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PHILIP BACON

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By a countably compact space we mean a topological space every countable open cover of which contains a finite subcover. It is known that a countably compact space is compact if it is either a Moore space or a paracompact space. In the first section of this note we introduce a class of topological spaces that includes all Moore spaces and all paracompact spaces but includes no space that is countably compact and not compact. In the second section we study the class of those spaces in which closed countably compact subsets are always compact.

1. Property L. According to Michael [13, p. 309] a collection $\boldsymbol{D}$ of subsets of a space X is *cushioned* in a collection $\boldsymbol{E}$ of subsets of X if there is a function $f: \boldsymbol{D} \rightarrow \boldsymbol{E}$ such that, for any subcollection $\boldsymbol{G}$ of $\boldsymbol{D}$, $(\bigcup \boldsymbol{G})^- \subset \bigcup (f\boldsymbol{G})$. We shall say that $\boldsymbol{D}$ is *weakly cushioned* in $\boldsymbol{E}$ if there is a function $f: \boldsymbol{D} \rightarrow \boldsymbol{E}$ such that, if $\boldsymbol{G}$ is a countable subcollection of $\boldsymbol{D}$ and, for each G in $\boldsymbol{G}$, $x(G)$ is a point of G, then $\{x(G): G \in \boldsymbol{G}\}^- \subset \bigcup (f\boldsymbol{G})$. If $\boldsymbol{E}$ is a collection of sets let $\omega(\boldsymbol{E})$ denote the collection of all countable (finite or infinite) unions of members of $\boldsymbol{E}$. A space X will be said to have property L if, whenever $\boldsymbol{E}$ is an open cover of X, there is a sequence $\boldsymbol{D}_1, \boldsymbol{D}_2, \cdots$ such that, for each n, $\boldsymbol{D}_n$ is a collection of subsets of X weakly cushioned in $\omega(\boldsymbol{E})$ and $\bigcup_{n=1}^{\infty} \boldsymbol{D}_n$ covers X.

THEOREM 1.1. *A countably compact space is compact if it has property L.*

Proof. Suppose X is a countably compact space with property L and $\boldsymbol{E}$ is an open cover of X. Let $\boldsymbol{D}_1, \boldsymbol{D}_2, \cdots$ be a sequence such that $\bigcup_{n=1}^{\infty} \boldsymbol{D}_n$ covers X and, for each n, $\boldsymbol{D}_n$ is weakly cushioned in $\omega(\boldsymbol{E})$. For each n, let $Z_n = \bigcup \boldsymbol{D}_n$ and let $f_n: \boldsymbol{D}_n \rightarrow \omega(\boldsymbol{E})$ be a function such that, if $\boldsymbol{G}$ is a countable subcollection of $\boldsymbol{D}_n$ and $x(G)$ is a point of G for each G in $\boldsymbol{G}$, then $\{x(G): G \in \boldsymbol{G}\}^- \subset \bigcup (f\boldsymbol{G})$.

Suppose that, for some n, Z_n is not a subset of any element of $\omega(\boldsymbol{E})$. Suppose $\{x_1, \cdots, x_k\}$ is a subset of Z_n and, for each i in $\{1, \cdots, k\}$, G_i is an element of $\boldsymbol{D}_n$ that contains x_i. Define $A_k = \bigcup_{i=1}^{k} f_n G_i$. Since A_k is in $\omega(E)$, there is a point x_{k+1} in $Z_n - A_k$. Let G_{k+1} be an element of $\boldsymbol{D}_n$ that contains x_{k+1}. Since $\bigcup_{i=1}^{k} G_i$ is a subset of A_k, G_{k+1} is not in $\{G_1, \cdots, G_k\}$. By induction there exist sequences $\{x_k\}_{k=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$ and $\{A_k\}_{k=1}^{\infty}$ such that for each k, G_k is an element of $\boldsymbol{D}_n$ different from G_j when j is not k, x_k is in $G_k \cap Z_n$, $A_k = \bigcup_{i=1}^{k} f_n G_i$

and x_{k+1} is in $Z_n - A_k$. Define $B = X - \{x_1, x_2, \cdots\}^-$. Since $\boldsymbol{D}_n$ is weakly cushioned in $\omega(\boldsymbol{E})$, $\{x_1, x_2, \cdots\}^- \subset \bigcup_{k=1}^\infty f_n G_k = \bigcup_{k=1}^\infty A_k$ and $\{B, A_1, A_2, \cdots\}$ covers X. Since X is countably compact, there is a k such that $X = B \cup A_k$. But x_{k+1} is in neither B nor A_k. This contradiction implies that, for each n, Z_n is contained in some element of $\omega(\boldsymbol{E})$. Since $\{Z_1, Z_2, \cdots\}$ covers X, X is in $\omega(\boldsymbol{E})$ and, by countable compactness, some finite subcollection of $\boldsymbol{E}$ covers X. This completes the proof.

Since a locally finite collection of subsets of a T_1-space is weakly cushioned in itself, a T_1-space X has property L if every open cover of X has a σ-locally finite refinement that covers X. Since a closure preserving collection (defined in [10, p. 822]) of closed sets is a cushioned refinement of itself, a space X has property L if every open cover of X has a σ-closure preserving closed refinement. In particular, F_σ-spaces [11, p. 796] have property L.

A topological space X is said to be *semi-stratifiable* if to each open set U of X there corresponds a sequence of closed sets $U_1, U_2, \cdots$ such that $U = \bigcup_{n=1}^\infty U_n$ and, whenever V is an open subset of an open set U, V_n is a subset of U_n. It is easily verified that, if $\boldsymbol{E}$ is an open cover of X, $\{U_n : U \in \boldsymbol{E}\}$ is cushioned in $\boldsymbol{E}$. Hence all semi-stratifiable spaces have property L. Among the semi-stratifiable spaces are the stratifiable spaces [6, p. 1], the developable spaces, including the Moore spaces [5, p. 176], the semi-metric spaces [9, p. 103], and the regular σ-spaces [15, p. 472]. It is already known that countably compact semi-stratifiable T_1-spaces are compact [8, p. 321, Corollary 4.5].

According to a definition of Arhangel'skii [3, p. 145], a space X is said to be *σ-paracompact* if, whenever $\boldsymbol{E}$ is an open cover of X, there is a sequence $\boldsymbol{D}_1, \boldsymbol{D}_2, \cdots$ of open covers of X such that, if $p \in U \in \boldsymbol{E}$, there is an integer n such that $\mathrm{St}(p, \boldsymbol{D}_n) \subset U$. (Here, $\mathrm{St}(p, \boldsymbol{D}_n)$ means $\bigcup \{D \in \boldsymbol{D}_n : p \in D\}$.) For each n, let Z_n denote the set of all points p of X such that $\mathrm{St}(p, \boldsymbol{D}_n)$ is a subset of some element of $\boldsymbol{E}$. Then $\{\{x\} : x \in Z_n\}$ is cushioned in $\{\mathrm{St}(x, \boldsymbol{D}_n) : x \in Z_n\}$ and so in $\boldsymbol{E}$. Hence σ-paracompact spaces in the sense of Arhangel'skii have property L. Clearly, fully normal spaces [16, p. 53] and developable spaces are of this kind.

A space X is said to be *meta-Lindelöf* [7, p. 796] if every open cover of X has a point-countable open refinement that covers X. If $\boldsymbol{D}$ is a point-countable collection of open sets covering a space X, then $\{\{x\} : x \in X\}$ is cushioned in $\{\mathrm{St}(x, \boldsymbol{D}) : x \in X\}$ and, therefore, cushioned in $\omega(\boldsymbol{D})$. Hence meta-Lindelöf spaces have property L. It has already been shown that countably compact meta-Lindelöf spaces are compact [1, p. 41, Proposition 3]. Among the meta-Lindelöf spaces are the Lindelöf spaces, all spaces with point-countable bases, the σ-paracompact spaces of Aull [4, p. 45], the screenable spaces [5, p. 176], the metacompact spaces [2, p. 142], and the paracompact spaces.

Suppose $\mathfrak{M}$ is an infinite cardinal. A space X is said to be $\mathfrak{M}$-*compact* if every open cover of X of cardinality $\leqq \mathfrak{M}$ contains a finite subcover. Let us say that a space has property $L(\mathfrak{M})$ if it satisfies the definition given for property L, provided the collection $\boldsymbol{E}$ occurring in that definition has cardinality $\leqq \mathfrak{M}$. A slight modification of the proof given for (1.1) shows that a countably compact space with property $L(\mathfrak{M})$ is $\mathfrak{M}$-compact. This strengthens a theorem of Morita [14, p. 228, Th. 1.8].

2. Isocompact spaces.

Call a topological space X isocompact if every closed countably compact subset of X is compact. Every closed subset of a space having property L has property L. Hence it follows from (1.1) that every space having property L is isocompact.

THEOREM 2.1. *If a space X is the union of a countable collection of closed isocompact subsets then X is isocompact.*

Proof. Suppose $X = \bigcup_{i=1}^{\infty} F_i$ where each F_i is closed and isocompact. Let M be a closed countably compact subset of X and $\boldsymbol{G}$ be an open cover of M. For each i, $M \cap F_i$ is a closed countably compact subset of F_i, and so is compact and covered by a finite subcollection $\boldsymbol{H}_i$ of $\boldsymbol{G}$. $\bigcup_{i=1}^{\infty} \boldsymbol{H}_i$ is a countable open cover of M and so contains a finite subcollection that covers M.

As a corollary of (2.1) we have

THEOREM 2.2. *Every F_σ subset of an isocompact space is isocompact.*

We say that a map (= continuous function) $f: X \rightarrow Y$ is *countably compact* {*compact*} if $f^{-1}(y)$ is countably compact {compact} for each point y in Y.

LEMMA 2.3. *If f is a closed countably compact {compact} map from a space X onto a countably compact {compact} space Y then X is countably compact {compact}.*

LEMMA 2.4. *If f is a map from a countably compact {compact} space X onto a space Y then Y is countably compact {compact}.*

THEOREM 2.5. *If f is a closed countably compact map from an isocompact space X onto a space Y then Y is isocompact.*

Proof. Let M be a closed countably compact subset of Y. Using (2.3), $f^{-1}M$ is closed and countably compact, hence compact. M is a closed subset of the compact set $ff^{-1}M$ and so is compact.

THEOREM 2.6. *If f is a closed compact map from a space X into an isocompact space Y then X is isocompact.*

Proof. Let M be a closed countably compact subset of X. Then fM is a closed countably compact subset of Y and so is compact. By (2.3), $f^{-1}fM$ is compact. Since M is closed in $f^{-1}fM$, M is compact.

LEMMA 2.7. *If X is a space and Y is a compact space, the canonical projection $\pi: X \times Y \rightarrow X$ is a closed map.*

From (2.6) and (2.7) we have

THEOREM 2.8. *The product of a compact space and an isocompact space is isocompact.*

THEOREM 2.9. *If X is an isocompact space and Y is an isocompact space each point of which has a closed and compact neighborhood then $X \times Y$ is an isocompact space.*

Proof. We may assume that each of X and Y is nonempty. Let $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ be the canonical maps. Suppose M is a closed countably compact subset of $X \times Y$ and q is a point of $Y - \pi_Y M$. Let K be a closed and compact neighborhood of q. Define $A = M \cap \pi_Y^{-1}K$. A is a closed countably compact subset of the product of the compact space K and the isocompact space X and so, by (2.8), is compact. A is a closed subset of the product of the compact space $\pi_X A$ and the space Y. By (2.7) $\pi_Y A$ is closed, that is, $K \cap \pi_Y M$ is closed. $K^0 - \pi_Y M$ is an open set containing q. Thus $\pi_Y M$ is closed. By (2.4) $\pi_Y M$ is countably compact. Since Y is isocompact, $\pi_Y M$ is compact. M is a closed countably compact subset of the product of the compact space $\pi_Y M$ and the isocompact space X. By (2.8) M is compact.

From (2.1) and (2.9) we have

THEOREM 2.10. *If X is an isocompact space and Y is an isocompact Hausdorff space that is a countable union of closed locally compact subsets then $X \times Y$ is an isocompact space.*

To say that a space X is *hereditarily isocompact* means, of course, that every subspace of X is isocompact or, equivalently, that *every* countably compact subset of X is compact. For example, all semi-stratifiable spaces are hereditarily isocompact. Isocompact spaces in which every countably compact subset is closed are hereditarily isocompact. Isocompact first countable T_3-spaces are of this kind.

THEOREM 2.11. *The product of an isocompact space and a hereditarily isocompact space is isocompact.*

Proof. Suppose X is isocompact, Y is hereditarily isocompact and M is a closed countably compact subset of $X \times Y$. By (2.4) $\pi_Y M$ is countably compact and is therefore compact. M is a subset of the product of a compact space $\pi_Y M$ and an isocompact space Y and so, by (2.8), is compact.

THEOREM 2.12. *The product of any collection of hereditarily isocompact spaces is isocompact.*

Proof. Let P be the product of a collection $\{X_i: i \in A\}$ of hereditarily isocompact spaces and for each i in A let $\pi_i: P \to X_i$ be the canonical projection. Suppose M is a closed countably compact subset of P. By (2.4), for each i, $\pi_i M$ is countably compact and, so, compact. Since M is a closed subset of the product of the compact spaces $\pi_i M$, M is compact.

From (2.12) it follows that any realcompact space (a space homeomorphic to a closed subset of a product of real lines) is isocompact. A Hausdorff space X is said to be *almost realcompact* if each maximal centered collection $\boldsymbol{M}$ of open subsets of X with $\bigcap \{U^-: U \in \boldsymbol{M}\} = \varnothing$ has the property that for some countable subcollection $\boldsymbol{D}$ of $\boldsymbol{M}$, $\bigcap \{U^-: U \in \boldsymbol{D}\} = \varnothing$ [9, p. 128].

THEOREM 2.13. *Every regular almost realcompact space is isocompact.*

Proof. Since any closed subset of a regular almost realcompact space is almost realcompact [9, p. 133, Th. 5], it will suffice to show that every regular countably compact almost realcompact space is compact. Suppose X is a regular countably compact almost realcompact space and $\boldsymbol{C}$ is a centered collection of closed subsets of X. Let $\boldsymbol{E}$ be the collection to which U belongs if and only if U is an open set containing some element of $\boldsymbol{C}$. Since $\boldsymbol{E}$ is centered, $\boldsymbol{E}$ is contained in some maximal centered collection $\boldsymbol{M}$ of open sets. Since X is countably compact, $\bigcap \{U^-: U \in \boldsymbol{D}\} \neq \varnothing$ for any countable subcollection $\boldsymbol{D}$ of $\boldsymbol{M}$. Since X is almost realcompact, there is a point p in $\bigcap \{U^-: U \in \boldsymbol{M}\}$. Suppose there is an element C of $\boldsymbol{C}$ that does not contain

p. Since X is regular, there is an open set U containing C whose closure does not contain p, which involves a contradiction. Hence p is in $\bigcap C$ and X is compact.

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