do not commute. Hence E is one-dimensional and A is a division ring.

Next we shall show that A is necessarily a Q-ring, from which the result follows by preceding. As A is totally disconnected A has a compact open subring D [7, Lemma 4]. If $D = J(D)$, the radical of D, then D and hence A are Q-rings. Assume therefore that $J(D) \subset D$. We shall show that $D/J(D)$ is a finite ring and hence is discrete.

The radical, $J(D)$, of D is closed [8, Th. 1], $D/J(D)$ is compact semisimple ring and thus $D/J(D)$ is topologically isomorphic to the Cartesian product of a family $(F_\gamma)_{\gamma \in \Gamma}$ of finite simple rings with identities $(f_\gamma)_{\gamma \in \Gamma}$ [11, Th. 16]. As $J(D)$ is topologically nilpotent [11, Th. 14], D is suitable for building idempotents [12, Lemma 4] (cf. [11, Lemma 12]). Suppose that Γ has more than one element, say $\{\alpha, \beta\} \subseteqq \Gamma$. Then there are nonzero orthogonal idempotents e_α, e_β in D such that $e_\alpha + J(D)$, $e_\beta + J(D)$ correspond, respectively, under the isomorphism to (f_γ^α), (f_γ^β) where $f_\gamma^\lambda = 0 \in F_\gamma$ if $\gamma \neq \lambda$ and $f_\lambda^\lambda = f_\lambda$. Let ϕ be the canonical mapping $x \to x + J(D)$ from D onto $D/J(D)$. As $(f_\gamma^\alpha) + (f_\gamma^\beta)$ annihilates the open neighborhood $\prod_{\gamma \in \Gamma} G_\gamma$ of zero where $G_\alpha = \{0\}$, $G_\beta = \{0\}$, and $G_\gamma = F_\gamma$ for $\gamma \neq \alpha, \beta$, we conclude that $\phi(e_\alpha + e_\beta)$ annihilates a neighborhood V of zero in $D/J(D)$. Consequently $U = \phi^{-1}(V)$ is a neighborhood of zero in D, and $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) \subseteqq J(D)$ (cf. [7, proof of Th. 11]). Therefore as $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) = U \cap (e_\alpha + e_\beta)A(e_\alpha + e_\beta)$, $(e_\alpha + e_\beta)U(e_\alpha + e_\beta)$ is a neighborhood of zero in $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ consisting of quasi-invertable elements, so $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a Q-ring. As $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is primitive [6, Proposition 1, p. 48] and is clearly closed, $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a locally compact, primitive Q-ring with property P, so $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a division ring. But it contains nonzero e_α, e_β satisfying $e_\alpha e_\beta = 0$, a contradiction. Thus Γ can contain only one element, so $D/J(D)$ is isomorphic to a finite ring. Hence $J(D)$, being closed in D, is open in D and thus in A, so A is a Q-ring.

LEMMA 7. *If A is an H_i-ring, $i = 1, 2, 3, 4$ and if A is a locally compact division ring, then A is a field.*

Proof. If A is discrete and is an H_i-ring ($i = 1, 2, 3, 4$) then A is commutative [3, Th. 2; 4, Th. 1; 3, Th. 1; 1, Lemma 1].

If A is not discrete, then A has a nontrivial absolute value giving its topology, and A is a finite-dimensional algebra over its center, on which the absolute value is nontrivial [10, Th. 8].

If A is an H_1-ring and x is nonzero in A, then there exists some nonzero z in the center of A such that $|z| < 1/|x|$. Thus $|xz| < 1$, so $\lim_n (xz)^n = 0$. Hence for any $y \in A$, $\lim_n (xz)^n y - y(xz)^n = 0$, so as $(xz)y - y(xz)$ is in the closure of $\{(xz)^n y - y(xz)^n : n \geqq 2\}$, $0 = (xz)y - y(xz) = z(xy - yx)$. Hence $xy = yx$, as $z \neq 0$. Thus A is commutative.

If A is an H_2-ring and if $x, y \in A$ satisfy $xy - yx \neq 0$, then there exists some nonzero z in the center such that $|z| < 1/|xy - yx|$. Thus

$|(xz)y - y(xz)| < 1$, so $\lim_n [(xz)y - y(xz)]^n = 0$. Hence $0 = (xz)y - y(xz) = (xy - yx)z$, so $xy - yx = 0$ as $z \neq 0$, a contradiction. Thus A is commutative.

Assume that A is an H_3-ring. As A is a division ring, A is either totally disconnected or connected [7, Th. 2].

Case 1. A is totally disconnected. Then the topology of A is given by a nonarchimedean absolute value. Suppose A is not commutative. Then as A is a finite-dimensional and hence an algebraic extension of its center C, there exists some $x \notin C$ having minimal degree $m > 1$ over C. Let y be arbitrary in A, and assume that for no $1 \leqq i \leqq m - 1$, does $x^i y = yx^i$. Hence $x^i y - yx^i \neq 0$, $1 \leqq i \leqq m - 1$, and we claim $\{x^i y - yx^i : 1 \leqq i \leqq m - 1\}$ is a linearly independent set over C. Suppose $\sum_{i=i}^{m-1} \beta_i(x^i y - yx^i) = 0$, where $\beta_i \in C$, and let $z = \sum_{i-i}^{m-1} \beta_i x^i$. Then $zy = yz$. By the definition of m, either $z \in C$ on z has degree $\geqq m$ over C. Suppose $z \notin C$. Then $C[x]$ has dimension m over C, so m is the degree of z as $z \in C[x]$. Therefore $C[x] = C[z]$, so as $zy = yz$, every element of $C[x]$ commutes with y, contrary to our assumption. Thus $z \in C$; let $-\beta_0 = z$. Then $\sum_{i=0}^{m-1} \beta_i x^i = 0$, so $\beta_i = 0$, $0 \leqq i \leqq m-1$ since $\{1, x, \cdots, x^{m-1}\}$ is linearly independent over C.

Since x is algebraic of degree m over the center C of A, there exist $\alpha_i \in C$, $0 \leqq i \leqq m - 1$, such that $x^m = \sum_{i=0}^{m-1} \alpha_i x^i$; thus for all $n \geqq m$, there exist $\alpha_{i,n} \in C$, $0 \leqq i \leqq m - 1$, such that $x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i$. We may also assume that $|x| > 1$, since all our assumption on x are true for any λx, $\lambda \in C^*$. We note that there is therefore some r such that $|x|^r \geqq |\alpha_i|$, $0 \leqq i \leqq m - 1$.

Since $x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i$,

$$x^n y - yx^n = \sum_{i=i}^{m-1} \alpha_{i,n}(x^i y - yx^i) ;$$

so $\lim_n x^n y - yx^n = 0$ if and only if $\lim_n \alpha_{i,n} = 0$, $1 \leqq i \leqq m - 1$.

Since $|x^n| \leqq \max\{|\alpha_{i,n}|\,|x|^i : 0 \leqq i \leqq m - 1\}$, if $|\alpha_{i,n}| < 1$, $1 \leqq i \leqq m - 1$, then $|x|^n \leqq |\alpha_{0,n}|$. Let r_0 be such that $|x|^{r_0} > |x| + 1$. Since $\lim_n \alpha_{i,n} = 0$, $1 \leqq i \leqq m - 1$, there exists $n_0 > r + r_0$ such that $|\alpha_{i,n}| < 1$, for all $n \geqq n_0$ and all i such that $1 \leqq i \leqq m - 1$. But for any $n > n_0$,

$$\begin{aligned} x^{n+1} &= \sum_{i=0}^{m-2} \alpha_i,\, x^{i+1} + \alpha_{m-1,n}\left(\sum_{i=0}^{m-1} \alpha_i x^i\right) \\ &= \alpha_{m-1,n}\alpha_0 + \sum_{i=1}^{m-1} [\alpha_{i-1,n} + (\alpha_{m-1,n})\alpha_i]x^i , \end{aligned}$$

so

$$\begin{aligned} |\alpha_{1,n+1}| &= |\alpha_{0,n} + \alpha_{m-1,n}\alpha_1| \geqq |\alpha_{0,n}| - |\alpha_{m-1,n}|\,|\alpha_1| \\ &\geqq |x|^n - |\alpha_1| \geqq |x|^{r+r_0} - |x|^r = |x^r|\,(|x|^{r_0} - 1) > 1 . \end{aligned}$$

a contradiction. Hence A is commutative.

Case 2. A is connected. Then the center C of A contains the real number field R, A is finite-dimensional over R, so the degree of each element of A over R is less than or equal to 2, and the topology is given by an absolute value. Suppose $x \notin C$. Then $\deg x = 2$; let $x^2 = \alpha_1 + \alpha_2 x$, and for each $n \geqq 2$, let $x^n = \alpha_{1,n} + \alpha_{2,n}x$, where $\alpha_{1,n}$, $\alpha_{2,n} \in R$. As before we may assume that $|x| > 1$. Let r be such that $|x|^r > \max\{|\alpha_1|, |\alpha_2|\}$. Let $y \in A$ be such that $xy \neq yx$. Then $0 = \lim_n (x^n y - yx^n) = \lim_n \alpha_{2,n}(xy - yx)$, so $\lim_n \alpha_{2,n} = 0$. Let $n_0 > r$ be such that $|\alpha_{2,n}| < 1$ for all $n \geqq n_0$. But if $n \geqq n_0$ is such that $|x|^n > 3\,|x|^r$, then

$$|x|^n = |\alpha_{1,n} + \alpha_{2,n}x| \leqq |\alpha_{1,n}| + |\alpha_{2,n}|\,|x| < |\alpha_{1,n}| + |x|\,,$$

so $|x^n| - |x| < |\alpha_1, n|$. As

$$x^{n+1} = \alpha_{1,n}x + \alpha_{2,n}(\alpha_1 + \alpha_2 x) = \alpha_{2,n}\alpha_1 + (\alpha_{1,n} + \alpha_{2,n}\alpha_2)x\,,$$

$$|\alpha_{2,n+1}| = |\alpha_{1,n} + (\alpha_{1,n})\alpha_2| \geqq |\alpha_{1,n}| - |\alpha_{2,n}|\,|\alpha_2|\,.$$

Hence $|\alpha_{2,n+1}| \geqq (|x|^n - |x|) - |x|^r \geqq 3\,|x|^r - |x|^r - |x|^r = |x|^r > 1$, a contradiction. Hence A is commutative.

Finally let A be an H_4-ring. If for all x and y, $\lim_n x^n y - yx^n = 0$, then A is an H_3-ring and so a field; so assume there are x and y in A such that $\lim_n x^n y - yx^n \neq 0$. Let $W = \{w \in A\colon \lim_n x^n w - wx^n = 0\}$. Clearly W is a division subring of A, and since $y \notin W$, W is a proper division subring. By hypothesis, for all $a \in A$ there is an $r \geqq 1$ such that $a^r \in W$; thus A is a field [2, Th. B].

THEOREM 4. *All H_i-rings that are locally compact and semisimple are commutative,* $i = 1, 2, 3, 4$.

Proof. P is a primitive ideal of such a ring A if and only if $P = (B\colon A)$ (by definition $(B\colon A) = \{x \in A\colon Ax \subseteqq B\}$) where B is a regular maximal to left ideal [5, Corollary to Proposition 2, p. 7]. Let $e \in A$ be such that $x - ex \in B$ for all $x \in A$. If $x \in (B\colon A)$, then $ex \in B$, so $x \in B$. Hence $(B\colon A) \subseteqq B$.

If B is closed, then $(B\colon A)$ is closed for if (x_α) is a directed set of elements of $(B\colon A)$ converging to x, then for all $a \in A$, $ax_\alpha \in B$, whence $ax = \lim ax_\alpha \in B$.

As A is semisimple, $(0) = \bigcap\{B\colon B$ is a closed regular maximal left ideal$\} \supseteqq \bigcap\{P\colon P$ is a closed primitive ideal$\}$ [8, Th. 1]. By Lemma 6 and 7, A/P is a field if P is a closed primitive ideal. Thus for all $x, y \in A$, $xy - yx \in P$, so $xy - yx \in \bigcap\{P\colon P$ is a closed primitive ideal$\} = (0)$.

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