Pacific Journal of Mathematics

ON A CLASS OF NÖRLUND MEANS AND FOURIER SERIES

H. P. DIKSHIT

Vol. 33, No. 3 May 1970

ON A CLASS OF NÖRLUND MEANS AND FOURIER SERIES

H. P. Dikshit

By considering a class of Nörlund means that covers as a subclass the corresponding (C) means, we obtain in the present paper, several results concerning absolute Nörlund summability and deduce from these the corresponding |C| results as special cases. What is indeed remarkable, is that a special case of our Theorem 2 improves an earlier result due to Bosanquet and Hyslop in dropping one of the two independent conditions used by them. Further, the proofs of some of our results are shorter and even more direct than the proofs given for the corresponding special cases by using equivalent Riesz means instead of (C) means.

1. Definitions and notations. Let $\sum_{n} v_{n}$ be a given infinite series with the sequence of partial sum $\{s_{n}\}$. We shall consider sequence to sequence transformations of the type

$$t_n = \sum\limits_{k=0}^{\infty} d_{nk} s_k$$
; $d_{nk} = 0$ for $k > n$;

in which the elements of the matrix $D = (d_{nk})$ are real or complex constants. t_n is called the *n*-th *D*-mean of $\{s_n\}$.

Let $\{p_n\}$ be a sequence of constants, real or complex and let $P_n = p_0 + p_1 + \cdots + p_n \neq 0, P_{-1} = p_{-1} = 0$. Then the matrix D defines a Nörlund matrix (N, p), if

$$d_{nk}=p_{n-k}/P_n$$
 , $n\geq k\geq 0$.

In the special case in which

$$(1.1) p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}, \alpha \neq -1, -2, \cdots,$$

the (N, p) mean reduces to the familiar (C, α) mean.

The (N, p)(C, 1) matrix is defined as the product of a (N, p) matrix with the (C, 1) matrix. Thus the (N, p)(C, 1) mean of $\{s_n\}$ is

$$t_{\scriptscriptstyle n} = rac{1}{P_{\scriptscriptstyle n}} \sum_{r=0}^{n} p_{n-r} rac{1}{r+1} \sum_{k=0}^{r} s_k$$
 .

Similarly, one defines the (C, 1)(N, p) mean [5].

Let f(t) be integrable (L) in $(-\pi, \pi)$ and periodic with period 2π . We assume as we may without any loss of generality that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_n A_n(t)$$
.

Then the conjugate series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_n B_n(t) .$$

We shall also consider the series

$$\sum_{n} rac{1}{n} \Bigl\{ \sum_{k=1}^{n} A_{k}(t) - s \Bigr\} = \sum_{n} A_{n}^{*}(t)$$
 ,

where s is an appropriate number, independent of n.

Throughout the present paper we write $L^{\alpha}(t)$ for the series $\sum_{n} n^{\alpha} A_{n}(t)$, $\widetilde{L}^{\alpha}(t)$ for $\sum_{n} n^{\alpha} B_{n}(t)$, $(\alpha \geq 0)$, $L^{*}(t)$ for $\sum_{n} A_{n}^{*}(t)$ and \mathscr{A} for the class $\{L^{\alpha}(t), \widetilde{L}^{\alpha}(t), L^{*}(t)\}$.

Let E_f be a point set in the interval $(-\pi, \pi)$ for each function f(t) and such that at every point $x \in E_f$, f(x) has a finite definite value and satisfies a prescribed condition of regularity.

DEFINITION 1. A method of summation $D=(\!(d_{nk})\!)$ is said to be $|A(x),E_f|$ -effective, if for each $x\in E_f$

$$\sum_{n=1}^{\infty} |t_n(A(x)) - t_{n-1}(A(x))| < \infty$$
 ,

symbolically, $\{t_n(A(x))\} \in BV$; where $t_n(A(x))$ denotes the *n*th *D*-mean of $A(x) \in \mathcal{A}$.

We write

$$egin{aligned} \phi(t) &= rac{1}{2} \{ f(x+t) + f(x-t) \} \; ; \; \phi^*(t) = \phi(t) - s \; ; \ &m{arPhi}_lpha(t) &= rac{1}{\Gamma(lpha)} \! \int_{\circ}^t \! (t-u)^{lpha-1} \! \phi(u) du, \; lpha > 0 \; ; \; m{arPhi}_\circ(t) = \phi(t) \; ; \ &m{arphi}_lpha(t) &= \Gamma(lpha+1) t^{-lpha} m{arPhi}_lpha(t), \; lpha \geqq 0 \; ; \ &m{\psi}(t) &= rac{1}{2} \{ f(x+t) - f(x-t) \} \; , \end{aligned}$$

 $\Psi_{\alpha}(t)$ and $\psi_{\alpha}(t)$ have similar meanings.

[x] denotes the greatest integer not greater than x.

By ' $F(t) \in BV(a, b)$ ', we mean that F(t) is a function of bounded variation in (a, b) and by ' $\{\lambda_n\} \in B$ ' that $\{\lambda_n\}$ is a bounded sequence.

K denotes a positive constant, not necessarily the same at each occurrence.

DEFINITION 2. For some $\alpha \ge 0$, the point x is said to be

(i)
$$|F_{\alpha}| - regular$$
, if $\phi_{\alpha}(t) \in BV(0, \pi)$,

$$(\ \text{ii} \) \quad | \ \widetilde{F}_{\alpha} \ | \ - \ regular, \ \ \text{if} \ \ \varPsi_{\alpha}(+0) = 0 \ \ \text{and} \ \ \int_{_0}^{\pi} t^{-\alpha} \ | \ d\varPsi_{\alpha}(t) \ | \le K,$$

(iii)
$$|F^{\alpha}| - regular$$
, if $\int_{0}^{\pi} t^{-\alpha} |d\phi(t)| \leq K$,

$$\text{(iv)} \quad |\,\widetilde{F}^\alpha\,|\,-\,regular, \,\, \text{if} \,\,\psi(+0)=0 \,\,\text{and} \,\, \int_0^\pi \!\! t^{-\alpha}\,|\,d\psi(t)\,| \leqq K,$$

$$(\mathrm{\,v\,}) \quad |F^*| - \mathit{regular}, \ \mathrm{if} \ \int_{0}^{\pi} t^{-1} \, |d\phi^*(t)| \leqq K,$$

(vi)
$$|\widetilde{F}^*| - regular$$
, if $\int_0^{\pi} t^{-1} |\psi(t)| dt \leq K$.

Denoting the set of |X| - regular points with respect to f(t) in $(-\pi, \pi)$ by E|X, f|, we know the following ([14], §13.24)

$$(1.2) E|\widetilde{F}^*, f| \not\subset E|\widetilde{F}_0, f| and E|\widetilde{F}_0, f| \not\subset E|\widetilde{F}^*, f|^1.$$

DEFINITION 3. A method of summation, which is $|A(x), E_f|$ - effective is said to be

- (i) $|F_{\alpha}|$ effective, if $A(x) = L^{0}(x)$ and $E_{f} = E|F_{\alpha}, f|$;
- (ii) $|\widetilde{F}_{lpha}|-\textit{effective}, \ ext{if} \ A(x)=\widetilde{L}^{\scriptscriptstyle 0}(x) \ \ ext{and} \ \ E_f=E|\widetilde{F}_{lpha}, \ f|$;
- (iii) $|\widetilde{F}^{\alpha}| \textit{effective}$, if $A(x) = L^{\alpha}(x)$ and $E_f = E|F^{\alpha}, f|$;
- (iv) $|\widetilde{F}^{lpha}|-e \mathit{ffective}, \ \mathrm{if} \ A(x)=\widetilde{L}^{lpha}(x) \ \mathrm{and} \ E_f=E\,|\,\widetilde{F}^{lpha},\,f\,|\,;$
- (v) $|F^*|$ effective, if $A(x) = L^*(x)$ and $E_f = E|F^*, f|$;
- (vi) absolute α -effective or $|\alpha|$ -effective, if it is effective in the sense of (i)-(iv) simultaneously.

The following notations will be used throughout. If for $n = 0, 1, 2, \cdots$

$$p_n > 0$$
, $p_{n+1}/p_n \le p_{n+2}/p_{n+1} \le 1$,

then we shall write $\{p_n\} \in M$. If $\{p_n\} \in M$ and for some α ,

$$(1.3) P_k \sum_{n=k}^{\infty} \frac{1}{n^{1-\alpha} P_n} \leqq Kk^{\alpha} , k=1, 2, \cdots$$

then we write $\{p_n\} \in M_{\alpha}$.

For a given series $v = \sum_{n} v_n$,

$$\sigma_n(v) = \sum_{k=0}^n p_{n-k} k v_k$$
 .

We also write

¹ I.e., $x \in E \mid \widetilde{F}^*, f \mid$ does not imply that $x \in E \mid \widetilde{F_0}, f \mid$ and conversely.

$$h(n, t) = rac{2}{\pi} \sum_{k=0}^{n} p_{n-k} \exp{(ikt)}$$
 ,
$$H(n, u) = rac{1}{\Gamma(1-lpha)} \int_{u}^{\pi} (t-u)^{-lpha} rac{d}{dt} h(n, t) dt$$
 ,

g(n, t) and $\widetilde{g}(n, t)$ for the imaginary and real parts respectively of h(n, t). J(n, u) (or $\widetilde{J}(n, u)$) is H(n, u) with h(n, t) replaced by g(n, t) (or $\widetilde{g}(n, t)$). Further

$$V(n, u) = rac{1}{\Gamma(1+lpha)} \int_0^u v^lpha rac{d}{dv} J(n, v) dv$$
 .

2. Introduction. Concerning $|\alpha| - effectiveness$ of the (C)-method we have the following result which is known to be the best possible in the sense that it breaks down if $\delta = 0$.

Theorem A. If $0<\alpha<1$, then $(C,\alpha+\delta)$ for any $\delta>0$ is $|\alpha|-effective$.

Starting with the proof of $|F_{\alpha}|-effectiveness$ of $(C,\alpha+\delta)$ given by Bosanquet [1] in 1936, Theorem A has been completed in stages by different authors. Thus $|\tilde{F}_{\alpha}|-effective$ part of Theorem A is due to Bosanquet and Hyslop ([2], Th. 2) and $|F^{\alpha}|$ and $|\tilde{F}^{\alpha}|-effective$ parts are due to Mohanty ([11], Th. 1 and Th. 2). It is somewhat peculiar to observe that Theorem A may be extended to cover the case $\alpha=0$, as far as $|F_{\alpha}|$ or $|F^{\alpha}|-effective$ parts are concerned but for $|\tilde{F}_{0}|$ or equivalently $|\tilde{F}^{0}|-effectiveness$ of the (C) method, Bosanquet and Hyslop ([2], Th. K, with $\alpha=0$) require an additional condition that x is $|\tilde{F}^{*}|-regular$ also, in the following.

THEOREM B. If x is $|\tilde{F}^*| - regular$, then the (C, δ) method is $|\tilde{F}_0| - effective$ for each $\delta > 0$.

The condition that x is $|\widetilde{F}^*| - regular$ is independent of the condition that x is $|\widetilde{F}_0| - regular$ in view of (1.2).

In Theorem 1 of the present paper we prove $|\alpha| - effectiveness$ $(0 < \alpha < 1)$ of a (N, p) method which covers the corresponding (C) method as a special subclass and deduce the various results of Theorem A as particular cases of our Theorem 1. What is indeed remarkable is that in Theorem 2, we have succeeded in extending Theorem 1 to the case $\alpha = 0$ by proving $|\widetilde{F}_0| - effectiveness$ of the (N, p) method, even without using the hypothesis that x is a $|\widetilde{F}^*| - regular$ point. Thus the following special case of Theorem 2 improves Theorem B in dropping the condition that x is a $|\widetilde{F}^*|$ -

regular point.

Theorem C. The $(C,\,\delta)$ method is $|\,\widetilde{F}_{\scriptscriptstyle 0}\,|-$ effective for every $\delta>0$.

Covering as a special case, an earlier result due to Mohanty and Mohapatra ([12], Th. 3) we prove in Theorem 3, $|F^*| - effectiveness$ of the (N, p) method and thus demonstrate that the |N, p| of $L^*(x)$, is a local property of its generating function [6].

It may be observed that the proofs of some of our theorems are shorter and even more direct than the proofs given in support of the corresponding special cases by using equivalent Riesz methods instead of the (C) — methods.

3. We prove the following.

Theorem 1. If 0<lpha<1 and $\{p_n\}\in M_lpha$, then $(N,\,p)$ is $|lpha\,|-effective$.

Theorem 2. If $\{p_n\} \in M_0$, then (N, p) is $|F_0|$ and $|\widetilde{F}_0|$ -effective.

THEOREM 3. If $\{p_n\} \in M_0$, then (N, p) is $|F^*| - effective$.

4. Some preliminary results. We need the following lemmas, of which Lemma 1 is the same as Theorem 6 of Das [3].

LEMMA 1. If $\{p_n\} \in M$, then a necessary and sufficient condition that $\{t_n(v)\} \in BV$, for a given series $\sum_n v_n$ is that

$$\sum_{n=1}^{\infty} rac{1}{nP_n} |\sigma_n(v)| \leq K$$
 ,

where $t_n(v)$ is the n-th (N, p) mean of $\sum_n v_n$.

LEMMA 2. If $\{p_n\}$ is a nonnegative monotonic nonincreasing sequence, then for any n and $0 \le a \le b$

$$\left|\sum_{k=a}^{b} p_k \exp i(n-k)t\right| \leq KP_{[1/t]},$$

uniformly in $0 < t \le \pi$.

Lemma 2 is given in McFadden [10].

LEMMA 3. If $\{p_n\}$ is a positive nonincreasing sequence, then

$$\mid H(n, \, u) \mid = egin{cases} O(n^{lpha}P_n) \;, & for \; all \quad u \;, \ O(n^{lpha}P_{\scriptscriptstyle [1/u]}) \;, & for \; \; u \geqq rac{1}{n} \;. \end{cases}$$

Proof. We write

$$\Gamma(1-\alpha)H(n, u) = \left\{\int_{u}^{u+(1/n)} + \int_{u+(1/n)}^{\pi} \left\{(t-u)^{-\alpha} \frac{d}{dt} h(n, t) dt\right\} \right\}$$

$$= H_1 + H_2,$$

say. If $u \ge 1/n$, then by Abel's Lemma and Lemma 2,

$$|H_1| \le K \!\! \int_u^{u+(1/n)} (t-u)^{-lpha} n P_{[1/t]} dt \le K n^{lpha} P_{[1/u]}$$
 ,

since $\{P_n\}$ is nondecreasing. Next, we have

$$egin{align} |H_2| &= \left| n^lpha \!\! \int_{u+(1/n)}^{\eta} \!\! rac{d}{dt} h(n,\,t) dt
ight|, \qquad u+rac{1}{n} < \eta < \pi \;, \ &\leq K n^lpha P_{\lceil 1/n
ceil} \;, \end{aligned}$$

by virtue of Lemma 2.

This proves the second part of the lemma. The other part follows by a similar reasoning when one observes that

$$\left|\sum_{k=0}^{n} p_{n-k} k \exp\left(ikt\right)\right| \leq K n P_n$$
.

Lemma 4. If $\{p_n\}$ is a positive monotonic nonincreasing sequence, then

$$\mid V(n,\,u)\mid =egin{cases} O(n^{lpha}u^{lpha}P_{n}) & for \ all \quad u \ ; \ O(n^{lpha}) + O(n^{lpha}u^{lpha}P_{[1/u]}) & for \quad u\geqqrac{1}{n} \ . \end{cases}$$

For the proof of Lemma 4, reference may be made to ([9], p. 265).

5. Proof of Theorem 1. (I) $|F_{\alpha}|$ - effectiveness: We have

$$nA_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \frac{d}{dt} \sin nt dt$$

and

$$\sigma_n(L^0(x)) = \int_0^\pi \!\! \phi(t) rac{d}{dt} g(n,\,t) dt$$
 .

As in ([1], proof of Theorem 1), on integration by parts, we get

$$\sigma_n(L^0(x)) = \Phi_{lpha}(\pi)J(n,\,\pi) - \phi_{lpha}(\pi)V(n,\,\pi) + \int_0^\pi V(n,\,u)d\phi_{lpha}(u)$$
 .

Thus, by Lemma 3

$$\sigma_{n}(L^{\scriptscriptstyle 0}(x)) = O(n^{\scriptscriptstyle lpha}) \, - \, \phi_{\scriptscriptstyle lpha}(\pi) \, V(n,\,\pi) \, + \, \int_{\scriptscriptstyle 0}^{\pi} V(n,\,u) d\phi_{\scriptscriptstyle lpha}(u) \; .$$

If in particular, we suppose $\phi(t) = 1$ for all t, in which case $\phi_{\alpha}(t) = 1$ for all t and $\sigma_n(L^0(x)) = 0$ for every n, we obtain

$$0 = O(n^{\alpha}) - \phi_{\alpha}(\pi) V(n, \pi)$$

and therefore

$$\sigma_{n}(L^{\scriptscriptstyle 0}(x)) = O(n^{\scriptscriptstyle lpha}) \, + \, \int_{\scriptscriptstyle 0}^{\scriptscriptstyle au} \! V(n,\, u) d\phi_{\scriptscriptstyle lpha}(u)$$
 .

Thus,

$$egin{aligned} &\sum_{n=1}^{\infty} rac{1}{n P_n} | \ \sigma_n(L^0(x)) \ | \ &\leq K \sum_{n=1}^{\infty} rac{1}{n^{1-lpha} P_n} + K \!\! \int_0^{\pi} \sum_{n=1}^{\infty} rac{1}{n P_n} | \ V(n, \ u) \ | \ | \ d\phi_{lpha}(u) \ | \leq K \ , \end{aligned}$$

since by hypothesis $\int_0^\pi |d\phi_{\alpha}(u)| \le K$ and by Lemma 4,

$$\textstyle \sum_{n=1}^{\infty} \frac{1}{n P_n} | \ V(n, \ u) \ | \ \leq K u^{\alpha} \sum_{n \leq 1/u} n^{\alpha - 1} \ + \ K \{ 1 \ + \ u^{\alpha} P_{\lfloor 1/u \rfloor} \} \sum_{n \geq 1/u} \frac{1}{n^{1 - \alpha} P_n} \ \leq \ K \ ,$$

by virtue of the hypothesis that $\{p_n\} \in M_{\alpha}$.

This completes the proof of $|F_{\alpha}|-e \textit{frective}$ part of Theorem 1, when one appeals to Lemma 1.

(II) $|\widetilde{F}_{\alpha}|$ - effectiveness: We have

$$nB_n(x) = -\frac{2}{\pi} \int_0^{\pi} \psi(t) \frac{d}{dt} \cos nt dt$$

and therefore

$$\sigma_n(\widetilde{L}^0(x)) = -\int_0^{\pi} \psi(t) \frac{d}{dt} \widetilde{g}(n, t) dt$$
.

As in ([2], proof of Theorem 2), we have

$$\begin{split} \sigma_{\scriptscriptstyle{n}}(\widetilde{L}^{\scriptscriptstyle{0}}(x)) &= - \! \int_{\scriptscriptstyle{0}}^{\scriptscriptstyle{\pi}} \! \frac{d}{dt} \widetilde{g}(n,\,t) \! \left\{ \! \frac{1}{\Gamma(1-\alpha)} \! \! \int_{\scriptscriptstyle{0}}^{t} \! (t-u)^{-\alpha} \! d\varPsi_{\scriptscriptstyle{\alpha}}(u) \! \right\} \! dt \\ &= - \! \frac{1}{\Gamma(1-\alpha)} \! \! \int_{\scriptscriptstyle{0}}^{\scriptscriptstyle{\pi}} \! d\varPsi_{\scriptscriptstyle{\alpha}}(u) \! \! \int_{\scriptscriptstyle{u}}^{\scriptscriptstyle{\pi}} \! (t-u)^{-\alpha} \! \frac{d}{dt} \widetilde{g}(n,\,t) dt \\ &= - \! \! \int_{\scriptscriptstyle{0}}^{\scriptscriptstyle{\pi}} \! \widetilde{J}(n,\,u) d\varPsi_{\scriptscriptstyle{\alpha}}(u) \; . \end{split}$$

 $|\widetilde{F}_{\alpha}|$ - effectiveness of the (N,p) mean now follows from Lemma 1 and the hypothesis that $\int_0^{\pi}\!u^{-\alpha}|\,d\varPsi_{\alpha}(u)\,|\le K$, when we observe that uniformly in $0< u\le \pi$

$$u^{\alpha} \sum_{n=1}^{\infty} \frac{1}{n P_n} |\tilde{J}(n, u)| \leq K u^{\alpha} \sum_{n \leq 1/u} n^{\alpha-1} + K u^{\alpha} P_{[1/u]} \sum_{n > 1/u} \frac{1}{n^{1-\alpha} P_n} \leq K,$$

by virtue of Lemma 3 and the hypothesis that $\{p_n\} \in M_{\alpha}$.

(III) $|F^{\alpha}|$ - effectiveness: Integrating by parts, we have

$$n^{lpha+1}A_n(x)=rac{2}{\pi}\!\int_0^\pi\!\!\phi(t)n^{lpha+1}\cos\,ntdt=-rac{2}{\pi}\!\int_0^\pi\!\!n^lpha\sin\,ntd\phi(t)$$
 .

Thus

$$\textstyle\sum_{n=1}^{\infty}\frac{1}{nP_{\scriptscriptstyle m}}|\;\sigma_n(L^{\scriptscriptstyle \alpha}(x))| \leq \int_0^{\pi}\Bigl\{\sum_{n=1}^{\infty}\frac{1}{nP_{\scriptscriptstyle m}}\left|\;\sum_{k=0}^{n}p_{n-k}k^{\scriptscriptstyle \alpha}\sin\,kt\;\right|\Bigr\}|\;d\phi(t)\;|\;.$$

 $|F^{\alpha}|-$ effectiveness, of the (N,p) mean now follows from Lemma 1 and the hypothesis that $\int_0^{\pi} t^{-\alpha} |d\phi(t)| \leq K$, when one observes that uniformly in $0 < t \leq \pi$

$$\begin{split} & t^{\alpha} \sum_{n=1}^{\infty} \frac{1}{nP_n} \bigg| \sum_{k=0}^{n} p_{n-k} k^{\alpha} \sin kt \, \bigg| \\ & \leq K t^{\alpha+1} \sum_{n \leq 1/t} n^{\alpha} + K t^{\alpha} \sum_{n \geq 1/t} \frac{1}{n^{1-\alpha}P_n} \max_{0 \leq \nu \leq n} \, \bigg| \sum_{k=0}^{\nu} p_k \sin \left(n-k\right) t \bigg| \\ & \leq K + K t^{\alpha} P_{\left[1/t\right]} \sum_{n \geq 1/t} \frac{1}{n^{1-\alpha}P} \leq K \; , \end{split}$$

by Abel's lemma, Lemma 2 and the hypothesis that $\{p_n\} \in M_\alpha$. (IV) $|\tilde{F}^\alpha| - effectiveness$: Integrating by parts and observing that $\psi(+0) = 0$, we have

$$egin{align} n^{lpha+1}B_n(x)&=rac{2}{\pi}\!\int_0^\pi\!\psi(t)n^{lpha+1}\sin\,ntdt\ &=-rac{2}{\pi}\psi(\pi)n^lpha\cos n\pi\,+rac{2}{\pi}\!\int_0^\pi\!n^lpha\cos\,ntd\psi(t)\;. \end{split}$$

Thus

$$\begin{split} \sum_{n=1}^\infty \frac{1}{nP_n} | \ \sigma_n(\tilde{L}^\alpha(x)) \ | \ & \leq \ | \ \psi(\pi) \ | \ \sum_{n=1}^\infty \frac{1}{nP_n} \ \Big| \ \sum_{k=0}^n p_{n-k} k^\alpha \cos k\pi \ \Big| \\ & + \ \int_0^\pi \Big\{ \sum_{n=1}^\infty \frac{1}{nP_n} \ \Big| \ \sum_{k=0}^n p_{n-k} k^\alpha \cos kt \ \Big| \ \Big\} | \ d\psi(t) \ | \ . \end{split}$$

 $|\widetilde{F}^{lpha}|-effectiveness$ now follows from Lemma 1 and the hypothesis

that $\int_0^{\pi} t^{-\alpha} |d\psi(t)| \le K$, when one observes that uniformly in $0 < t \le \pi$

$$egin{aligned} t^{lpha} \sum_{n=1}^{\infty} rac{1}{nP_n} \Big| \sum_{k=0}^n p_{n-k} k^{lpha} \cos kt \Big| \ & \leq K t^{lpha} \sum_{n \leq 1/t} n^{lpha-1} + K t^{lpha} \sum_{n > 1/t} rac{1}{n^{1-lpha} P_n} \max_{0 \leq
u \leq n} \Big| \sum_{k=0}^{
u} p_k \cos (n-k) t \Big| \ & \leq K + K t^{lpha} P_{[1/t]} \sum_{n > 1/t} rac{1}{n^{1-lpha} P_n} \leq K \;, \end{aligned}$$

by Lemma 2 and the hypothesis that $\{p_n\} \in M_{\alpha}$.

(V) $|F^*|-effectiveness$: This follows from the result of Theorem 3, when one observes that $\{p_n\}\in M_\alpha$, $\alpha<0$, implies $\{p_n\}\in M_0$.

This completes the proof of Theorem 1.

6. Proof of Theorem 2. It may be observed that the proof of $|F^{\alpha}|$ - effectiveness, given in the preceding section remains valid even for the case $\alpha=0$ and therefore the (N,p) method is $|F^{\alpha}|$ or equivalently $|F_{\alpha}|$ - effective.

In order to prove $|\widetilde{F}_{\scriptscriptstyle 0}|$ - effectiveness, we observe that on integration by parts we get

$$egin{align} B_{\scriptscriptstyle n}(x) &= rac{2}{\pi} \int_{\scriptscriptstyle 0}^{\pi} \psi(t) \sin nt dt = -rac{2}{\pi} \int_{\scriptscriptstyle 0}^{\pi} rac{1-\cos nt}{n} d\psi(t) \ &= rac{2}{\pi} \int_{\scriptscriptstyle 0}^{\pi} rac{\cos nt}{n} d\psi(t) \; , \end{split}$$

since $\psi(\pi) = \psi(0) = 0$.

Thus, we have (cf. [13])

$$rac{\pi}{2}\sigma_{\scriptscriptstyle n}(\widetilde{L}^{\scriptscriptstyle 0}(x)) = egin{cases} -\int_{\scriptscriptstyle 0}^{\pi}\Bigl\{\sum_{k=0}^{n}p_{\scriptscriptstyle n-k}(1-\cos kt)\Bigr\}d\psi(t) \;, \ & \qquad \qquad \ \int_{\scriptscriptstyle 0}^{\pi}\Bigl\{\sum_{k=0}^{n}p_{\scriptscriptstyle n-k}\cos kt\Bigr\}d\psi(t) \;, \end{cases}$$

and

$$egin{aligned} &\sum_{n=1}^{\infty}rac{1}{nP_n}ig|\,\sigma_n(\widetilde{L}^0(x))\,ig|\ &\leq \int_0^\piig|\,d\psi(t)\,ig|igg\{\sum_{n\leq 1/t}rac{1}{nP_n}\,igg|\sum_{k=0}^n p_{n-k}(1-\cos kt)igg|\,+\,\sum_{n\geq 1/t}rac{1}{nP_n}\,igg|\sum_{k=0}^n p_{n-k}\cos ktigg|igg\}\ &=\int_0^\piig|\,d\psi(t)\,ig|\{\Sigma_1+\Sigma_2\}\,\,, \end{aligned}$$

say. $|\widetilde{F}_{\scriptscriptstyle 0}| - \textit{effectiveness}$ of the (N,p) method now follows from Lemma 1 and the hypothesis that $\int_{\scriptscriptstyle 0}^{\pi} \! |d\psi(t)| \leq K$, when one observes that uniformly in $0 < t \leq \pi$

$$\Sigma_1 \leq K t^2 \sum_{n \leq 1/t} n \leq K$$
,

since $|1 - \cos kt| \le k^2 t^2$ and

$$\Sigma_2 \leq KP_{[1/t]} \sum_{n>1/t} \frac{1}{nP_n} \leq K$$
 ,

by virtue of Lemma 2 and the hypothesis that $\{p_n\} \in M_0$. This completes the proof of Theorem 2 (cf. [7]).

7. Proof of Theorem 3. We have ([14], § 13.2)

$$nA_{\scriptscriptstyle n}^*(x) = rac{1}{\pi} \int_{\scriptscriptstyle 0}^{\scriptscriptstyle au} \! \phi^*(t) \Bigl\{ \sin \Bigl(n \, + \, rac{1}{2} \Bigr) t / \sin rac{1}{2} t \Bigr\} dt$$
 .

Thus

$$egin{aligned} &\sum_{n=1}^{\infty} rac{1}{n P_n} | \ \sigma_n(L^*(x)) \ | \ & \leq \int_0^{\pi} \Bigl\{ \sum_{n=1}^{\infty} rac{1}{n P_n} \Bigl| \sum_{k=0}^n p_{n-k} \sin \Bigl(k + rac{1}{2} \Bigr) t \Bigr| \Bigr\} rac{| \ \phi^*(t) \ |}{t} rac{t}{\sin rac{1}{2} t} dt \ . \end{aligned}$$

 $|F^*|-e ext{\it effectiveness}$ of the (N,p) method now follows from Lemma 1 and the hypothesis that $\int_0^{\pi} t^{-1} \, |\phi^*(t)| \, dt \leq K$, when we observe that uniformly in $0 < t \leq \pi$

$$egin{align} \sum_{n=1}^\infty rac{1}{nP_n} \left| \sum_{k=0}^n p_{n-k} \sin\left(k+rac{1}{2}
ight) t
ight| \ & \leq Kt \sum_{n \leq 1/t} 1 + KP_{[1/t]} \sum_{n \geq 1/t} rac{1}{mP_n} \leq K$$
 ,

by virtue of Lemma 2 and the hypothesis that $\{p_n\} \in M_0$. This completes the proof of Theorem 3.

8. Remarks. Corresponding to our Theorem 2, we have an earlier result of Hille and Tamarkin ([8], Th. II) for ordinary (N, p) summability of $L^{0}(x)$ which states that under certain condition on $\phi(t)$ the hypothesis

$$\left\{\frac{1}{P_n}\sum_{k=0}^n\frac{P_k}{k+1}\right\}\in B$$

is both necessary and sufficient for (N,p) summability of $L^{\circ}(x)$, if $\{p_n\}$ is a positive monotonic nonincreasing sequence. The intrinsic character of the hypothesis $\{p_n\}\in M_{\circ}$ of Theorem 2, emerges from

the above result when one observes that the condition (8.1) implies that

$$P_n \sum_{k=n}^{\infty} \frac{1}{kP_k} \leq K$$
, $n=1, 2, \cdots$.

This follows from a recent paper of the author ([4], p. 168).

The claim that the corresponding (C) method results, reduce to special cases of our theorems, follows when we observe that

$$\left\{egin{pmatrix} n+eta-1\ eta-1 \end{pmatrix}
ight\}\in M_{\scriptscriptstylelpha},\, 1\geqqeta>lpha\geqq 0$$
 ,

and appeal to a well known inclusion relation for the absolute (C)-method.

Recently $|F_1|$ - effectiveness of (N, p)(C, 1) and (C, 1)(N, p) methods have been proved by the present author.

REFERENCES

- L. S. Bosanquet, The absolute Cesàro summability of a Fourier series, Proc. London Math. Soc. 41 (1936), 517-528.
- 2. L. S. Bosanquet and J. M. Hyslop, On the absolute summability of the Allied series of a Fourier series, Math. Z. 42 (1937), 489-512.
- 3. G. Das, Tauberian theorems for absolute Nörlund summability, Proc. London Math. Soc. (3) 19 (1969), 357-384.
- H. P. Dikshit, Absolute summability of a Fourier series by Nörlund means, Math. Z. 102 (1967), 166-170.
- 5. ———, Summability of a sequence of Fourier coefficients by a triangular matrix transformation, Proc. Amer. Math. Soc. 21 (1969), 10-20.
- 6. ———, Absolute summability of a series associated with a Fourier series, Proc. Amer. Math. Soc. 22 (1969), 316-318.
- 7. ———, Absolute summability of the conjugate series of a Fourier series by Nörlund means, Math. Ann. **184** (1970), 106-112.
- 8. E. Hille and J. D. Tamarkin, On the summability of Fourier series I, Trans. Amer. Math. Soc. **34** (1932), 757-783.
- 9. N. Kishore and S. N. Bhatt, Absolute Nörlund summability of a Fourier series, Indian J. Math. 9 (1967), 259-267.
- 10. L. McFadden, Absolute Nörlund summability, Duke Math. J. 9 (1942), 168-207.
- 11. R. Mohanty, The absolute Cesàro summability of some series associated with a Fourier series and its allied series, J. London Math. Soc. 25 (1950), 63-67.
- 12. R. Mohanty and S. Mohapatra, On the absolute convergence of a series associated with a Fourier series, Proc. Amer. Math. Soc. 7 (1956), 1049-1053.
- 13. T. Pati, On the absolute Nörlund summability of the conjugate series of a Fourier series, J. London Math. Soc. 38 (1963), 204-213.
- 14. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, New York, 1957.

Received October 22, 1969.

UNIVERSITY OF JABALPUR JABALPUR, INDIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON Stanford University Stanford, California 94305

RICHARD PIERCE University of Washington Seattle, Washington 98105 J. DUGUNDJI Department of Mathematics University of Southern California Los Angeles, California 90007

RICHARD ARENS University of California Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. Yoshida

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

AMERICAN MATHEMATICAL SOCIETY CHEVRON RESEARCH CORPORATION TRW SYSTEMS NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. 36, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 33, No. 3

May, 1970

Charles A. Akemann, Approximate units and maximal abelian	543			
C^* -subalgebras	343			
cellular hulls	551			
John Logan Bryant and De Witt Sumners, On embeddings of 1-dimensional				
compacta in a hyperplane in E^4	555			
H. P. Dikshit, On a class of Nörlund means and Fourier series	559			
Nancy Dykes, Generalizations of realcompact spaces	571			
Hector O. Fattorini, Extension and behavior at infinity of solutions of certain linear operational differential equations	583			
Neal David Glassman, Cohomology of nonassociative algebras				
Neal Hart, Ulm's theorem for Abelian groups modulo bounded groups				
Don Barker Hinton, Continuous spectra of second-order differential	635			
operators	641			
Donald Gordon James, On Witt's theorem for unimodular quadratic forms.	0.1			
II	645			
Melvin F. Janowitz, <i>Principal multiplicative lattices</i>	653			
James Edgar Keesling, On the equivalence of normality and compactness in				
hyperspaces	657			
Adalbert Kerber, Zu einer Arbeit von J. L. Berggren über ambivalente				
Gruppen	669			
Keizō Kikuchi, <i>Various m-representative domains in several complex</i>				
variables	677			
Jack W. Macki and James Stephen Muldowney, <i>The asymptotic behaviour</i>				
of solutions to linear systems of ordinary differential equations	693			
Andy R. Magid, Locally Galois algebras	707			
T. S. Ravisankar, On differentiably simple algebras	725			
Joseph Gail Stampfli, <i>The norm of a derivation</i>	737			
Francis C.Y. Tang, On uniqueness of central decompositions of groups	749			
Robert Charles Thompson, Some matrix factorization theorems. I	763			
Robert Charles Thompson, Some matrix factorization theorems, II	811			