

# Pacific Journal of Mathematics

**ON THE HIGMAN-SIMS SIMPLE GROUP OF ORDER  
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In a recent paper D. G. Higman and C. C. Sims announced their construction of a new simple group  $H_{100}$  of order 44,352,000. The group  $H_{100}$  is obtained as a rank 3 permutation group of degree 100 with subdegrees 1,22 and 77; and the stabilizer of a point is isomorphic to the Mathieu simple group  $M_{22}$ . Shortly after their announcement of the new simple group, Graham Higman constructed a simple group of the same order as a doubly transitive group of degree 176 and with stabilizer of a point isomorphic to  $PSU(3,5^2)$ .

The purpose of this paper is to show that the two groups mentioned above are isomorphic, and in fact, that there is exactly one (up to isomorphism) simple group of order 44,352,000.

**THEOREM.** *Let  $G$  be a nonabelian simple group of order 44,352,000. Then  $G$  is isomorphic to the Higman-Sims group  $H_{100}$ .*

Throughout this paper,  $G$  will denote a nonabelian simple group of order  $44,352,000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ . The notation will be standard; see for instance [13]. Further, a Sylow  $p$ -subgroup of  $G$  will be denoted by  $G_p$ ,  $A_n$  and  $S_n$  will denote the alternating group and symmetric group on  $n$  letters respectively; and  $F_{20}$  will denote a Frobenius group of order 20. The word "character" always refers to an irreducible character of  $G$  afforded by an irreducible representation of  $G$  in the complex number field. If the integer  $n$  divides the  $|G|$  of  $G$ , we will denote this by  $n \mid |G|$ .

In the proof of the theorem, the following results are of fundamental importance.

**RESULT 1 (R. Brauer [2], Theorem 11).** Let  $G$  be a group such that  $p \mid |G|$ , but  $p^2 \nmid |G|$ ,  $p$  a prime. If the  $p$ -block  $B_1(p)$  contains the principal character  $1_G$  of  $G$ , then  $B_1(p)$  has  $(p-1)/t$  (irreducible complex) characters which are  $p$ -conjugate only to themselves and one family of  $t$   $p$ -conjugate characters, where  $t$  denotes the number of conjugate classes of elements of order  $p$ . Further, the degrees  $\chi_i(1)$  of the irreducible characters  $\chi_i$  of  $B_1(p)$ , satisfy the following congruences:

$$\chi_i(1) = z_i \equiv \delta_i = \pm 1 \pmod{p},$$

if  $\chi_i$  is  $p$ -conjugate only to itself and

$$\chi_i(1) = z_i \equiv \frac{\delta_i}{t} = \pm \left(\frac{1}{t}\right) \pmod{p},$$

if  $\chi_i$  belongs to the family of  $p$ -conjugate characters. If  $1_G = \chi_1, \chi_2, \dots, \chi_{q+1}$  represent the different families of  $B_1(p)$ , ( $q = (p-1)/t$ ), then

$$(1) \quad 1 + \delta_2 z_2 + \dots + \delta_{q+1} z_{q+1} = 0.$$

The next result is very well-known. (For a proof, see [8] (2.15), or any book on group theory).

**RESULT 2.** Let  $K_1, K_2, K_3$  be any three conjugate classes of elements of a finite group  $G$ . Let  $x_3 \in K_3$  and let  $a(x_1, x_2; x_3)$  be the number of ordered pairs  $(x_1, x_2)$ ,  $x_1 \in K_1$  and  $x_2 \in K_2$ , such that  $x_1 \cdot x_2 = x_3$ . Then, if  $\chi_1, \dots, \chi_t$  are all the irreducible complex characters of  $G$ , we have

$$(2) \quad a(x_1, x_2, x_3) |C_G(x_1)| |C_G(x_2)| = |G| \sum_{i=1}^t \frac{\chi_i(x_1) \chi_i(x_2) \overline{\chi_i(x_3)}}{\chi_i(1)}.$$

Professor D. Wales has communicated the following result to the authors.

**RESULT 3.** Let  $G$  be a primitive permutation group of degree 100 with stabilizer of a point  $H$  isomorphic to the Mathieu simple group  $M_{22}$  and the orbits of  $H$  are of length 1, 22 and 77. Then  $G$  is isomorphic to the Higman-Sims simple group of order 44,352,000.

1. **Determination of the Sylow  $p$ -normalizers for  $p = 11, 7$  and 5.** In this section, we will determine the Sylow 11-, 7- and 5- normalizers. Unfortunately the amount of numerical work required to show that the Sylow 11-subgroup is self-centralizing is too large to enable us to present the proof here. However, the methods and results used are similar to the examples of this kind of work given in [12] and [14]. In addition to the results given in [12] and [14], we also need some results of R. Brauer on the defect group of a block (see [3], [4] and [7], §86, 87, also [5], Theorem 2 and Theorem 3). Thus combining these methods we are able to show the following result.

**LEMMA 1.1.** *The Sylow 11-normalizer of  $G$  is a Frobenius group of order 55.*

Using equation (1), it is not difficult to show that the number of conjugate classes of elements of order 7 is one, i.e.,  $|N_G(G_7): C_G(G_7)| = 6$ . Then, with Lemma 1.1 and more numerical work, we have that  $B_1(11) \cap B_1(7) = \{1, 3200\}$ , where the numbers in the brackets are the degrees of the irreducible characters which lie in both the principal

11-block,  $B_1(11)$  and the principal 7-block  $B_1(7)$ . If, in equation (2), we take  $x_1 = x_2 = \nu$ , where  $\nu$  is an element of order 7 and  $x_3$  is an element of order 11, then we have that  $|C_G(\nu)| \mid 2.3.7$ . Hence by Sylow theorems, we have

LEMMA 1.2. *The Sylow 7-normalizer of  $G$  is a Frobenius group of order 42.*

We now get the following possibilities for equation (1) for  $B_1(11)$  and  $B_1(7)$ :

*Possibilities for equation (1) for  $B_1(11)$ :*

$$(I) \quad 1 - 3200 - 175 + 1750 + 2520 - 896^* = 0.$$

$$(II) \quad 1 - 3200 - 175 + 1750 + 1750 - 126^* = 0.$$

(In case (I), the two 11-conjugate characters have the common degree 896 and in case (II), they have the common degree 126).

*Possibilities for equation (1) for  $B_1(7)$ :*

$$(A) \quad 1 + 3200 - 825 - 2750 - 1056 + 22 + 1408 = 0.$$

$$(B) \quad 1 + 3200 - 825 - 2750 + 22 + 22 + 330 = 0.$$

$$(C) \quad 1 + 3200 - 825 - 2750 - 55 + 330 + 99 = 0.$$

$$(D) \quad 1 + 3200 - 825 - 3520 - 1056 + 1408 + 792 = 0.$$

Now let  $\chi_2$  be the (unique) character of  $G$  with degree 3200. Then  $\chi_2$  lies in a 5-block  $B_2(5)$  say, of defect 1. If  $D$  is the defect group of  $B_2(5)$ , then  $D$  has order 5 and  $0_5(C_G(D)) = D$  (see [7]).

Suppose we have case (II) for  $B_1(11)$ . If we put  $\chi_3(1) = 175$ ,  $\chi_4(1) = \chi_5(1) = 1750$ ,  $\chi_6(1) = \bar{\chi}_6(1) = 126$  ( $\bar{\chi}_6$  is the complex conjugate of  $\chi_6$ ), we see that  $\chi_4(d) = \chi_5(d) = 0$ ,  $\chi_2(d) = -\chi_3(d)$  and hence  $\chi_6(d) = \overline{\chi_6(d)} = 1$  for any element  $d$  in  $G$ ,  $d$  of order 5. Now take  $d \in D$ , and since  $\chi_2(d) \neq 0$ , we have  $\chi_2(d) = -\chi_3(d) \equiv 0 \pmod{5}$ .

We now use Result 2 and put  $d = x_1 = x_2$  and take  $x_3$  to be an element of order 11. Since the left-hand side of (2) is nonnegative,  $|\chi_2(d)| \leq 10$ , and we get that in all cases,  $|C_G(d)| \mid 2^2.5^2$ . As  $\langle d \rangle = D$ , we have that  $0_5(C_G(D)) > D$ , a contradiction. Hence, case (II) for  $B_1(11)$  is not possible.

So, throughout the rest of the paper, we are in case (I) for  $B_1(11)$  and we put

$$\chi_2(1) = 3200, \chi_3(1) = 175, \chi_4(1) = 1750, \chi_5(1) = 2520$$

and  $\chi_6(1) = \bar{\chi}_6(1) = 896$ .

Since all the characters of  $B_1(11)$  are rational-valued, except  $\chi_6$  and  $\bar{\chi}_6$  which are rational-valued only on 11-regular elements, we obtain

a congruence modulo 5 for  $\chi(x)$ ,  $\chi \in B_1(11)$  and  $x$  any element of order 5. Put  $x = x_1 = x_2$  and  $x_3 = s$ , where  $s$  is an element of order 11, and equation (2) becomes

$$\begin{aligned} a(x, x; s) |C_G(x)|^2 &= |G| \sum_{i=1}^6 \frac{(\chi_i(x))^2 \overline{\chi_i(s)}}{\chi_i(1)} \\ &= |G| \left( 1 - \frac{(\chi_2(x))^2}{3200} - \frac{(\chi_3(x))^2}{175} + \frac{(\chi_5(x))^2}{2520} - \frac{(x_6(x))^2}{896} \right) \\ &= 2^2 \cdot 5^2 \cdot 11(40336 - 243\alpha^2 - 29\gamma^2 - 32\gamma), \end{aligned}$$

where  $\alpha = \chi_2(x) = -\chi_3(x)$ ,  $\gamma = \chi_6(x)$ . Note that  $\chi_5(x) = \chi_6(x) - 1 = \gamma - 1$  by the orthogonality relation. Since the left-hand side of the above formula is nonnegative, we see that

$|\alpha| < 15$  and so  $\alpha = 0, \pm 5$ , or  $+10$ , and if

$$\begin{aligned} \alpha = 0, & \quad -34 \leq \gamma \leq 36 \\ \alpha = \pm 5, & \quad -34 \leq \gamma \leq 31 \\ \alpha = \pm 10, & \quad -24 \leq \gamma \leq 21. \end{aligned}$$

By the orthogonality relation,  $1 + 5\alpha^2 + \gamma^2 + (\gamma - 1)^2 \leq |C_G(x)|$ , and because  $(|C_G(x)|, 77) = 1$ , we get the following possibilities for any element  $x$ , of order 5, in  $G$ :

- (i)  $\alpha = 0, \gamma = -4, |C_G(x)| \mid 2^4 \cdot 5^3$
- (ii)  $\alpha = 0, \gamma = 1, |C_G(x)| \mid 2 \cdot 3 \cdot 5^2$
- (iii)  $\alpha = \pm 5, \gamma = 1, |C_G(x)| \mid 2^2 \cdot 3 \cdot 5^2$ .

If  $\langle d \rangle = D$  is the defect group of the 5-block  $B_2(5)$  (of defect 1) which contains  $\chi_2$ , then  $\chi_2(d) = \alpha \neq 0$ , and so  $\alpha = \pm 5$  and  $|C_G(d)| \mid 2^2 \cdot 3 \cdot 5^2$ . As  $0_5(C_G(\langle d \rangle)) = D$ , it is immediate that  $C_G(d)/D \cong A_5$ . It follows now that  $C_G(G_5) = Z(G_5)$ , and that  $G_5$  is nonabelian and by a result of B. Huppert ([10], S. 8.6),  $G_5$  is of exponent 5. If  $Z(G_5) = \langle \rho \rangle$ , where  $\rho^5 = 1$ , then  $|C_G(\rho)| \mid 2^4 \cdot 5^3$ , and as  $\langle \rho \rangle \triangleleft N_G(G_5)$ , we have by Sylow theorems  $|N_G(G_5)| = 5^3$  or  $2^4 \cdot 5^3$ . The first possibility is impossible by a theorem of H. Wielandt ([10], S. 8.1), and so  $N_G(G_5) = S \cdot G_5$  where  $S$  is a 2-group of order 16. Since  $Z(S)$  is a cyclic group of order 4, we have that  $N_G(\langle d \rangle) \cong F_{20} \times A_5$  where  $F_{20}$  is a Frobenius group of order 20, and so  $d$  is conjugate to all its powers.

We shall now proceed to rule out possibilities (B), (C) and (D) for  $B_1(7)$ . In the cases (B) and (C), we put  $x_1 = x_2 = d$  and  $x_3 = \nu$ , where  $\nu^7 = 1$ , in equation (2). In both these cases, there are only a small number of possibilities for the values of the characters in  $B_1(7)$  on the element  $d$ , and in all cases we get that  $|C_G(d)| < 2^2 \cdot 3 \cdot 5^2$ , a contradiction; so cases (B) and (C) are not possible for  $B_1(7)$ . Case (D) is immediately ruled out by summing the squares of the degrees so far determined,

(noting that the 5-block of defect 1,  $B_2(5)$  which contains  $\chi_2$  has the following degrees:  $B_2(5) = \{3200, 175, 825, 1925, 1925\}$ ); as this sum is 45, 496,  $297 > |G|$ .

Hence, for the rest of the paper we are in case (A) for  $B_1(7)$  and we put  $B_3(3)$  to be the 3-block of defect 1 containing the characters  $\chi_7$  and  $\chi_9$  of degrees 825 and 1056 respectively. Then

$$B_3(3) = \{825, 1056, 231\} .$$

Let  $B_2(5)$  be the 5-block of defect 1 containing  $\chi_2$ ,  $\chi_3$ , and  $\chi_7$ . Then

$$B_2(5) = \{3200, 175, 825, 1925, 1925\} .$$

Further, we put  $\chi_8(1) = 2750$ ,  $\chi_{10}(1) = 22$  and  $\chi_{11}(1) = 1408$ . Then using Result 2 for  $x_1 = x_2 = \rho$  and  $x_3 = \nu$ , where  $\langle \rho \rangle = Z(G_5)$  and  $\nu^7 = 1$ , we again get a few possibilities for  $\chi(\rho)$ ,  $\chi \in B_1(7)$ , and this gives that  $|C_G(\rho)| \mid 2^3 \cdot 5^3$ . Hence  $C_G(\rho)$  is precisely of order  $4 \cdot 5^3$  and  $C_G(\rho)$  is a semi-direct product of a cyclic group of order 4 and  $G_5$ . It follows immediately that the Sylow 2-subgroup  $S$  of  $N_G(G_5)$  is a quasi-dihedral group of order 16 (i.e.,  $S = \langle a, b \mid a^8 = 1 = b^2, bab = a^5 \rangle$ ). Now let  $l \in G_5 \setminus \langle \rho \rangle$ , and  $l \not\sim_{N_G(G_5)} d$ , then  $l$  has precisely 80 conjugates in  $N_G(G_5)$ , and so  $C_G(l) \cap N_G(\langle \rho \rangle) = \langle l \rangle \times \langle \rho \rangle$ . Summing the character  $\chi_6$  on  $G_5$ , we see that  $\chi_6(l) = 1$  and as  $C_G(\rho) \cap C_G(l) = \langle l \rangle \times \langle \rho \rangle$ , and  $|C_G(l)| \mid 2 \cdot 3 \cdot 5^3$ , we have  $|C_G(l)| = 5^3$ , i.e.,  $C_G(l) = \langle l \rangle \times \langle \rho \rangle$ . In particular  $l \not\sim_G d$ . The 5-structure of  $G$  is now completely determined and we summarize these results in the following lemma:

**LEMMA 1.3.** *The group  $G$  has precisely 3 conjugate classes of elements of order 5 with representatives  $d, \rho$  and  $l$ . A Sylow 5-subgroup  $G_5$  of  $G$  is nonabelian of exponent 5 and  $N_G(G_5)$  is a semi-direct product of  $G_5$  and a quasi-dihedral group of order 16. Also,  $Z(G_5) = \langle \rho \rangle$  and  $\rho$  has centralizer of order  $4 \cdot 5^3$ ,  $|C_G(l)| = 5^3$ , and  $C_G(d) \cong \langle d \rangle \times A_5$ . Finally,  $N_G(\langle d \rangle)$  is isomorphic to the direct product of a Frobenius group of order 20 by  $A_5$ .*

**2. The 3-structure.** Let  $\langle c \rangle$  be a Sylow 3-subgroup of  $C_G(d)$ . Then

$$\begin{aligned} C_G(c) \cap C_G(d) &= \langle d \rangle \times \langle c \rangle, N_G(\langle c \rangle) \cap C_G(d) \\ &\cong S_3 \times \langle d \rangle, N_G(\langle c \rangle) \cap N_G(\langle d \rangle) \cong S_3 \times F_{20} . \end{aligned}$$

It now follows that  $C_G(c)/\langle c \rangle \cong S_5 \cdot E$ , where  $E = \langle 1 \rangle$  or  $E$  is an elementary 2-group of order 16, and  $E \triangleleft C_G(c)$ . Since  $N_G(\langle c \rangle) > C_G(c)$ ,  $G_3$  is then elementary abelian; and so  $C_G(c) \cong \langle c \rangle \times S_5 \cdot E$ .

Suppose  $|E| = 16$ . Let  $X$  denote a Sylow 2-subgroup of  $N = N_G(\langle c \rangle)$ , and let  $G_2$  be a Sylow 2-subgroup (of  $G$ ) containing  $X$ . If  $E \triangleleft G_2$ , then  $|N_G(E):N| = 2$ , and then  $2^4 \mid |N_G(\langle d \rangle)|$  (by the Frattini

argument), clearly a contradiction. We may suppose therefore that  $N_G(E) = N$ . As  $X/E$  is not elementary, there is an involution  $t \in Z(G_2) \cap E$ . It then follows that  $|C_G(t):C_N(t)| = 2$  or  $10$ , but in either case,  $\langle c \rangle \leq C(0_2(C_G(t)))$ . As  $C(E) \cap N = E \times \langle c \rangle$ ,  $|C_G(G_3)| \mid 2^3 \cdot 3^2$ . Also, if  $0_2 = 0_2(C_G(t))$ , then  $0_2 \leq E$ . If  $\langle c \rangle$  is a Sylow 3-subgroup of  $C(0_2)$ , we have a contradiction by the Frattini argument. So a Sylow 3-subgroup  $G_3$  of  $C(0_2)$  is of order 9. By the Frattini argument,  $5 \nmid |C_G(t)$ . So  $C_G(t)$  is a soluble group of order  $2^9 \cdot 3^2$ . From the structure of  $C_G(c)$ , we must have  $|0_2| = 4$  but then  $G_3 \triangleleft C_G(t)$ , which contradicts the fact that  $|C_G(G_3)| \mid 2^3 \cdot 3^2$ . We have proved:

**LEMMA 2.1.** *If  $\langle c \rangle$  is a Sylow 3-subgroup of  $C_G(d)$ , then  $C_G(c) \cong \langle c \rangle \times S_5$  and  $N_G(\langle c \rangle) \cong S_3 \times S_5$ .*

Put  $N_G(\langle c \rangle) = A \times B$ , where  $A \cong S_3$  and  $B \cong S_5$ . Let  $\pi$  be an involution in  $A$ . Since  $A$  is a maximal subgroup of  $V$ , where  $V \cong A_5$  and  $C_G(d) = \langle d \rangle \times V$ , it follows that if  $\langle \lambda \rangle$  is a Sylow 3-subgroup of  $B$ , then  $C_G(\lambda) \cap C_G(d) = A$ . We have  $C_G(\pi) \cap N_G(\langle d \rangle) = W \times \langle \pi, \tau \rangle$ , where  $W \cong F_{20}$  and  $\langle \pi, \tau \rangle$  is a 4-group. It now follows by Sylow that  $|C_G(\pi)| = 2^5 \cdot 3 \cdot 5$ ,  $2^9 \cdot 3 \cdot 5$  or  $2^6 \cdot 3^2 \cdot 5$ . In the first case,  $\langle \pi, \tau \rangle \triangleleft C_G(\pi)$ , which contradicts  $C_G(d) \cap C_G(\lambda)$ .

If  $|C_G(\pi)| = 2^9 \cdot 3 \cdot 5$ , then  $0_2 = 0_2(C_G(\pi))$  is elementary abelian of order 64 and  $C_G(\pi)/0_2 \cong S_5$ . (If  $0_2$  were nonabelian, then  $C_G(d) \cap C_G(\lambda) \neq \langle \pi \rangle$ .) Let  $f$  be an element of order 3 in  $C_G(d)$  such that  $f \in N_G(\langle \pi, \tau \rangle)$ . Now  $N_G(\langle \pi, \tau \rangle) \geq 0_2 \cdot W$ , and  $N_G(\langle \pi, \tau \rangle) \geq \langle f \rangle$ . Since  $C_G(\langle \pi, \tau \rangle) \leq C_G(\pi)$ , we have that  $0_2 = 0_2(C_G(\langle \pi, \tau \rangle))$ , and hence  $N_G(0_2) > C_G(\pi)$ . However, from Lemma 1.1 and the fact that  $A_7$  has no elements of order 15, we have that  $|N_G(0_2)| = 2^9 \cdot 3^2 \cdot 5$ ; thus  $N_G(0_2)/0_2 \cong \langle f \rangle \cdot S_5$  where  $0_2 \cdot \langle f \rangle$  is a normal subgroup of  $N_G(0_2)$ , again contradicting the structure of  $C_G(d) \cap C_G(\lambda)$ .

The order of  $C_G(\pi)$  is thus  $2^6 \cdot 3^2 \cdot 5$  and it follows that  $C_G(\pi)/\langle \pi \rangle \cong \text{Aut}(A_6)$ . Hence  $N_G(G_3)/C_G(G_3)$  is a semi-dihedral group of order 16 and hence all elements of order 3 are conjugate in  $G$ .

Let  $z$  be the unique involution of a Sylow 2-subgroup  $Z$  (of order 4) of  $C_G(\rho)$  where  $\langle \rho \rangle = Z(G_5)$ . Since  $\langle \rho \rangle \not\sim_G \langle d \rangle$ ,  $z \not\sim_G \pi$ , and so  $|C_G(z)| = 2^5 \cdot 3 \cdot 5$  or  $2^9 \cdot 3 \cdot 5$ . In the first case  $C_G(z)/Z \cong S_5$ , but since the Sylow 2-subgroup of  $N_G(\langle \rho \rangle)$  is quasi-dihedral,  $z \in \mathfrak{U}^2(X)$ , where  $X$  is a Sylow 2-subgroup of  $C_G(z)$ . However as  $X/Z \cong D_8$ , where  $D_8$  is a dihedral group of order 8,  $\mathfrak{U}^2(X) \leq Z$  which gives a contradiction. Hence  $|C_G(z)| = 2^9 \cdot 3 \cdot 5$  and if  $E = 0_2(C_G(z))$ ,  $E$  is a 2-group of order 64 and  $C_G(z)/E \cong S_5$ . Because  $C_G(\rho) \cap C_G(c) \cap C_G(z) = Z \cdot \langle \rho \rangle$ ,  $Z \leq E$  and  $Z \triangleleft C_G(z) = C$ .

As  $\pi \in V$ , where  $C_G(d) = \langle d \rangle \times V$  and  $V \cong A_5$ ,  $\pi \notin E$ , but  $\pi \in V \cdot E$ , where  $V \cdot E/E \cong A_5$ . In any case,  $N_G(\langle c \rangle) = A \times F$ , where  $A \cong S_3$  and

$F \cong D_8$ , with  $F \cap E = Z$ . Thus  $C_c(\pi) \cdot E$  is a Sylow 2-subgroup of  $C = C_c(z)$ , and hence  $|C_E(\pi)| \leq 8$ .

If  $E$  is abelian, it is of type (4, 2, 2, 2, 2) and

$$C_c(c) \cap E = C_c(\rho) \cap E = Z. \text{ Also, } |C_c(\pi) \cap \Omega_1(E)| \geq 8,$$

and as  $C_c(\pi) \geq Z$ , we have a contradiction. Hence  $E$  is nonabelian and thus  $E$  is a central product of two quaternion groups  $Q_1$  and  $Q_2$ , and the cyclic group  $Z$  of order 4. We have proved:

LEMMA 2.2. *The group  $G$  has only one conjugate class of elements of order 3,  $C_c(G_3) = \langle \pi \rangle \times G_3$ , where  $\pi$  is an involution, and  $N_c(G_3)/C_c(G_3)$  is a semidihedral group of order 16. If  $z$  is the involution in  $C_c(\rho)$ , where  $\langle \rho \rangle = Z(G_5)$ , then  $C = C_c(z)$  is an extension of a nonabelian 2-group  $E$  of order 64, (which is a central product of two quaternion groups and a cyclic group of order four) by the symmetric group  $S_5$  on 5 letters. Finally,  $C_c(\pi)/\langle \pi \rangle \cong \text{Aut}(A_5)$ .*

3. Determination of all degrees of irreducible characters of  $G$ . We are now in a position to apply the exceptional character theory to the group  $G$  with respect to the subgroup  $H = N_c(\langle c \rangle)$ , where  $c$  is any element of order 3. As "special classes" (in the sense of Wong [15]), we take all roots of  $c$ . As  $H \cong S_3 \times S_5$ , the character table of  $H$  is determined from the character tables of  $S_3$  and  $S_5$ . Put  $H = A \times B$ , where  $A \cong S_3$  and  $B \cong S_5$ .

In the above notation, our special classes of  $H$  are the conjugate classes in  $H$  with representatives  $c, ct, cz, cw$  and  $cd$ . As usual,  $B_c(3)$  will denote a 3-block of  $G$  and  $b_c(3)$ , a 3-block of  $H$ . The group  $H$

Character Table of  $A \cong S_3$

Order	Element	$\theta_1$	$\theta_2$	$\theta_3$
1	1	1	1	2
2	$\pi$	1	-1	0
3	$c$	1	1	-1

Character Table of  $B \cong S_5$

Order	Element	$\zeta_1$	$\zeta_2$	$\zeta_3$	$\zeta_4$	$\zeta_5$	$\zeta_6$	$\zeta_7$
1	1	1	1	4	4	6	5	5
2	$z$	1	1	0	0	-2	1	1
2	$t$	1	-1	-2	2	0	-1	1
4	$w$	1	-1	0	0	0	1	-1
3	$\lambda$	1	1	1	1	0	-1	-1
6	$\lambda t$	1	-1	1	-1	0	-1	1
5	$d$	1	1	-1	-1	1	0	0



has three 3-blocks:

$$\begin{aligned} b_1(3) &= \{1_H, \theta_2, \theta_3, \zeta_B, \zeta_3\theta_2, \zeta_3\theta_3, \zeta_6, \zeta_6\theta_2, \zeta_6\theta_3\} \\ b_2(3) &= \{\zeta_2, \zeta_2\theta_2, \zeta_2\theta_3, \zeta_4, \zeta_4\theta_2, \zeta_4\theta_3, \zeta_7, \zeta_7\theta_2, \zeta_7\theta_3\} \\ b_3(3) &= \{\zeta_5, \zeta_5\theta_2, \zeta_5\theta_3\} . \end{aligned}$$

Here  $1_H = \theta_1 \cdot \zeta_1$ , and  $b_1(3), b_2(3)$  are the 3-blocks of defect 2 of  $H$ ; and  $b_3(3)$  the unique 3-block of defect 1 of  $H$ . We denote by  $b_i(3)^G$ , the block of  $G$  which corresponds to the block  $b_i(3)$  of  $H$ , using Brauer's block correspondence (see [4]). By [15], Theorem 6 (or [4], S.2E)

$$\begin{aligned} b_1(3)^G &= B_1(3) \\ b_2(3)^G &= B_2(3) \\ b_3(3)^G &= B_3(3) = \{825, 1056, 231\} , \end{aligned}$$

where  $B_3(3)$  is the only other 3-block of defect 2 of  $G$  besides the principal 3-block  $B_1(3)$ . If  $D$  denotes the union of special classes of  $H$ , we take the following basis for the module of all generalized characters of  $H$  which vanish on  $H \setminus D$ :

$$\begin{aligned} \varphi_1 &= (1_H - \zeta_3)\Sigma \\ \varphi_2 &= (1_H + \zeta_6)\Sigma , \\ \varphi_3 &= (\zeta_2 - \zeta_4)\Sigma , \\ \varphi_4 &= (\zeta_2 + \zeta_7)\Sigma , \end{aligned}$$

and

$$\varphi_5 = \zeta_5 \Sigma ,$$

where

$$\Sigma = 1_H + \theta_2 - \theta_3 .$$

Note that  $\varphi_1$  and  $\varphi_2$  are expressed as a linear combination of irreducible characters occurring only in  $b_1(3)$ . Similarly  $\varphi_3$  and  $\varphi_4$ ; and  $\varphi_5$  are expressed only as a linear combination of characters occurring in  $b_2(3)$ , and  $b_3(3)$  respectively. Let  $\varphi_i^*$  denote the corresponding induced characters of  $\varphi_i (i = 1, 2, \dots, 5)$ .

The induced characters  $\varphi_1^*, \varphi_2^*$  can be expressed as a linear combination of the irreducible characters of  $B_1(3)$  and if  $\chi \in B_1(3)$ , then  $\chi$  appears as a constituent of  $\varphi_1^*$  or  $\varphi_2^*$  (see [15], Ths. 7 and 9). Similar statements can be made for  $\varphi_3^*, \varphi_4^*$  and  $B_2(3)$ , and  $\varphi_5^*$  and  $B_3(3)$ .

Finally, if  $\chi$  is any irreducible character of  $G$ , let  $n_i = \langle \chi, \varphi_i^* \rangle$ , ( $i = 1, 2, \dots, 5$ ). Then

$$(3) \quad \begin{aligned} \chi(\sigma) &= -n_1\zeta_3(\sigma) + n_2\zeta_6(\sigma) - n_3\zeta_4(\sigma) + n_4\zeta_7(\sigma) \\ &\quad + n_5\zeta_5(\sigma), \text{ for any } \sigma \in D . \end{aligned}$$

Since  $(\varphi_i^*, \varphi_j^*)_G = (\varphi_i, \varphi_j)_H$  we have:

$$\begin{aligned} (\varphi_1^*, \varphi_1^*) &= (\varphi_2^*, \varphi_2^*) = (\varphi_3^*, \varphi_3^*) = (\varphi_4^*, \varphi_4^*) = 6, \\ (\varphi_1^*, \varphi_2^*) &= (\varphi_3^*, \varphi_4^*) = 3, \\ (\varphi_1^*, \varphi_3^*) &= (\varphi_2^*, \varphi_3^*) = (\varphi_1^*, \varphi_4^*) = (\varphi_2^*, \varphi_4^*) = 0, \\ (\varphi_5^*, \varphi_5^*) &= 3 \end{aligned}$$

and  $(\varphi_i^*, \varphi_i^*) = 0$  for  $i = 1, 2, 3, 4$ .

Further, by the Frobenius reciprocity law,

$$(\varphi_1^*, 1_G) = (\varphi_2^*, 1_G) = 1 \quad \text{and} \quad (\varphi_3^*, 1_G) = (\varphi_4^*, 1_G) = 0.$$

From these values, it follows that

$$(4) \quad \begin{cases} \varphi_1^* = 1_G + \sum_{i=1}^5 \varepsilon_i X_i \\ \text{and} \\ \varphi_2^* = 1_G + \varepsilon_1 X_1 + \varepsilon_2 X_2 + \eta_1 Y_1 + \eta_2 Y_2 + \eta_3 Y_3 \end{cases}$$

where  $X_i$  and  $Y_i$  are distinct non-principal irreducible characters of  $G$ .

So far, the degrees of 15 irreducible characters of  $G$  have been determined:

$\chi_1 = 1_G$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\bar{\chi}_6 = \chi_7$	$\chi_8$	$\chi_9$
1	3200	175	1750	2520	896	896	825	1056

$\chi_{10}$	$\chi_{11}$	$\chi_{12}$	$\chi_{13}$	$\chi_{14}$	$\chi_{15}$
2750	22	1408	231	1925	1925

Using (3) and the fact that  $(\chi |_{G_3}, 1_{G_3})$  is an integer where  $G_3$  is a Sylow 3-subgroup of  $G$ , we get  $\chi_{13}(c) = 6$  and so

$$\varphi_5^* = \chi_{13} + \chi_8 - \chi_9.$$

It follows that  $\chi_2(cd) = \chi_8(cd) = -\chi_3(cd) = -\chi_{14}(cd) = -\chi_{15}(cd) = 1$ . Further,  $\chi_{11}(c) = 4$  and hence, if  $\chi_{11} \in B_1(3)$  then  $(\varphi_1^* \cdot \chi_{11}) = -1$  and  $(\varphi_2^*, \chi_{11}) = 0$ . By obtaining a congruence modulo 9 for the above characters on the element  $c$ , we see that if  $\chi_i \in B_1(3)$ , then  $\chi_i$  occurs in precisely one of  $\varphi_1^*$  or  $\varphi_2^*$  if  $i = 2, 3, 4, 6, 7, 10, 11$  and 12; and in both  $\varphi_1^*$  and  $\varphi_2^*$  if  $i = 1, 14$  and 15, using the block-intersection lemma of Brauer-Tuan ([6], Lemma 3), and since  $\chi_2(cd) = 1$ , we have the following possibilities for  $B_1(3), B_2(3)$ :

- (a)  $B_1(3) = \{1, 3200, 1750, 22, 1408, 2750, \dots\}$ ,  
 $B_2(3) = \{175, 896, 896, \dots\}$
- (b)  $B_1(3) = \{1, 3200, 175, 1750, 896, 896, 22, 1408, 2750\}$ ,

- $$B_2(3) = \{1925, \dots\}$$
- (c)  $B_1(3) = \{1, 1750, 896, 896, 2750, 1408, \dots\},$   
 $B_2(3) = \{3200, 175, 22, \dots\}$
- (d)  $B_1(3) = [1, 1750, 896, 896, 2750, 22, \dots],$   
 $B_2(2) = \{3200, 175, 1408, \dots\}.$

Note that  $\chi_6$  and  $\chi_7$  both lie in the same block, and

$$(\varphi_i^*, \chi_6) = (\varphi_i^*, \chi_7), \quad (i = 1, \dots, 4)$$

as  $\varphi_i^*$  is rational-valued. Further,  $\chi_4(cd) = \chi_{10}(cd) = 0$  and since both  $\chi_4, \chi_{10}$  always lie in  $B_1(3)$ ,  $\chi_4$  and  $\chi_{10}$  must be constituents of  $\varphi_2^*$ ; and as  $\chi_6(cd) = \chi_7(cd) = 1$ ,  $\chi_6$  and  $\chi_7$  either both occur in  $\varphi_1^*$  or both in  $\varphi_3^*$ .

By summing the squares of the degrees so far determined, any other irreducible character of  $G$  must have degree  $77k$ , where  $k \leq 20$ . In particular, if  $\chi \in B_1(3)$  or  $B_2(3)$ , then  $(k, 3) = 1$  and  $k \leq 20$ .

Case (b) for  $B_1(3)$  and  $B_2(3)$  is immediately ruled out using (4).

In case (c) for  $B_1(3)$  and  $B_2(3)$ , first assume that neither  $\chi_{14}$  nor  $\chi_{15}$  appears in  $\varphi_1^*$ . Then  $\varphi_3^*(1) = 3200 - 175 - 22 - 1925 + 847 = 0$ . But  $847 = 7 \times 11^2$  which is not possible. Hence either  $\chi_{14}$  or  $\chi_{15}$  appears in  $\varphi_1^*$  with nonzero multiplicity and we have:

$$\begin{aligned} \varphi_1^*(1) &= 1 + 896 + 896 - 1408 - 1925 + 1540 = 0, \\ \varphi_2^*(1) &= 1 - 1705 + 2750 - 616 - 1925 + 1540 = 0, \\ \varphi_3^*(1) &= 3200 - 175 - 22 - 1925 + \delta_1 z_1 + \delta_2 z_2 = 0, \end{aligned}$$

and

$$\varphi_4^*(1) = -1925 + \delta_1 z_1 + \delta_2 z_2 + \sum_{i=1}^3 \kappa_i y_i = 0,$$

where  $\delta_1, \delta_2, \kappa_i (i = 1, 2, 3)$  are equal to  $\pm 1$  and  $z_1, z_2, y_1, y_2, y_3$  are degrees of irreducible characters of  $G$ . No matter what values the  $z_i$  takes, at least two of the  $y_i$  take the value 1232 or one of them takes the value 1540. In any case, the sum of the squares of the degrees so far determined is greater than  $|G|$ .

In case (d) for  $B_1(3)$  and  $B_2(3)$ , we have;

$$\begin{aligned} \varphi_1^*(1) &= 1 + 896 + 896 - 22 - 1925 + 154 = 0, \\ \varphi_2^*(1) &= 1 - 1750 + 2750 + 770 - 1925 + 154 = 0, \\ \varphi_3^*(1) &= 3200 - 175 - 1408 - 1925 + \sum_{i=1}^2 \delta_i z_i = 0, \end{aligned}$$

and

$$\varphi_4^*(1) = -1925 + \sum_{i=1}^2 \delta_i z_i + \sum_{i=1}^3 \kappa_i y_i = 0,$$

in the same notation as above. We may take  $(\varphi_1^*, \chi_{14}) = -1$ , and then

on the element  $ct$  of order 6, we get:

$$\chi_2(ct) = -2, \chi_3(ct) = 2, \chi_8(ct) = 0, \chi_{14}(ct) = -1$$

and  $\chi_{15}(ct) = 1$ . Summing over the 5-block of defect 1,  $B_2(5)$ , and using [1], Corollary 4, we get

$$-2 + 0 = -2 \neq 2 + 1 - 1 = 2,$$

and a contradiction and so case (a) is the only possibility for  $B_1(3)$  and  $B_2(3)$ .

In case (a), we have:

$$\varphi_1^*(1) = 1 + 3200 - 22 - 1408 - 1925 + 154 = 0,$$

$$\varphi_2^*(1) = 1 - 1750 + 2750 + 770 - 1925 + 154 = 0,$$

$$\varphi_3^*(1) = 896 + 896 - 175 - 1925 + \sum_{i=1}^2 \delta_i z_i = 0,$$

and

$$\varphi_4^*(1) = -1925 + \sum_{i=1}^2 \delta_i z_i + \sum_{j=1}^3 \kappa_j y_j = 0.$$

Thus  $\sum_{i=1}^2 \delta_i z_i = 308 = 4 \times 77$  and  $\sum_{j=1}^3 \kappa_j y_j = 1617 = 21 \times 77$ , which gives a number of possibilities for  $z_i$  and  $y_j$ . However, if

$$\chi \in B_2(3) \cup B_2(3) \cup B_3(3), \text{ then } 9 \times 77 \mid \chi(1).$$

By summing the squares of degrees and using this fact, we get a unique decomposition for  $\varphi_3^*$  and  $\varphi_4^*$ :

$$\varphi_3^*(1) = 896 + 896 - 175 - 1925 + 154 + 154 = 0,$$

and

$$\varphi_4^*(1) = 770 + 770 + 77 - 1925 + 154 + 154 = 0.$$

There are now either 2 or 5 irreducible characters left to be determined (in the first case  $G$  would have one irreducible character of degree 693 and one of degree 1386; while in the second possibility  $G$  would have five irreducible characters of degree 693); and so  $G$  has either 24 or 27 irreducible characters. Using the orthogonality relations for the element  $l$  of order 5 (see Lemma 1.3) and with centralizer,  $C_G(l)$  of order 25, it follows immediately that  $G$  has only 24 irreducible characters and hence 24 conjugate classes of elements. The character table of  $G$  can now be completed, except for some classes of 2-elements which have not as yet been determined.

The partially completed character table of  $G$  shows that the characters  $\chi_6$  and  $\chi_7$  vanish on all 2-elements except the involution  $\pi$

(defined in Lemma 2.2). A result of Frobenius and Schur (see [8], (3.5)) shows that  $\pi$  is not the square of any element of order four in  $G$ .

**4. Completion of proof of theorem.** Let  $z$  be the central involution with centralizer  $C_c(z) = C$ , as in Lemma 2.2. Then  $C/0_2(C) \cong S_3$ , and  $0_2(C)$  is a central product of two quaternion groups  $Q_1$  and  $Q_2$  and a cyclic group  $Z$  of order four. We may take  $\langle c \rangle$  and  $P = \langle \rho \rangle$  to be a Sylow 3- and Sylow 5-subgroup of  $C$  respectively (for definitions of  $\langle c \rangle$  and  $\langle \rho \rangle$ , see Lemmas 1.3 and 2.1). Let  $E = 0_2(C)$ , then

$$C_c(c) \cap E = C(P) \cap E = Z, |N_c(P)| = 2^4 \cdot 5$$

with  $N_c(P)$  having as a Sylow 2-subgroup a quasidihedral group of order 16; and  $N_c(\langle c \rangle) \cong S_3 \times D_8$ , where  $D_8$  is a dihedral group of order 8. The action of  $P$  and  $\langle c \rangle$  on  $E$  shows that  $|C:C'| = 2$  and hence  $C'/E \cong A_5$ . We may suppose that the involution  $\pi$  lies in  $C' \setminus E$ . From the structure of  $C_c(c)$ , if  $Z = \langle \omega \rangle$ , where  $\omega^4 = 1$ , then  $C_c(\omega) = C'$ . Further, if  $u$  is any element of order four in  $E \setminus Z$ , then  $|C_c(u)| = 2^8$ ; as  $u$  must have exactly 30 conjugates in  $C$  and as  $\mathfrak{O}'(E) = \langle z \rangle$ . Similarly, if  $t$  is an involution in  $E \setminus Z$ , then  $|C_c(t)| = 2^8$ .

So far, we have determined the order of 20 of the 24 conjugate classes of elements of  $G$ . If  $K_1, K_2, K_3$  and  $K_4$  denote the remaining four classes of 2-elements of  $G$ , then using the previous lemmas and summing the order of the conjugate classes so far determined, we have the following possibilities:

	$ C_c(x_1) $	$ C_c(x_2) $	$ C_c(x_3) $	$ C_c(x_4) $
(1)	$2^7$	$2^3$	$2^4$	$2^7$
(2)	$2^6$	$2^4$	$2^4$	$2^4$
(3)	$2^6$	$2^3$	$2^5$	$2^5$

where  $x_i \in K_i, i = 1, 2, 3, 4$ .

It now follows that for  $t \in E \setminus Z, t$  an involution, then  $t \underset{G}{\sim} z$ . If  $x$  is an involution in  $C' \setminus E$ , then by Sylow theorems, we may take  $x \in N(P) \cap C'$ . However the Sylow 2-subgroup of  $N_c(P)$  contains only two involutions  $x$  and  $xz$  which are conjugate in  $N_c(P)$ , and so  $x \underset{G}{\sim} \pi$ ; and hence  $C' \setminus E$  has only one class of involutions. A transfer theorem of J. G. Thompson ([13], Lemma 5.38) now shows that  $G$  has precisely two conjugate classes of involutions with representatives  $z$  and  $\pi$ .

As before, we may take  $\pi \in N_c(\langle c \rangle) \setminus E \cap N(\langle c \rangle)$ . Put  $N_c(\langle c \rangle) = \langle c \rangle \cdot \langle \pi \rangle \times K$ , where  $K$  is a dihedral group of order 8. Denote the involutions of  $K$  by  $z, \tau, \tau z, \tau \omega^{-1}$  and  $\tau \omega$ , where  $Z = \langle \omega \rangle$  is as above and  $Z$  is the unique cyclic group of order four in  $K$ . From the structure of  $C_c(c)$ , we may take  $\tau \underset{G}{\sim} \tau z \underset{G}{\sim} z$  and  $\pi \underset{G}{\sim} \tau \omega \underset{G}{\sim} \tau \omega^{-1}$ . Further,  $[\langle c \rangle, E]$  is an extra-special 2-group of order 32 and we write  $[\langle c \rangle, E] = Q_1 \vee Q_2$ , where  $Q_1$  and  $Q_2$  are quaternion groups of order 8

and  $Q_1 \wr Q_2$  denotes the central product of  $Q_1$  and  $Q_2$ . Put  $Q_1 = \langle \alpha_1, \alpha_2 \rangle$ ,  $Q_2 = \langle \alpha_3, \alpha_4 \rangle$  ( $\alpha_i, i = 1, \dots, 4$  are elements of order 4) and note that  $\langle \pi \rangle \cdot K \cong N_C(Q_1 \wr Q_2)$ .

As  $|C_E(\pi)| \leq 8$ , we may choose the  $\alpha_i$  in such a way that  $\alpha_1^r = \alpha_1^{-1}$ ,  $\alpha_2^r = \alpha_2^{-1}$ ,  $\alpha_3^r = \alpha_1\alpha_3$  and  $\alpha_4^r = \alpha_2\alpha_4$ . Hence  $C_E(\pi)$  is an abelian group of order 8 and type (4,2), and  $C_E(\pi) = Z \times \langle \alpha_1, \alpha_2 \rangle$ . Similarly, as  $c \in C_G(\tau)$  and  $\tau \in N_C(Q_1 \wr Q_2)$ ; and  $\tau\omega \sim \pi$ , we get  $\alpha_1^r = \alpha_2$  and  $\alpha_3^r = \alpha_4$  (again with appropriate choice of the  $\alpha_i$ ). Since  $\pi$  is not the square of any element of order four, an easy computation now gives that  $\mathcal{U}^1(C_G(\pi)) = \langle \omega\alpha_1\alpha_2\pi \rangle$ . A result of Gaschutz ([10], S. 17.4) shows that  $C_G(\pi) \cong \langle \pi \rangle \times \text{Aut}(A_6)$ .

If we put  $v = \omega\pi$ , then  $v$  is of order four and  $C_C(v) = C_C(\pi)$ . Certainly,  $v^2 = z$  and as  $\alpha_1\tau \in C_C(v)$ , we have that  $|C_C(v)| = |C_G(v)| = 2^6$ . The action of  $\pi$  on  $E$  gives that the coset  $E\pi$  has precisely eight elements of order four whose square is  $z$ , and so  $C' \setminus E$  contains one conjugate class of elements of order four (with representative  $v$ ) whose square is  $z$ . The action of  $\tau$  and  $c$  on  $E$  shows that  $C \setminus C'$  has no element of order four whose square is  $z$ .

*The group  $G$  has therefore precisely three conjugate classes of elements of order four, with representatives  $\omega, u$  and  $v$ .*

The orthogonality relations enable us to complete the character table on all but the last three classes of  $G$ . It can be shown by an easy computation that there are no elements of order 16. In any case, from the orthogonality relations the character  $\chi_{11}$  of degree 22 vanishes on the remaining 3 conjugate classes of 2-elements.

We give below only part of the character table of  $G$ , namely, the value of the irreducible character  $\chi_{11}$  of degree 22 on certain elements. Note that for any element  $x$  of order eight,  $\chi_{11}(x) = 0$ , from the orthogonality relation.

Element (s)	Order	$\chi_{11}$
1	1	22
$c$	3	4
$cz$	6	0
$\rho$	5	-3
$l, d$	5	2
$z$	2	6
$w$	4	-6
$u, v$	4	2
$x$	8	0

Using the same notation as above,  $C_E(\tau) = \langle \alpha_1\alpha_2, \alpha_3\alpha_4, z \rangle$ , an elementary abelian group of order 8. If we put  $F = \langle \tau \rangle \times C_E(\tau)$ , then  $F$  is an elementary abelian group of order 16, and further,  $\langle E, c, \pi \rangle \leq$

$N_c(F)$ . Hence  $F$  contains precisely one class of involutions in  $G$  (since  $\tau \sim_{\mathcal{C}} z$ ) and  $|N_c(F)| = 2^8 \cdot 3$ , as  $E \cap F \triangleleft C$ . Also,  $C_G(F) = C_c(F) = F$ .

The coset  $E\tau$  contains precisely 16 involutions, and so, if  $\kappa$  is an involution in  $C \setminus C'$ , then  $\kappa \sim_{\mathcal{C}} \tau$  or  $\kappa \sim_{\mathcal{C}} \tau\omega$ . Now, let  $J$  be any elementary abelian subgroup of order 16 of  $C$ , such that  $J$  contains only involutions conjugate to  $z$ . Then  $|J \cap E| = 8$ , and if  $\kappa \in J \setminus E$ , then  $\kappa \in C \setminus C'$  and hence  $\kappa \sim_{\mathcal{C}} \tau$ . Thus  $C$  has precisely one class of elementary abelian subgroup of order 16 containing only involutions conjugate to  $z$ . The structure of  $A_8$  gives the following result:

LEMMA 4.1. *If  $F$  is an elementary abelian subgroup of order 16 in  $C$ , and if all involutions in  $F$  are conjugate to  $z$ , then  $N_G(F) \setminus F \cong S_8$ .*

Clearly,  $z$  has 15 conjugates in  $N_G(F)$ , and so  $N_G(F)' = M$  is of index 2 in  $N_G(F)$ , and  $M \setminus F \cong A_8$ . Since  $\langle c \rangle \leq M$  and  $|C_M(c) \cdot F \setminus F| = 9$ ,  $\omega \in M$ . Thus  $|E : M \cap E| = 2^2$  and there must be an involution  $t \in E \cap M \setminus F \cap E$ . Further,  $|C_E(t)| = 4$  and as the normalizer of a Sylow 5-subgroup of  $M$  is dihedral of order 10, there must be precisely one class of involutions in  $M \setminus F$  and they are all conjugate to  $z$  in  $G$ .

If  $\mu$  is an element of order five in  $M$ , then  $N_M(\langle \mu \rangle) = \langle \mu \rangle \times \langle i \rangle$ , for some involution  $i$  in  $M \setminus F$ . Since  $i \sim_{\mathcal{C}} z$ ,  $\mu$  must be conjugate to  $l$  or  $d$  in  $G$ , since if  $\mu \sim_{\mathcal{C}} \rho$ , then  $z \sim_{\mathcal{C}} \pi$ .

Since  $c$  normalizes  $[c, E] \cap M$ , and

$$|[c, E] \cap M| \geq 16, [c, E] = E \cap M.$$

As  $|\langle \pi, \omega \rangle \cap M| = 4$ , and  $\pi, \omega \notin M$ ,  $\langle \pi\omega \rangle = \langle \pi, \omega \rangle \cap M$ , and  $\langle \pi\omega \rangle \cdot [c, E] \cdot F/F \cong D_8$  and is a Sylow 2-subgroup of  $M/F$ . Let  $T/F = \langle \pi\omega \rangle \cdot [c, E] \cdot F/F$ . Since the action of  $\pi, \tau$  on  $E$  has been completely determined above, the action of  $T/F$  on  $F$  is now completely determined. If we regard  $F$  as a vector space with basis  $z, \tau, \alpha_1\alpha_2$  and  $\alpha_3\alpha_4$ , the action of  $T$  on  $F$  is given by:

$$t_0 = \alpha_1\alpha_3\alpha_4z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad t_1 = \alpha_4\alpha_1\alpha_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$t_2 = \pi\omega\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

where  $t_0, t_1$  and  $t_2$  are involutions,  $\langle t_0, t_1, t_2 \rangle \cong D_8$  where  $D_8$  is a dihedral group of order 8 with  $Z(\langle t_0, t_1, t_2 \rangle) = \langle t_0 \rangle$  and  $T = \langle t_0, t_1, t_2 \rangle \cdot F$ . Note that  $(t_1 t_2)^2 = t_0$  and  $|C_M(t_1 t_2)| = 2^8$ , and  $\langle \tau t_1 t_2 \rangle$  of order 8 is self-centralizing in  $M$ . As  $|C_M(t_0):C_M(t_1 t_2)| = 4$ ,  $t_1 t_2 \not\sim \omega$ , also if  $x$  is any element of order four such that  $x^2 \in F$ , then  $x \not\sim \omega$  as  $\omega \notin M$ .

The group  $M$  has therefore 195 involutions (all conjugate to  $z$  in  $G$ ), 1260 elements of order four, all conjugate in  $G$  to either  $u$  or  $v$ , 720 elements of order eight, 800 elements of order three, 480 elements of order six and 2,304 elements of order five. Summing  $\chi_{11}$  over the group  $M$ , we find that  $(\chi_{11}, 1)_M = 2$ . Let  $U = V \cdot G_3$  be a Sylow 3-normalizer in  $M$ , where  $V$  is a cyclic group of order four and  $G_3$  is a Sylow 3-subgroup of  $M$ . By an easy computation,  $(\chi_{11}, 1)_U = 4$ . Now take a quaternion group  $W$  of order eight,  $W \leq N_G(G_3)$ , and  $V < W$ . Since  $C \setminus C'$  contains no elements of order four whose square is  $z$ ,  $W$  contains only elements of order four which are conjugate to  $u$  or  $v$ . If  $L = W \cdot G_3$ , then we find that  $(\chi_{11}, 1)_L = 3$ . Since  $2 + 3 > 4$ ,  $\langle L, M \rangle = H$  is a proper subgroup of  $G$ , and  $|H| = 2^7 \cdot 3^2 \cdot 5 \cdot k$ , where  $k$  is an integer,  $k > 1$ .

If  $N$  is a normal subgroup of  $H$  and  $1 < N \leq M$ , then  $N = F$  or  $N = M$ . Since  $H = N_G(F)$  contradicts the structure of  $S_6$ ,  $|H:M| > 10$ . From the degrees of the irreducible characters of  $G$ , we see that if  $\bar{G}$  is a subgroup of  $G$  with  $|G:\bar{G}| \leq 200$ , then  $|\bar{G}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ ,  $2^9 \cdot 3^2 \cdot 5^3 \cdot 7$  or  $2^9 \cdot 5^2 \cdot 7 \cdot 11$ ; that is  $|G:\bar{G}| = 100, 176$  or  $45$  respectively.

The following are therefore the only possibilities for  $|H|$ : (a)  $2^7 \cdot 3^2 \cdot 5^3$ , (b)  $2^7 \cdot 3^2 \cdot 5^2 \cdot 7$ , (c)  $2^7 \cdot 3^2 \cdot 5 \cdot 11$ , (d)  $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ , (e)  $2^8 \cdot 3^2 \cdot 5 \cdot 11$ , (f)  $2^8 \cdot 3^2 \cdot 5 \cdot 7$ , (g)  $2^9 \cdot 3^2 \cdot 5^2$ , (h)  $2^9 \cdot 3^2 \cdot 5 \cdot 7$ .

In the cases (a) to (d),  $H$  must be a simple group and a result of Z. Janko [11], shows that we are in case (d) and then  $H \cong M_{22}$ . Cases (f) and (g) can be eliminated by Sylow theorems and the structure of the Sylow 7- and 11-normalizers. If we are in case (g), we may take  $z$  to be a central involution in  $H$  and then  $N_H(F) = N_G(F)$ . However, now  $|H:N_H(F)| = 10$  which contradicts the order of  $A_{10}$ . Finally, suppose we are in case (h). If  $H_5$  is a Sylow 5-subgroup of  $H$ , then as  $H$  is simple by the Frattini argument  $|N_G(H_5)| = 2^4 \cdot 5^2$ . This is a contradiction, as no subgroup of order  $5^2$  has normalizer divisible by 16.

We have thus shown that case (d) is the only possibility for the order of  $H$  and so  $H$  is isomorphic to the Mathieu simple group  $M_{22}$ . The degrees of the irreducible characters of  $G$  yield the following result:

**LEMMA 4.2.** *The group  $G$  is a primitive permutation group of degree 100 with stabilizer of a point  $H$  isomorphic to the Mathieu*



*simple group  $M_{22}$  and the orbits of  $H$  are of length 1,22 and 77.*

With Result 3 and Lemma 4.2, the theorem is proved.

The authors would like to thank Professor D. Wales for his kind permission to quote Result 3.

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Received January 8, 1969.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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