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Necessary and sufficient conditions for a linearly compact simplex K to be uniquely decomposable at a face are given. If P is a cone having the Riesz decomposition property and if $-f, g$ are subadditive homogeneous functions on P with $f \geq g$ then it is shown that there is an additive homogeneous function h on P with $f \geq h \geq g$. If P is a lattice cone for the dual space of an ordered Banach space X and if $-f, g$ are also w^* -continuous then, under certain conditions, it is possible to choose $h \in X$; a consequence of this result is Andô's theorem, that an ordered Banach space has the Riesz decomposition property if its dual space is a lattice. A nonmeasure theoretic proof of Edwards' separation theorem for compact simplexes is also deduced from these results.

Let K be a linearly compact simplex in a real vector space E . Without loss of generality we will assume that K is contained in a hyperplane $e^{-1}(1)$ and that $E = \text{lin } K$, where $\text{lin } K$ denotes the linear hull of K . Then it is well known that E is a vector lattice relative to the cone with base K , and that $\text{co } (K \cup -K)$ is the closed unit ball for a norm making E a pre- AL -space. In fact if K is compact for a locally convex Hausdorff topology on E then E is the Banach dual space of $A(K)$, the space of all affine continuous functions on K with supremum norm (cf. [5]). (We refer to [10] as a general reference for the lattice theory and terminology that is used.)

The set K is said to be *decomposable* at a face F if there exists a *complementary* face F' of K such that $F \cap F' = \phi$ while $\text{co } (F \cup F') = K$. If a complementary face to F exists then it is evident that it is uniquely determined; moreover, in this case, Alfsen [1] has shown that the decomposition is *unique* in the sense that each $k \in K$ has a unique decomposition $k = \lambda x + (1 - \lambda) y$ with $x \in F$, $y \in F'$ and $0 \leq \lambda \leq 1$. Alfsen has also given a necessary and sufficient condition for K to be decomposable at F ; we give here other necessary and sufficient conditions which are perhaps more closely tied to the order and norm structure of E .

THEOREM 1. *Let K be a linearly compact simplex and F, F' disjoint faces of K . Then F and F' are complementary faces for a*

(necessarily unique) decomposition of K if and only if E is the order-direct sum of $\text{lin } F$ and $\text{lin } F'$. Consequently, if E is complete in its norm then K is uniquely decomposable at F if and only if F is norm-closed.

Proof. Since F is a face of K it is easily verified that $\text{lin } F$ is a lattice ideal in E and that $(\text{lin } F) \cap K = F$. If E is the order-direct sum of $\text{lin } F$ and $\text{lin } F'$ then each $x \in K$ has a unique decomposition $x = y + z$ with $y, z \geq 0$, $y \in \text{lin } F$, $z \in \text{lin } F'$; hence $K = \text{co}(F \cup F')$, and the decomposition is unique.

Suppose conversely that $K = \text{co}(F \cup F')$. Then, since $E = \text{lin } F + \text{lin } F'$, it will follow that E is the order-direct sum of $\text{lin } F$ and $\text{lin } F'$ if we prove that $\text{lin } F' = (\text{lin } F)^\perp = \{y \in E: |x| \wedge |y| = 0, \forall x \in \text{lin } F\}$ (cf. [10, p. 38]). Since $(\text{lin } F)^\perp$ is a lattice ideal the set $G = K \cap (\text{lin } F)^\perp$ is a face of K disjoint from F , and hence $G \subseteq F'$. However if $x \in F' \setminus G$ then there exists a $y \in F$ such that $x \wedge y = z \neq 0$; but, since F and F' are faces of K and $x = z + (x - z)$, $y = z + (y - z)$, this implies that $z/\|z\| \in F \cap F'$ which is impossible. Therefore we have $(\text{lin } F)^\perp = \text{lin } G = \text{lin } F'$.

If E is complete in its norm then it is an AL -space. If F is norm-closed then the continuity of the lattice operations in E shows that $\text{lin } F$ is also norm-closed, and hence is a band. Therefore, by a theorem of Riesz (cf. [10, p. 39]), $\text{lin } F$ has an order-direct complement in E , and so K is uniquely decomposable at F . If, conversely, K is uniquely decomposable at F then there exists a natural affine function f on K such that $F = f^{-1}(0)$, $F' = f^{-1}(1)$. The function f has an obvious extension to a continuous linear functional g on E and, since $F = K \cap g^{-1}(0)$, it follows that F is norm-closed.

If K is a compact simplex then E is certainly a Banach space, and so the following result is immediate.

COROLLARY. *If K is a compact simplex and F a face of K , then K is uniquely decomposable at F if and only if F is norm-closed.*

The corollary generalizes Alfsen's result that a compact simplex is uniquely decomposable at each closed (i.e., compact) face. When K is an arbitrary compact convex set Alfsen and Andersen [2] characterize the decomposable faces of K . However it is not true that every linearly compact simplex is decomposable at every norm-closed face, as the following example shows.

Example. Let K denote the continuous nonnegative functions f on

$[0, 1]$ such that $\int_0^1 f(t)dt = 1$, and let $F = \{f \in K: \int_0^{1/2} f(t)dt = 0\}$.

Then K is a base for the lattice cone in $C[0, 1]$, and hence is a linearly compact simplex, and it is clear that F is a face of K . The norm in $C[0, 1]$ associated with K is the $L_1[0, 1]$ -norm, and hence F is norm-closed. Suppose that there exists a face F' complementary to F in K . Then, since $f(1/2) = 0$ for all $f \in F$, there exists a $u \in F'$ such that $u(1/2) > 0$. However it is easy to decompose u nontrivially $u = \lambda g + (1-\lambda)h$ with $g \in F$, $h \in F'$ and $0 < \lambda < 1$. Since F' is a face of K it follows that $g \in F' \cap F$, which is a contradiction. Therefore K is not decomposable at the norm-closed face F .

It has been shown by Asimow [4] that the state space of a function algebra is decomposable at every extreme point, and so such a property does not characterize simplexes among compact convex sets; this property does however characterize simplexes among finite-dimensional compact convex sets as the following slightly more general result shows.

PROPOSITION. *If K is a compact convex set which is decomposable at each extreme point x , and such that each complementary face $\{x\}'$ is closed, then K is a finite-dimensional simplex.*

Proof. If the set K_e of extreme points of K is infinite then there exists an accumulation point $u \in K$. For each $x \in K_e$ the set K_e consists of x together with the extreme points of the closed set $\{x\}'$. Consequently $u \in \{x\}'$ for all $x \in K_e$. Therefore the intersection of the faces $\{x\}'$ forms a closed face F of K which is not empty, since $u \in F$. However if y is an extreme point of F then $y \in K_e$, and also $y \in \{y\}'$ which is impossible. Hence K_e is finite.

If K_e has m points and K has dimension n then, for each $x \in K_e$, it is clear that $\{x\}'$ has $m-1$ extreme points and has dimension $n-1$, and $\{x\}'$ has a similar decomposition property to K . Reducing in this way we see eventually that $m = n + 1$, that is K is an n -dimensional simplex.

If K is a compact simplex then the above result shows that not all faces $\{x\}'$ can be closed. For example, for the simplex $\{x \in l_1: x \geq 0, \|x\| \leq 1\}$ all but one of the faces $\{x\}'$ are closed, while for the simplex of probability measures on $[0, 1]$ none of the faces $\{x\}'$ are closed.

2. We prove an analogue for linearly compact simplexes of Edwards' separation theorem [6], which characterizes compact simplexes; this is a corollary of the following result.

THEOREM 2. *Let P be a cone possessing the Riesz decomposition property, and let $-f, g$ be subadditive homogeneous functionals on P with $f \geq g$. Then there exists an additive homogeneous functional h on P such that $f \geq h \geq g$.*

Proof. If we define h on P by

$$h(x) = \inf \left\{ \sum_{i=1}^n f(x_i) : x = \sum_{i=1}^n x_i, x_i \in P \right\}$$

then it is clear that $f \geq h \geq g$, and hence h is finite-valued. Moreover, h is positive-homogeneous and subadditive. If $x = y + z$ with $x, y, z \in P$, and if $\varepsilon > 0$ choose $x_i \in P$ such that $x = \sum_{i=1}^n x_i$ and $\sum_{i=1}^n f(x_i) \leq h(x) + \varepsilon$. Then there exist $a_{ij} \in P$ such that $\sum_{i=1}^n a_{i1} = y$, $\sum_{i=1}^n a_{i2} = z$, $a_{i1} + a_{i2} = x_i$ for $i = 1, 2, \dots, n$. We have

$$h(x) \geq \sum_{i=1}^n f(x_i) - \varepsilon \geq \sum_{i=1}^n f(a_{i1}) + \sum_{i=1}^n f(a_{i2}) - \varepsilon \geq h(y) + h(z) - \varepsilon,$$

so that h is additive and homogeneous.

In the corollary below K will denote a linearly compact subset of E , again contained in a hyperplane $e^{-1}(1)$ and such that $\text{lin } K = E$. By $A^b(K)$ we will denote the Banach space of all bounded real-valued affine functions on K with the supremum norm. If $\text{co}(K \cup -K)$ is linearly bounded then its Minkowski functional is a norm in E and $A^b(K)$ is simply the Banach dual space of E for this norm. In the particular case when K is compact for some locally convex Hausdorff topology on E , $A^b(K)$ is the second dual space of $A(K)$.

COROLLARY. *The following statements are equivalent.*

- (i) K is a linearly compact simplex.
- (ii) $\text{co}(K \cup -K)$ is linearly compact and, if $-f, g$ are bounded convex functions on K with $f \geq g$, there exists an $h \in A^b(K)$ such that $f \geq h \geq g$.

Proof. (i) \rightarrow (ii). That $\text{co}(K \cup -K)$ is linearly compact was proved in [5, Th. 2]. If P is the cone generated by K as a base then P is a lattice-cone. If f and g are extended homogeneously to the rest of P then the existence of the required $h \in A^b(K)$ follows from the theorem.

(ii) \rightarrow (i). If $u_1, u_2, v_1, v_2 \in A^b(K)$ and $u_1, u_2 \leq v_1, v_2$ then, putting $g(x) = \max [u_1(x), u_2(x)]$, $f(x) = \min [v_1(x), v_2(x)]$ for all $x \in K$, we obtain a function $h \in A^b(K)$ such that $u_1, u_2 \leq h \leq v_1, v_2$. The w^* -compactness of order intervals in $A^b(K)$ now shows that $A^b(K)$ is a vector lattice, in fact an AM -space. Therefore E is an AL -space and, in

particular, each $\eta \in E$ has a unique decomposition $\eta = \eta_1 - \eta_2$ with $\eta_i \geq 0$ and $\|\eta\| = \|\eta_1\| + \|\eta_2\|$, namely for $\eta_1 = \eta^+$, $\eta_2 = \eta^-$. Since, by hypothesis, $\text{co}(K \cup -K)$ is the closed unit ball of E it follows that E is a sublattice of E . Therefore K is a linearly compact simplex.

It is perhaps surprising that the linear compactness condition on $\text{co}(K \cup -K)$ cannot be dropped, as the following two simple examples show.

Examples. (i) Let E be the linear subspace of l_1 spanned by those elements with only finitely many nonzero coordinates, together with the two elements $\{2^{-n}\}$, $\{(-3)^{-n}\}$, and let $K = \{x \in E : x \geq 0, \|x\|_1 \leq 1\}$. If $S = \{x \in E : \|x\|_1 \leq 1\}$ then it is obvious that for each $\varepsilon > 0$ we have $\text{co}(K \cup -K) \subseteq S \subseteq (1 + \varepsilon) \text{co}(K \cup -K)$. If $x = \{(-3)^{-n}\}$ then $x^+ \notin E$, so that $2x \in S$ but $2x \notin \text{co}(K \cup -K)$. Therefore $\text{co}(K \cup -K)$ is not linearly closed; in other terminology E has a $(1 + \varepsilon)$ -generating cone for all $\varepsilon > 0$ but not a 1-generating cone. However, a straightforward verification shows that E has the Riesz decomposition property and hence, by Theorem 2, K has the separation property stated in part (ii) of the Corollary. However K is not a simplex.

(ii) Let K denote the polynomials p nonnegative on $[0, 1]$ and such that $\int_0^1 p(x) dx = 1$. It is clear that $\text{co}(K \cup -K)$ is not linearly compact because the polynomials do not constitute a sublattice of $L_1[0, 1]$. It is true, but less obvious, that $\text{lin } K$ has the Riesz decomposition property (cf. [7]). We are grateful to Professor W. A. J. Luxemburg for bringing this fact and reference to our notice.

By an *ordered Banach space* we shall mean a partially ordered Banach space which has a closed, normal, generating cone. If X is an ordered Banach space then so is X^* (cf. [8]). The following lemma now follows from a result of Kadison ([9, Lemma 4.3]).

LEMMA 1. *Let X be an ordered Banach space and let*

$$K = \{f \in X^* : f \geq 0, \|f\| \leq 1\},$$

equipped with the w^ -topology. Then X is order and topologically isomorphic to*

$$A_0(K) = \{f \in A(K) : f(0) = 0\}.$$

LEMMA 2. *Let C be a cone in a vector space V , let p be a function homogeneous on C and let f be a function affine on V such that $f(x) \leq p(x)$ for all $x \in C$. Then the linear function $g = f - f(0)$ satisfies $g(x) \leq p(x)$ for all $x \in C$.*

Proof. It is simple to check that g is linear on V . Suppose that there is a point $x \in C$ such that $g(x) > p(x)$. Then if $\varepsilon = -f(0)$ and $\delta = f(x) - p(x)$ we have $\varepsilon \geq 0$ and $g(x) - p(x) = \delta + \varepsilon > 0$. Hence there exists an $r \geq 1$ such that $r(\delta + \varepsilon) > \varepsilon$, and we have

$$f(x) = r^{-1}f(rx) + (1 - r^{-1})f(0) .$$

Therefore

$$f(rx) - p(rx) = r(f(x) - p(x)) + (r - 1)\varepsilon = r(\delta + \varepsilon) - \varepsilon > 0 ,$$

which gives a contradiction.

The following theorem is the main result of this section and is a topological version of Theorem 2.

THEOREM 3. *Let X be an ordered Banach space such that the dual cone P^* is a lattice cone in X^* , and let $-f, g$ be w^* -continuous sub-additive homogeneous functionals on P^* with $f \geq g$. If either (i) $f = u_1 \wedge u_2$, $g = v_1 \vee v_2$, where $u_1, u_2, v_1, v_2 \in X$, or (ii) the dual cone in X^{**} possesses an interior point, then there exists an $h \in X$ such that $f \geq h \geq g$.*

Proof. If $K = \{x \in P^*: \|x\| \leq 1\}$ then Lemma 1 shows that we can assume that $X = A_0(K)$, and it is sufficient to find an $h \in X$ such that $f(x) \geq h(x) \geq g(x)$ for all $x \in K$.

Let G denote the w^* -closed convex hull of the graph of f in $K \times R$ and define $\hat{f}(x) = \sup \{u(x) : u \in A(K), u \leq f\}$ for all $x \in K$. A straightforward calculation shows that $\hat{f}(x) \leq \inf \{r : (x, r) \in G\}$ for each $x \in K$. If $\mu < \inf \{r : (x, r) \in G\}$ then by separating (x, μ) from G we obtain a $v \in A(K)$ such that $v \leq f$ while $v(x) > \mu$; therefore $\hat{f}(x) = \inf \{r : (x, r) \in G\}$. Given $\varepsilon > 0$, for each $x \in K$ let N_x be a w^* -compact convex neighbourhood of x such that $|f(x) - f(y)| < \varepsilon$ for each $y \in N_x$, and let $K \subseteq \bigcup_{i=1}^n N_{x_i}$. For each $x \in K$ we therefore have

$$(x, \hat{f}(x)) \in \text{co} \bigcup_{i=1}^n \{N_{x_i} \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon]\} ,$$

and so we can write $x = \sum_{i=1}^n \lambda_i y_i$, $\hat{f}(x) = \sum_{i=1}^n \lambda_i r_i$ with $y_i \in N_{x_i}$ and $r_i \in [f(x_i) - \varepsilon, f(x_i) + \varepsilon]$ for each i . If we now define for each $x \in P^*$

$$\bar{f}(x) = \inf \left\{ \sum_{i=1}^n f(x_i) : x_i \in P^*, \sum_{i=1}^n x_i = x \right\}$$

then, for each $x \in K$,

$$\bar{f}(x) \leq \sum_{i=1}^n \lambda_i f(y_i) \leq \sum_{i=1}^n \lambda_i f(x_i) + \varepsilon \leq \sum_{i=1}^n \lambda_i r_i + 2\varepsilon = \hat{f}(x) + 2\varepsilon .$$

Therefore $\bar{f}(x) \leq \hat{f}(x)$ for each $x \in K$. If $\alpha > 0$ and if $\hat{f}_\alpha(x) = \sup \{u(x) : u \in A(\alpha K), u \leq f\}$ then the argument shows that $\bar{f}(x) \leq \hat{f}_\alpha(x)$ for all $x \in \alpha K$; in particular $\hat{f}_\alpha(0) = 0 = \bar{f}(0)$.

If condition (i) holds then we have

$$\bar{f}(x) = \inf \{u_1(x_1) + u_2(x_2) : x = x_1 + x_2, x_i \in P^*\}.$$

Since P^* is a normal cone we can choose $\alpha > 0$ such that $\|x_1\| + \|x_2\| \leq \alpha \|x\|$ whenever $x = x_1 + x_2$ with $x_i \in P^*$.

If condition (ii) holds and if ζ is an interior point of the dual cone in X^{**} then the order interval $[-\zeta, \zeta]$ is the unit ball for an equivalent norm in X^{**} , and hence X^* has an equivalent norm which is additive on P^* . Therefore there exists an $\alpha > 0$ such that $\sum_{i=1}^n \|x_i\| \leq \alpha \|\sum_{i=1}^n x_i\|$ whenever $x_i \in P^*$.

Now let $x = \sum_{i=1}^n x_i \neq 0$, where $x_i \in P^*$, and with $n = 2$ if (i) holds. If $\lambda = \sum_{i=1}^n \|x_i\|$, and if $y_i = 0$ when $x_i = 0$, $y_i = \lambda x_i / \|x_i\|$ when $x_i \neq 0$, then $y_i \in \alpha K$ for each i . Since \hat{f}_α is convex on αK we have

$$\begin{aligned} \hat{f}_\alpha(x) &= \hat{f}_\alpha\left(\sum_{i=1}^n \frac{\|x_i\|}{\lambda} y_i\right) \leq \sum_{i=1}^n \frac{\|x_i\|}{\lambda} \hat{f}_\alpha(y_i) \leq \sum_{i=1}^n \frac{\|x_i\|}{\lambda} f(y_i) \\ &= \sum_{i=1}^n f(x_i). \end{aligned}$$

In case (ii) this inequality gives $\hat{f}_\alpha(x) \leq \bar{f}(x)$ for each $x \in K$, while in case (i) we have $\hat{f}_\alpha(x) \leq u_1(x_1) + u_2(x_2)$ which again gives $\hat{f}_\alpha(x) \leq \bar{f}(x)$; in either case therefore we have proved that $\bar{f}(x) = \hat{f}_\alpha(x)$ for each $x \in K$. If we define $\|f\| = \sup \{|f(x)| : x \in K\}$, and $\|\bar{f}\|$ similarly, then we have $|\bar{f}(x)| \leq \sum_{i=1}^n \|f\| \|x_i\| \leq \alpha \|f\| \|x\|$ for each $x \in K$, so that $\|\bar{f}\| \leq \alpha \|f\|$.

By Theorem 2 \bar{f} is additive on P^* and the above argument shows that \bar{f} is w^* -l.s.c. on βK for each $\beta > 0$. The set $\{x \in P^* : \bar{f}(x) \leq r\}$ is convex and its intersection with each multiple of the unit ball of X^* is w^* -closed; hence \bar{f} is w^* -l.s.c. on P^* . If we write \bar{g} for $-(-\bar{g})$ then \bar{g} is w^* -u.s.c. on P^* and is additive, homogeneous and satisfies $g \leq \bar{g} \leq \bar{f} \leq f$. If $\varepsilon > 0$ and $r > \alpha$ then, by separating the sets $\{(x, t) \in P^* \times R : t > \bar{f}(x)\}$ and $\{(y, s - \varepsilon/r) \in K \times R : s \leq \bar{g}(y)\}$ and applying Lemma 2, we obtain a $w_\varepsilon \in X$ such that $w_\varepsilon \leq f$ and $w_\varepsilon(x) > g(x) - \varepsilon/r$ for all $x \in K$. Hence if $z_\varepsilon = (g - w_\varepsilon) \vee 0$, z_ε is homogeneous, subadditive and w^* -continuous on P^* with $\|z_\varepsilon\| < \varepsilon/r$. The above argument shows that \bar{z}_ε is w^* -u.s.c. on P^* and that $\|\bar{z}_\varepsilon\| \leq \alpha \|z_\varepsilon\| < \varepsilon$. Since the set $K \times \{\varepsilon/r\}$ is disjoint from the w^* -closed cone $\{(x, t) : x \in P^*, t \leq \bar{z}_\varepsilon(x)\}$ the separation theorem gives a $p_\varepsilon \in X$ such that $p_\varepsilon \geq \bar{z}_\varepsilon \geq g - w_\varepsilon$, 0 and $\|p_\varepsilon\| \leq \varepsilon$.

Using the procedure of the preceding paragraph choose $f_i, g_i \in X$

such that $f_1 \leq f$, $g_1 \geq 0$, $g \leq f_1 + g_1$ and $\|g_1\| < 1/2$, in particular we have $f \wedge (f_1 + g_1) \geq g \vee f_1$. By induction there exist sequences $\{f_n\}$ and $\{g_n\}$ in X such that (a) $g_n \geq 0$, $\|g_n\| < 2^{-n}$, (b) $g \vee f_n \leq f_{n+1} + g_{n+1}$, (c) $f_{n+1} \leq f \wedge (f_n + g_n)$. Properties (b) and (c) give $-g_{n+1} \leq f_{n+1} - f_n \leq g_n$ so that $\|f_{n+1} - f_n\| < 2^{-n}$. Therefore $\{f_n\}$ converges to $h \in X$ such that $h \leq f$ by (c), and $h \geq g$ by (b).

COROLLARY 1 (Andô [3]). *If X is an ordered Banach space such that X^* is a lattice for the dual ordering then X has the Riesz decomposition property.*

COROLLARY 2 (Edwards [6]). *If K is a compact simplex and if $-f$, g are u. s. c. convex functions on K with $f \geq g$ then there exists an $h \in A(K)$ such that $f \geq h \geq g$.*

Proof. By truncating if necessary we may assume that f and g are bounded, say $|f(x)|, |g(x)| \leq \lambda$ for all $x \in K$. First suppose that the strict inequality $f > g$ holds; then the set $G = \{(x, t) : \lambda \leq t \leq g(x)\}$ is compact in $K \times R$ and is a subset of the convex set $H = \{(y, s) : s < f(y)\}$ which is relatively open in $K \times R$. Therefore, taking the convex hull of a finite covering of G by compact convex neighbourhoods in H , we see that H contains the closed convex hull of G . Hence for each $x \in K$ there is an $f_x \in A(K)$ and a neighbourhood U_x of x such that $g < f_x$ while $f_x(y) < f(y)$ for all $y \in U_x$. If $K \leq \bigcup_{i=1}^n U_{x_i}$ and if $f' = f_{x_1} \wedge \dots \wedge f_{x_n}$ then f' is continuous and concave on K with $g < f' < f$. Similarly we can construct a continuous convex function g' on K such that $g < g' < f' < f$. The functions $-f'$, g' have natural extensions to w^* -continuous subadditive homogeneous functions on the positive cone P^* of $A(K)^*$ such that $g' \leq f'$, and so Theorem 3 gives an $h' \in A(K)$ such that $g < g' \leq h' \leq f' \leq f$.

In the general case $f \geq g$ there exists an $h_1 \in A(K)$ such that $f + 1 > h_1 > g - 1$. By considering the functions $(f \wedge h_1) + 1/2$ and $(g \vee h_1) - 1/2$ we similarly obtain an $h_2 \in A(K)$ such that $f + 1/2 > h_2 > g - 1/2$ while $\|h_2 - h_1\| < 1/2$. Proceeding in this way we obtain a sequence $\{h_n\}$ which converges in $A(K)$ to h such that $g \leq h \leq f$.

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