INTEGRATION WITH RESPECT TO VECTOR MEASURES

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The purpose of this paper is to develop a theory of integration with respect to measures into a locally convex, Hausdorff linear topological space, using a linear functional approach.

Section 1 presents some basic facts about such measures, chiefly through the study of their p-semi-variations (Definition 1.2). Devices of this sort have been considered by other authors (see [1], [4]), but chiefly to give expressions for the norms of linear operators defined by vector measures. We consider the continuity properties of the p-semi-variation, and define regularity in terms of the p-semi-variation.

The integration theory is developed in § 2. Although the integral is defined in terms of linear functionals, it is in no sense a weak integral. The dominated (strong) convergence theorem is proven under the additional assumption that the limit function is integrable, and it is shown that this is true whenever the range space of the measure is sequentially complete.

In § 3 integral representations of weakly compact operators from $C(S)$, $C_0(T)$, $C_c(T)$ and $C(T)_\beta$ into a locally convex, Hausdorff space are given. We used these representations to show that the above spaces satisfy a strengthened rersion of the Dunford-Pettis Property-specifically that a weakly compact operator on these spaces maps weakly Cauchy sequences into convergent sequences, without any assumption about the completeness of the range of the operator.

1. Throughout the first two sections (S, Σ) denotes a measurable space, $(X, \mathcal{T})$ a complex, locally convex Hausdorff linear topological space with dual X^* and μ an additive set function from Σ into X. If $x^* \in X^*$ and p is a semi-norm on X we will write $x^* \leq p$ whenever $|x^*(x)| \leq p(x)$ for all $x \in X$. The following theorem, stated here without proof, was first proved by Pettis [6] for normed spaces and Grothendieck [5] for locally convex spaces.

THEOREM 1.1. *If μ is countably additive in the weak topology, then μ is countably additive in $\mathcal{T}$.*

DEFINITION 1.2. *If p is a semi-norm on X, then the p-semivariation of μ is the function from Σ into the extended reals defined by* $||\mu||_p(E) = \sup_{x^* \leq p} v(x^*\mu, E)$, *where $v(x^*\mu, \cdot)$ is the scalar variation of*

$x^*\mu.$

It follows immediately that $\| \mu \|_p(\cdot)$ is monotone, subadditive and that $p[\mu(E)] \leqq \| \mu \|_p(E) \leqq 4 \sup_{F \subset E} p[\mu(F)]$ for each $E \in \Sigma$. If μ is a measure then $\| \mu \|_p(\cdot)$ is countably subadditive and real valued, since the range of μ is bounded.

THEOREM 1.3. *If μ is a measure, p a continuous semi-norm and (E_n) a convergent sequence in Σ, then $\| \mu \|_p(\lim_n E_n) = \lim_n \| \mu \|_p(E_n)$.*

Proof. We first establish a special case. Suppose (E_n) is a decreasing sequence in Σ with empty intersection and that, for some $\varepsilon > 0$, $\| \mu \|_p(E_n) > \varepsilon$ for each n. Let $n_1 = 1$. For some $x^* \leqq p$ and $n_2 > n_1$, $v(x^*\mu, E_{n_1}) > \varepsilon$ and $v(x^*\mu, E_{n_2}) < \varepsilon/2$. Then also $4 \sup_{F \subset E_{n_1} \backslash E_{n_2}} p[\mu(F)] \geqq v(x^*\mu, E_{n_1} \backslash E_{n_2}) > \varepsilon/2$, so for some $F_1 \subset E_{n_1} \backslash E_{n_2}$ we have

$$p[\mu(F_1)] > \frac{\varepsilon}{8} .$$

Continuing in this manner there is an increasing sequence (n_k) of positive integers and a sequence (F_k) in Σ such that $F_k \subset E_{n_k} \backslash E_{n_{k+1}}$ and $p[\mu(F_k)] > \varepsilon/8$ for each k. This contradicts the countable additivity of μ since the F_k's are pairwise disjoint.

If (E_n) is in Σ and has limit E, then $\| \mu \|_p(E) = \lim_n \| \mu \|_p(E_n)$ since the inequality

$$\| \| \mu \|_p(E) - \| \mu \|_p(E_n) \| \leqq \| \mu \|_p[\bigcup_{k \geqq n}(E \backslash E_k)] + \| \mu \|_p[\bigcup_{k \geqq n}(E_k \backslash E)]$$

holds.

COROLLARY 1.4. *If μ is a measure and (E_n) is a convergent sequence in Σ, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.*

Proof. Let $E = \lim_n E_n$. The corollary follows since

$$\lim_n(E \backslash E_n) = \lim_n(E_n \backslash E) = \varnothing$$

and $p[\mu(E) - \mu(E_n)] < \| \mu \|_p(E \backslash E_n) + \| \mu \|_p(E_n \backslash E)$ for each n and semi-norm p.

DEFINITION 1.5. Suppose S is a topological space. μ is *regular* (in $\mathscr{T}$) if, for each $E \in \Sigma$ $\varepsilon > 0$ and continuous semi-norm p on X, there is a relatively compact set K in Σ whose closure is contained in E and a set G in Σ whose interior contains E such that

$$\| \mu \|_p(G \backslash K) < \varepsilon .$$

If μ is bounded and regular, then $x^*\mu$ is bounded and regular for each $x^* \in X^*$. This implies that $x^*\mu$ is countably additive for each $x^* \in X^*$, which in turn implies that μ is a measure. Also, regularity in the weak topology is equivalent to the regularity of $x^*\mu$ for each $x^* \in X^*$.

THEOREM 1.6. *If μ is regular in the weak topology and μ is a measure, then μ is regular in $\mathscr{T}$.*

Proof. If μ is not inner regular there is an $E \in \Sigma$, a positive ε and a $\mathscr{T}$-continuous semi-norm p such that $\| \mu \|_p(E \backslash K) > \varepsilon$ for each relatively compact K in Σ with $\bar{K} \subset E$. Let $K_1 = \varnothing$. There is an $x_1 \leqq p$ and a relatively compact set K_2 in Σ such that

$$K_1 \subset K_2, \; \bar{K}_2 \subset E, \; v(x_1\mu, E \backslash K_1) > \varepsilon$$

and $v(x_1\mu, E \backslash K_2) < \varepsilon/2$. Since $\| \mu \|_p(E \backslash K_2) > \varepsilon$ we may continue this process and obtain a sequence (x_n) of functionals dominated by p and an increasing sequence (K_n) of relatively compact sets in Σ such that $\bar{K}_n \subset E$, $v(x_n\mu, E \backslash K_n) > \varepsilon$ and $v(x_n\mu, E \backslash K_{n+1}) < \varepsilon/2$ for each n. Let $K = \bigcup_{n \geqq 1} K_n$. Since $\lim_n(K \backslash K_n) = \varnothing$ there is an m such that

$$\| \mu \|_p(K \backslash K_m) < \frac{\varepsilon}{2} \; .$$

But then $v(x_m\mu, E \backslash K_m) \leqq v(x_m\mu, E \backslash K_{m+1}) + \| \mu \|_p(K \backslash K_m) < \varepsilon$. The outer regularity of μ is proven similarly.

2. Throughout this section μ is a fixed measure from Σ into $(X, \mathscr{T})$ and $\mathcal{C}$ the complex numbers.

DEFINITION 2.1. A function $f \colon S \to \mathcal{C}$ is *μ-integrable* if
(1) f is $x^*\mu$-integrable for each $x^* \in X^*$, and
(2)   for each $E \in \Sigma$ there is an element of X, denoted by

$$\int_E f(t)\mu(dt) \; ,$$

such that $x^* \int_E f(t)\mu(dt) = \int_E f(t)x^*\mu(dt)$ for each $x^* \in X^*$.

Since $(X, \mathscr{T})$ is Hausdorff the integral is well-defined. No assumption is made about the completeness of $(X, \mathscr{T})$. $\mathscr{T}$ enters into the definition of the integral only in that it determines X^*, so that in any of the results of this section $\mathscr{T}$ may be replaced by any topology in the Mackey spectrum of $(X, \mathscr{T})$.

It is easy to see that the integral has the following properties:

(1)   The integral is linear.

(2)   Every simple function $\sum_{i \leq n} a_i \chi_{E_i}$ is μ-integrable and

$$\int_E \Big(\sum_{i \leq n} a_i \chi_{E_i}\Big)(t)\mu(dt) = \sum_{i \leq n} a_i \mu(E \cap E_i)$$

for $E \in \Sigma$.

(3)   If f is bounded and μ-integrable, then

$$p\Big[\int_E f(t)\mu(dt)\Big] \leq \|\mu\|_p(E) \cdot \sup_{s \in S} |f(s)|$$

for each $E \in \Sigma$ and continuous semi-norm p.

(4)   If f is μ-integrable and T is a continuous linear operator from X into a locally convex Hausdorff space Y, then f is $T\mu$-integrable and $\int_E f(t)T\mu(dt) = T\int_E f(t)\mu(dt)$ for $E \in \Sigma$.

THEOREM 2.2.   (1)   *If f is μ-integrable, then the set function on Σ defined by $\Phi(E) = \int_E f(t)\mu(dt)$ is a measure,*

$$\|\Phi\|_p(E) = \sup_{x^* \leq p} \int_E |f(t)| v(x^*\mu, dt) \ \text{and} \ \lim_{\|\mu\|_p(E) \to 0} \|\Phi\|_p(E) = 0$$

for each continuous semi-norm p.

(2)   *Let (f_n) be a sequence of μ-integrable functions which converge pointwise to f on S and g be a μ-integrable function such that $|f_n| \leq |g|$ for each n. f is μ-integrable if X is sequentially complete.   If f is μ-integrable then*

$$\int_E f(t)\mu(dt) = \lim_n \int_E f_n(t)\mu(dt)$$

uniformly with respect to $E \in \Sigma$.

Proof.   The set function Φ in (1) is a measure by Pettis's Theorem, and the expression for its p-semi-variation is correct since

$$v(x^*\Phi, E) = \int_E |f(t)| v(x^*\mu, dt)$$

for each x^*.   Clearly $\|\Phi\|_p(E) = 0$ whenever $\|\mu\|_p(E) = 0$, so in light of Theorem 1.3 the usual proof by contradiction that zero-zero and $\varepsilon - \delta$ absolute continuity are equivalent in finite measure spaces establishes that $\lim_{\|\mu\|_p(E) \to 0} \|\Phi\|_p(E) = 0$.

To prove (2) we first show that

$$\Big(\int_E f_n(t)\mu(dt)\Big)$$

is Cauchy uniformly with respect to $E \in \Sigma$. Let p be a continuous seminorm, $\varepsilon > 0$,

$$\Phi(F) = \int_F g(t)\mu(dt)$$

and $E_n = \{s \in S : |f(s) - f_n(s)| \geq \varepsilon\}$. If $E \in \Sigma$ and $x^* \leq p$, then f is $x^* \mu$-integrable by the dominated convergence theorem for scalar measures and for each n

$$\left| \int_E (f - f_n)(t)x^*\mu(dt) \right| \leq \varepsilon \| \mu \|_p(E \backslash E_n) + 2\| \Phi \|_p(E \cap E_n) .$$

Thus

$$p\left[\int_E f_n(t)\mu(dt) - \int_E f_m(t)\mu(dt) \right]$$
$$\leq 2\varepsilon \| \mu \|_p(S) + 2\| \Phi \|_p(E_n) + 2\| \Phi \|_p(E_m)$$

for all n and m. The sequence is Cauchy since $\lim_n \| \Phi \|_p(E_n) = 0$.

The first assertion of (2) follows from this Cauchy condition and the dominated convergence theorem, and the second is true since

$$\left(\int_E f_n(t)\mu(dt) \right)$$

is uniformly Cauchy in $\mathscr{T}$ with respect to E and converges weakly to $\int_E f(t)\mu(dt)$.

It follows that every bounded measurable function is μ-integrable if X is sequentially complete. The last two theorems of this section give characterizations of μ-integrable functions when X is sequentially complete.

LEMMA 2.3. *Let λ be a complex valued measure on (S, Σ) and (f_n) a sequence of λ-integrable functions such that*
(1) *(f_n) converges to f pointwise on S, and*

(2) *$\left(\int_E f_n(t)\lambda(dt) \right)$ is Cauchy for each $E \in \Sigma$. Then f is λ-integrable*

and $\int_E f(t)\lambda(dt) = \lim_n \int_E f_n(t)\lambda(dt)$ uniformly with respect to $E \in \Sigma$.

Proof. The proof follows the standard argument. For each m define λ_m on Σ by $\lambda_m(E) = \int_E f_m(t)\lambda(dt)$. Each λ_m is $v(\lambda)$-continuous and by (2) $\lim_m \lambda_m(E)$ exists for each $E \in \Sigma$. By the Vitali-Hahn-Saks Theorem $\lim_{v(\lambda,E)\to 0} \lambda_m(E) = 0$ uniformly in m. Let $\varepsilon > 0$ and

$$E_n = \{s \in S : |f_n(s) - f(s)| \geq \varepsilon\}$$

for each n.   By (1) $\lim_n E_n = \varnothing$, so $\sup_{n \geq n_0} \sup_m v(\lambda_m, E_n) < \varepsilon$ for some n_0.   Also, for $n \geq n_0$,

$$\int | f(t) - f_n(t) | v(\lambda, dt)$$

$$\leq \varepsilon v(\lambda, S \backslash E_n) + \liminf_m \int_{E_n} | f_m(t) - f_n(t) | v(\lambda, dt)$$

$$\leq \varepsilon[v(\lambda, S) + 2] .$$

This inequality establishes the lemma.

THEOREM 2.4.   *Suppose X is sequentially complete and f is a complex valued function on S.   The following are equivalent:*
   (1)   *f is μ-integrable.*
   (2)   *There is a sequence (f_n) of bounded measurable functions which converges pointwise to f and for which $\left(\int_E f_n(t) \mu(dt) \right)$ is Cauchy uniformly with respect to $E \in \Sigma$.*
   (3)   *There is a sequence (f_n) of simple functions which converges pointwise to f and for which $\left(\int_E f_n(t) \mu(dt) \right)$ is Cauchy for each $E \in \Sigma$.*

Proof.   For each n let $E_n = \{s \in S \colon | f(s) | \leq n\}$ and $f_n = f \chi_{E_n}$.   If f is μ-integrable then (f_n) satisfies condition (2) by Theorem 2.2.   (2) clearly implies (3).   To see that (3) implies integrability, let $E \in \Sigma$ and $x_E = \lim_n \int_E f_n(t) \mu(dt)$.   For $x^* \in X^*$, an application of Lemma 2.3 with $\lambda = x^* \mu$ shows that

$$x^*(x_E) = \lim_n \int_E f_n(t) x^* \mu(dt) = \int_E f(t) x^* \mu(dt) .$$

Notice that if f and (f_n) satisfy condition (2) or (3) of Theorem 2.4, then, by Lemma 2.3,

$$\int_E f(t) \mu(dt) = \lim_n \int_E f_n(t) \mu(dt)$$

for each $E \in \Sigma$.   Also, a reformation of (3) with the word Cauchy replaced by convergent is equivalent to (1) without sequential completeness.

We next consider the case in which X is normed.   Here we need only one semi-variation of μ, that with $p(x) = \| x \|$.   The dual of X^* under its natural norm topology will be denoted by X^{**}.

DEFINITION 2.5.   Suppose X is a normed space.   A function $f \colon S \to \phi$ has a *generalized integral* with respect to μ if f is $x^* \mu$-

integrable for each $x^* \in X^*$. If f is such a function, then $\int_E f d\mu$ is the linear form on X^* defined by $\left(\int_E f d\mu\right)x^* = \int_E f(t) x^* \mu(dt)$.

If f is μ-integrable, then $\int_E f du$ is the image of $\int_E f(t)\mu(dt)$ under the natural map from X into X^{**}. $\int_E f du$ is always in X^{**}, since it is the pointwise limit of a sequence of the integrals of simple functions.

THEOREM 2.6.   *Suppose X is a Banach space and $f: S \to \mathcal{C}$ has a generalized integral. The following are equivalent:*
   (1)   *f is μ-integrable.*
   (2)   *The set function Φ from Σ into X^{**} defined by*

$$\Phi(E) = \int_E f d\mu$$

is measure in the norm topology on X^{**}.
   (3) $\lim_{||\mu||(E) \to 0} \left|\left|\int_E f d\mu\right|\right| = 0.$

Proof.   (1) implies (3) by Theorem 2.2.   (2) is immediate from (3). For each n let $F_n = \{s \in S: |f(s)| \leq n\}$ and $f_n = f\chi_{E_n}$. For n and m positive integers and $E \in \Sigma$,

$$\left|\left|\int_E f_m(t)\mu(dt) - \int_E f_m(t)\mu(dt)\right|\right| \leq ||\Phi||(S \backslash E_n) + ||\Phi||(S \backslash E_m) ,$$

so that (f_n) satisfies condition (2) of Theorem 2.4 if Φ is a measure.

3.   Below S is a compact Hausdorff space, Σ the Borel sets of S, $C(S)$ the Banach space (under supremum norm) of continuous complex valued functions on S and $(X, \mathcal{T})$ a locally convex, Hausdorff linear topological space.

THEOREM 3.1.   *Let $A: C(S) \to X$ be a weakly compact linear operator. There is a measure $\mu: \Sigma \to X$ such that*
   (1)   *μ is regular,*
   (2)   *the closed absolutely convex hull of $\mu[\Sigma]$ is weakly compact,*
   (3)   *every bounded Borel function on S is μ-integrable,*
   (4)   *$Af = \int f(t)\mu(dt)$ for $f \in C(S)$, and*
   (5)   *$A^* x^* = x^* \mu$ for $x^* \in X^*$.*
Conditions (1) and (4) define μ uniquely. $||A|| = ||\mu||(S)$ whenever X is normed.   If μ is a measure on Σ which satisfies (1), (2), and

(3), *then* (4) *defined a weakly compact operator which satisfies* (5).
If X is complete, then (2) *and* (3) *follow from* (1).

Proof. Since the dual of $C(S)$ may be identified with the bounded
regular Borel measures on $\varSigma$, the equation

$$g^{\wedge}(\lambda) = \int g(t)\lambda(dt)$$

defines an element of $C(S)^{**}$ for each bounded Borel function g.
Since A is weakly compact, A^{**}, the algebraic adjoint of A^*, maps
$C(S)^{**}$ into X. For $E \in \varSigma$ let $\mu(E) = A^{**}(\chi_E^{\wedge})$. For each $x^* \in X^*$,
$x^*\mu = A^*x^*$ is a regular measure, so μ is a regular measure. Since
A^{**} maps the unit ball of $C(S)^{**}$ into a weakly compact subset of X,
condition (3) is satisfied. If $\sum_{i \leqq n} a_i \chi_{E_i}$ is a simple Borel function, then

$$A^{**}\left[\left(\sum_{i \leqq n} a_i \chi_{E_i}\right)^{\wedge}\right] = \sum_{i \leqq n} a_i A^{**}(\chi_{E_i}^{\wedge}) = \int \left(\sum_{i \leqq n} a_i \chi_{E_i}\right)(t)\,\mu(dt) \ .$$

Thus $x^*(A^{**}g^{\wedge}) = \int g(t)x^*\mu(dt)$ holds for each bounded Borel function
g and $x^* \in X^*$. Finally, $\|A\| = \|A^*\| = \|\mu\|(E)$ if X is normed.

Conversely, suppose μ satisfies (1), (2), (3). The operator A de-
fined by (4) is continuous since $p[Af] \leqq \|f\| \|\mu\|_p(S)$ for each con-
tinuous semi-norm p on X. Also, by the regularity of μ, $A^*x^* = x^*\mu$
for $x^* \in X^*$. Let U be the polar of the closed, absolutely convex
hull of $\nu[\varSigma]$. U is a neighborhood of zero in the Mackey topology on
X^*, and, for $x^* \in U$, $\|A^*x^*\| \leqq 4$. Thus A^* is continuous with the
Mackey topology on X^* and the norm topology on $C(S)^*$-this implies
that A is weakly compact.

If X is complete and μ is regular, then $Af = \int f(t)\mu(dt)$ defines a
continuous linear operator for $C(S)$ into X such that $A^*x^* = x^*\mu$ for
$x^* \in X^*$. To see that (2) holds, it is sufficient to show that A is
weakly compact-equivalently, that A^* maps equicontinuous sets into
weakly relatively compact sets. Let V be an open neighborhood of
zero in X generated by a semi-norm p. For $x^* \in V^{\circ}$ and $E \in \varSigma$,

$$|x^*\mu(E)| \leqq \|\mu\|_p(E) \ ,$$

so the countable additivity of $A^*[V^{\circ}]$ is uniform. This together with
norm boundedness implies that $A^*[V^{\circ}]$ is relatively weakly compact.

In [1] Bartle, Dunford and Schwartz have given a similar integral
representation for weakly compact operators from $C(S)$ into a Banach
space. Grothendieck [5] has noted that there is a one-to-one cor-
respondence between the weakly compact operators from $C(S)$ into a
complete, locally convex, Hausdorff space X and the X-valued measures
on the Baire sets of S, although he did not give an integral repre-

sentation of such operators.

Let T be a locally compact Hausdorff space. $C_0(T)[C_c(T)]$ is the Banach space under supremum norm of complex valued functions on T which vanish at infinity [have compact support]. $C(T)_\beta$ is the space of bounded, continuous, complex valued functions on T topologized by the semi-norms $p_\varphi(f) = \sup_{t \in T} |\varphi(t)f(t)|$, where $\varphi \in C_0(T)$. A weakly compact operator $A: C_0(T) \to X$ can be represented by integration with respect to an X-valued measure since A can be extended to a weakly compact operator on the space of continuous functions over the one point compactification of T. Weakly compact operators on $C_c(T)$ have such a representation since they can be extended to $C_0(T)$. The bounded sets of $C(T)_\beta$ are precisely the uniformly bounded sets and each element of $C(T)_\beta^*$ can be identified with a bounded regular Borel measure on T [2], so the proof of Theorem 3.1 generalizes immediately for $C(S)_\beta$.

COROLLARY 3.2. *Let A be a weakly compact operator from one of $C(S)$, $C_0(T)$, $C_c(T)$ or $C(T)_\beta$ into X. A maps weakly Cauchy sequences into convergent sequences.*

Proof. If (f_n) is weakly Cauchy in any of the above spaces, then (f_n) is uniformly bounded and converges pointwise to a bounded Borel function f. Let μ be the measure determining A. f is μ-integrable and, by Theorem 2.1, $\int f(t)\mu(dt) = \lim_n Af_n$.

This is proven in [5] for $C(S)$ under the assumption that X is complete.

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