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SOLVABILITY OF CERTAIN p-SOLVABLE LINEAR GROUPS OF FINITE ORDER

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DAVID L. WINTER

Let p be an odd prime. Let G be a finite p-solvable group which does not have a normal p-Sylow subgroup. Let G have a faithful, irreducible representation of degree n over the complex number field. It is proved that if $n = p - 1, p$ or $p + 1, G$ is solvable.

Until now most of the general structure theorems on finite linear groups of degree n over the complex field have been limited to the case $n < p - 1$ where p is a prime divisor of the group order (for example, [5], [8], [3], [4]). In order to obtain suitable results for $n \geqq p - 1$, it is necessary (as it was for $n < p - 1$) to first have results for the class of p-solvable linear groups. Such results are obtained here for $n = p - 1$, p and $p + 1$ in §'s 3, 4 and 5, respectively.

2. Notation and preliminary results.

All groups considered are of finite order. All group representations occurring are representations by linear transformations over the complex numbers and all characters mentioned are characters of such representations. p will always denote a fixed odd prime. A group is called *p-closed* if it has a normal p-Sylow subgroup and *p-nilpotent* if it has a normal p-complement. $Z(H)$ denotes the center of the group H. Z will sometimes be used in place of $Z(G)$.

The following easily verified result is referred to as the *Frattini argument*.

2.1. *If H is a normal subgroup of G and P is a Sylow p-subgroup of H, then $G = N(P)H$.*

2.2. ([7], p. 253) *If the Sylow p-subgroup P of G is abelian, then the maximal p-factor group of G is isomorphic to $P \cap Z(N(P))$.*

2.3. *Let G be a p-solvable group which has a Sylow p-subgroup P of order p. If P is self-centralizing, then G is solvable.*

Indeed, by p-solvability $PO_{p'}(G) \lhd G$ and by the Frattini argument $G = N(P)O_{p'}(G)$. Because P acts fixed-point-free on $O_{p'}(G)$, the latter group is nilpotent by a result of Thompson and (2.3) follows.

The following statement is an immediate consequence of Schur's lemma.

2.4. *Let X be a faithful representation over the field of complex numbers of the finite group G. If H is a subgroup of G such that $X|H$ is irreducible, then $C(H) \leqq Z(G)$.*

2.5. ([10], (2.1)) *Let p be an odd prime and let G be a finite p-solvable group. Suppose G has a faithful representation X over the complex number field all of whose irreducible constituents have degree not exceeding $p-1$. Then G is p-closed unless p is a Fermat prime and X has an irreducible constituent of degree $p-1$.*

We omit the proof of the following elementary result.

2.6. *Let H be a normal subgroup of G of prime index p. Let χ be an ordinary irreducible character of G such that $\chi|H$ is reducible. Then $\chi|H$ is a sum of p distinct irreducible characters of H which are conjugate in G.*

It will be convenient to have (*) denote the following set of conditions.

(*) *Let p be an odd prime and let G be a finite p-solvable group with p-Sylow subgroup P which is not normal in G. Let G have a faithful, irreducible representation X of degree n over the complex number field with character χ.*

3. In this section we prove

THEOREM 1. *If G satisfies (*) and $n = p-1$, then p is a Fermat prime and G/Z has order $2^s p$ for some s. In particular, G is solvable.*

A preliminary step is needed.

3.1. *The conclusions of Theorem 1 hold if it is also assumed that $G = PN$ where $|P| = p$ and N is a normal p-complement of G.*

Proof. Let $B = C(P) \cap N$. We may assume $(\det \chi)(w) = 1$ for $w \in P$, multiplying χ by a suitable linear character of G/N if necessary ([6], Th. 2). Then by ([10], (2.3)) $\chi|P \times B = \rho\Psi + \lambda$ or $\chi|P \times B = \rho\Psi - \lambda$ where Ψ, λ are characters of PB/P and ρ is the character of the regular representation of PB/B. Since $\chi(1) = p-1$, it is easily verified that the second case must occur and $\Psi = \lambda$ is a linear character. Then $\chi | B = (p-1)\lambda$ and therefore $B = Z$.

Let q be an odd prime divisor of $|N|$. Since G is p-nilpotent, there is a q-Sylow subgroup Q of N normalized by P. Applying (2.5) to the odd order group PQ, we get that $Q \leqq B = Z$. Thus G/Z has order $2^s p$ for some s. Finally, p is a Fermat prime by (2.5).

Now let G be a counterexample to Theorem 1 of minimal order. Because $n < p$, $\chi | P$ is a sum of linear characters and P is therefore abelian. Hence $P \leqq C(O_p(G)) \lhd G$. If $C(O_p(G)) \neq G$, then $\chi | C(O_p(G))$ is reducible by (2.4). By (2.5) $C(O_p(G))$ is p-closed which implies G is p-closed. This is a contradiction and therefore $O_p(G) \leqq Z$. By [10], $|P \colon O_p(G)| = p$. Suppose $O_p(G) \neq \langle 1 \rangle$. From (2.2) it follows that G has a normal subgroup H of index p. If H is not p-closed, then $\chi | H$ is irreducible by (2.5). Therefore $Z(H) \leqq Z$ and we get a contradiction by applying the induction hypothesis to H. Therefore H is p-closed and it follows that $H = O_p(G) \times N$ where N is a normal p-complement of G. Since $O_p(G) \leqq Z$, $\chi | O_p(G) = (p-1)\lambda$ for some linear character λ of $O_p(G)$. Let μ be a linear constituent of $\chi | p$. Then $\mu | O_p(G) = \lambda$. Consider μ as a linear character of G/N. Then $\bar{\mu}\chi$ is a faithful irreducible character of $G/O_p(G)$ of degree $p-1$. The induction hypothesis now yields a contradiction because $|\bar{\mu}\chi| = |\chi|$ implies $Z(G/O_p(G)) = Z(G)/O_p(G)$. This proves that $O_p(G) = \langle 1 \rangle$ and $|P| = p$.

By p-solvability, $PO_{p'}(G) \lhd G$ and by the Frattini argument $G = N(P)PO_{p'}(G) = N(P)O_{p'}(G)$. $N(P)$ normalizes the normal p-complement V of $C(P)$ and therefore $V \leqq O_{p'}(G)$. Furthermore, $G/PO_{p'}(G) \cong N(P)/C(P)$ is cyclic of order dividing $p-1$. Since p is a Fermat prime by (2.5), $|G \colon PO_{p'}(G)|$ is a power of 2. Because $PO_{p'}(G)$ is not p-closed, $\chi | PO_{p'}(G)$ is irreducible by (2.5) and this implies $Z(PO_{p'}(G)) \leqq Z$. The proof of Theorem 1 is now completed by applying (3.1) to $PO_{p'}(G)$.

4. The purpose of this section is to prove the following result.

THEOREM 2. *If G satisfies (*) and $n = p$, then G is solvable.*

For the proof, assume Theorem 2 is false and let G denote a counterexample of minimal order.

4.1. *G has a normal series $O_p(G) < N_1 < P_1 \leqq G$ where $N_1/O_p(G) = O_{p'}(G/O_p(G))$, $P_1/N_1 = O_p(G/N_1)$ has order p and $|G \colon P_1|$ is relatively prime to p.*

This is clear from the definitions and the fact that $|P \colon O_p(G)| = p$ by [10].

4.2. *$O_p(G) \nleqq Z$. In particular, $O_p(G) \neq \langle 1 \rangle$.*

Proof. Suppose $\langle 1 \rangle \neq O_p(G) \leqq Z$. Then P is abelian and by (2.2), G has a normal subgroup H of index p. If H is not p-closed, $\chi | H$ is irreducible by Clifford's theorem and (2.5) and then minimality of $|G|$ yields a contradiction. Therefore H is p-closed and we must have $H = O_p(G) \times N$ where N is a normal p-complement of G. A contradiction can now be obtained by applying the induction hypothesis to $G/O_p(G)$ as in the proof of Theorem 1.

Therefore if (4.2) is false, $O_p(G) = \langle 1 \rangle$ and $|P| = p$. In this case consider $PO_{p'}(G) \lhd G$. $PO_{p'}(G)$ cannot be p-closed and therefore, $\chi | PO_{p'}(G)$ is irreducible. This implies that $\chi | O_{p'}(G)$ is a sum of p distinct conjugate linear characters. Hence $O_{p'}(G)$ must be abelian. By the Frattini argument, $G = N(P)PO_{p'}(G) = N(P)O_{p'}(G)$. Since $|P| = p$, $C(P) = P \times V$ for some group $V \leqq O_{p'}(G)$. It follows that $N(P)$ is solvable and hence G is solvable, proving (4.2).

4.3. *X is primitive and $O_p(G)$ is nonabelian.*

Proof. If X is imprimitive, the underlying vector space is a direct sum of p subspaces of dimension 1 which are permuted transitively by the action of G. If K is the normal subgroup of G stabilizing all the subspaces, then K is abelian and G/K is isomorphic to a subgroup of the symmetric group S_p. Since P is not contained in K, it follows from (2.3) that G/K is solvable. This implies that G is solvable, a contradiction. Therefore X is primitive.

If $O_p(G)$ were abelian, primitivity of X would force $O_p(G) \leqq Z$, contrary to (4.2).

Since we are interested only in the solvability of G, it may be assumed, by a method of Blichfeldt ([1], p. 14), that X is unimodular. Now a result of Brauer ([2], (5C)) yields that $G/O_p(G) \cong SL(2, p)$. If $p > 3$, G is not p-solvable and if $p = 3$, G is solvable. These are contradictions and the proof of Theorem 2 is complete.

5. In this section the following theorem is proved.

THEOREM 3. *If G satisfies* (*) *and $n = p + 1$, then p is a Mersenne prime and G is solvable.*

In the first step a special case is treated.

5.1. *Let G be a finite 3-solvable group which has a faithful irreducible representation of degree $n = 4$ over the complex number* *field. If G is not 3-closed, then G is solvable.*

Proof. Let q be a prime with $q \geqq 11$. Then $(q-1)/2 \geqq 5 > n$ and so G has a normal abelian q-Sylow subgroup by [5]. Suppose G does not have a normal abelian 7-Sylow subgroup. Then by [8], G/Z is isomorphic to PSL $(2, 7)$ or A_7 and so G is not 3-solvable. Hence if F is the maximal normal nilpotent subgroup of G, the only possible prime divisors of $|G: F|$ are 2, 3 and 5. Since G/F is 3-solvable, it must be solvable and therefore G is solvable.

Suppose Theorem 3 is false and let G be a counterexample of minimal order. A contradiction is obtained after a series of steps. By [9] it is sufficient to prove that G is solvable.

5.2. *X is a primitive representation of G.*

Proof. Suppose X is imprimitive. Let V be the underlying vector space and let $V_1, \cdots, V_r$ be the subspaces which form a system of imprimitivity for G. Let K be the normal subgroup of G stabilizing all V_i. Then G/K is isomorphic to a subgroup of S_r.

$\chi | K$ is a sum of r constituents all of the same degree $(p+1)/r$ which is less than $p-1$ unless $p = 3$ and $r = 2$. By (5.1) the latter case does not occur. Therefore by (2.5), K is p-closed and consequently $p \| G: K|$ and $r > p$. It follows that $r = p + 1$. Therefore the dimension of each V_i is 1, $\chi | K$ is a sum of linear characters and K is abelian. G/K is solvable by (2.3) and therefore G is solvable.

5.3. *It may be assumed that the following does not hold: $G = PN$ where N is a normal p-complement of G and $|P| = p$.*

Assume on the contrary that $G = PN$ as in (5.3). A contradiction proving (5.3) is obtained after a number of steps. By a method of Blichfeldt ([1], p. 14) we may assume χ is unimodular for this proof.

5.3.1. *Let $B = C(P) \cap N$. Then $\chi | P \times B = \rho\Psi + \lambda$ where Ψ and λ are linear characters of PB/P and ρ is the character of the regular representation of PB/B. B is abelian.*

Proof. By ([10], (2.3)), $\chi | P \times B = \rho\Psi + \lambda$ or $\chi | P \times B = \rho\Psi - \lambda$ where Ψ and λ are characters of PB/P with λ irreducible and ρ is the character of the regular representation of PB/B. It is easily verified that the first case must occur and Ψ and λ are linear characters. Here we use the fact that $\chi | P \times B$ is a linear combination of irreducible characters of $P \times B$ with nonnegative coefficients and

$p > 3$ by (5.1). B is abelian because $\chi | B = p(\Psi | B) + (\lambda | B)$ is a sum of linear characters.

5.3.2. *G contains no proper normal subgroup of index prime to p.*

Proof. Let H be such a subgroup. Then H cannot be p-closed. Therefore by Clifford's theorem, (2.5) and (5.1), $\chi | H$ is irreducible. By minimality of $|G|$, H is solvable and p is a Mersenne prime. By the Frattini argument $G = N(P)H = (P \times B)H = BH$, B being abelian implies that G is solvable and (5.3.2) is proved.

5.3.3. *Let H be a subgroup of G such that $\chi | H$ contains an irreducible constituent of degree p. Then $H \cap N$ is abelian.*

Proof. By assumption $\chi | H = \chi_1 + \chi_2$ where χ_1 is irreducible and χ_2 is linear. Because $p \nmid |H \cap N|$, $\chi_1 | H \cap N$ must be reducible and by (2.6), $\chi_1 | H \cap N$ is a sum of linear characters. Therefore $\chi | H \cap N$ is a sum of linear characters implying $H \cap N$ is abelian.

5.3.4. *Let Q be a Sylow q-subgroup of G for some prime $q \neq p$. Then Q is not contained in B.*

Proof. Suppose on the contrary that $Q \leqq B$. Then $P \leqq C(Q) \lhd N(Q)$. If $P \lhd N(Q)$, then $N(Q) \leqq N(P) = P \times B$. This implies that $N(Q)$ is abelian and that G has a normal q-complement by Burnside's transfer theorem. This contradicts (5.3.2) and therefore $N(Q)$ and, consequently, $C(Q)$ are not p-closed.

By (2.5), $\chi | N(Q)$ contains an irreducible constituent χ_1 of degree at least $p - 1$. If $\chi_1(1) = p$, $N(Q) \cap N$ is abelian by (5.3.3). By Burnside's theorem, N has a normal q-complement N_1 which is a characteristic subgroup of N. Therefore $N_1 \lhd G$ and PN_1 is a group. If $P \lhd PN_1$, then $N_1 \leqq B$ and N_1 is therefore abelian. This yields that G is solvable. Therefore PN_1 is not p-closed and $\chi | PN_1$ contains an irreducible constituent φ of degree at least $p - 1$. If $\varphi(1) = p - 1$, then $\varphi | N_1$ is irreducible. This implies by Clifford's theorem that $\chi | N_1$ is a sum of irreducible characters of degree $p - 1$. $\chi(1) = p + 1$ implies $p = 3$, a contradiction. If $\varphi(1) = p$, then N_1 is abelian by (5.3.3) and G is solvable.

Suppose now that $\chi_1(1) = p - 1$ and at first that $\chi | N(Q) = \chi_1 + \chi_2$ where χ_2 is irreducible. $N(Q) \neq C(Q)$ because G does not have a normal q-complement. $\chi_1 | C(Q)$ must be irreducible by (2.5) and (5.1), and therefore $\chi_2 | C(Q) = \lambda_1 + \lambda_2$ where λ_1 and λ_2 are linear characters conjugate in $N(Q)$ which do not agree on Q. Indeed, otherwise we would have $|\chi_i(x)| = \chi_i(1)$, $i = 1, 2$ for $x \in Q$ and this would imply $Q \leqq Z(N(Q))$. However, by (5.3.1) $\chi | Q$ contains at most two distinct characters. $\chi_1 | Q$ contains exactly one linear character because $\chi_1 | C(Q)$ is irreducible. Therefore $\chi_1 | Q = (p-1)\lambda_i$, $i = 1$ or $i = 2$. But this contradicts Clifford's theorem which states that $\chi_1 | Q$ must contain both λ_1 and λ_2.

Suppose now that $\chi | N(Q) = \chi_1 + \chi_2 + \chi_3$ where χ_1 is irreducible of degree $p - 1$ and $\chi_2(1) = \chi_3(1) = 1$. By the complete reducibility of $X | N(Q)$, $Z(N(Q)) = \{x \in N(Q) \,||\, \chi_1(x)| = p - 1]$. By Theorem 1, P normalizes but does not centralize some Sylow 2-subgroup S of $N(Q)$. Therefore by (2.5) $\chi_1 | S$ is irreducible. This yields that $Z(S) \leqq Z(PS) \leqq B$. Let μ be the linear character of $Z(S)$ such that $\chi_1 | Z(S) = (p-1)\mu$. By (5.3.1), μ must have multiplicity at least p as a constituent of $\chi | Z(S)$. Therefore $\chi_i | Z(S) = \mu$ for $i = 2$ or 3. It follows that $S' \cap Z(S) = \langle 1 \rangle$ because $S' \leqq \ker \chi_2 \cap \ker \chi_3$ and χ is faithful. This is possible only if S is abelian and $\chi_1(1) = p - 1 = 1$, which is a contradiction.

The only remaining case is $\chi | N(Q)$ irreducible. If this holds, $\chi | Q$ is a sum of distinct (since $N(Q) \neq C(Q)$) linear characters each occurring with the same multiplicity. This is contradictory to (5.3.1) and (5.3.4) is proved.

5.3.5. *p is a Mersenne prime and not a Fermat prime. Let q be any odd prime divisor of $|N|$ and let Q be a Sylow q-subgroup of N normalized by P. Then Q is abelian and $\chi | N(Q)$ is irreducible.*

Proof. By (5.3.4) PQ is not p-closed and by (2.5), $\chi | PQ$ contains an irreducible constituent of degree at least $p - 1$. PQ having odd order implies $\chi | PQ$ must have an irreducible constituent of degree p. By (5.3.3), Q is abelian and $\chi | N(Q)$ must contain an irreducible constituent χ_1 of degree at least p. *If* $\chi_1(1) = p$, $N(Q) \cap N$ is abelian and we obtain a contradiction (as in the second paragraph of the proof of (5.3.4)). Therefore $\chi | N(Q)$ is irreducible and $\chi | Q$ is a sum of distinct (by (5.3.4)) linear characters. $N(Q) \neq G$ for otherwise the primitivity of X would be contradicted. Minimality of $|G|$ yields that p is a Mersenne prime. p is not also a Fermat prime since $p \neq 3$.

5.3.6. *Let q be an odd prime divisor of $|N|$. Then q divides $|B|$.*

Proof. Let Q be a Sylow q-subgroup of G normalized by P. By (5.3.4), PQ is not p-closed. Because PQ has odd order $\chi | PQ = \chi_1 + \chi_2$ where the χ_i are irreducible of degree p and 1, respectively. Let K be the kernel of χ_2. Then $Q \nleqq K$ because by (5.3.5) and Clifford's theorem $\chi | Q$ is the sum of conjugate characters and χ is faithful.

Multiplying χ_2 by a nonprincipal linear character of PQ/Q if necessary, we may assume $P \cap K = \langle 1 \rangle$. Then χ_2 is a faithful linear character of $PQ/K \cap Q$ and therefore this group is cyclic and P centralizes $Q/K \cap Q$. It follows that $Q = (B \cap Q)(K \cap Q)$ ([6], Lemma 3 (c)), proving (5.3.6).

5.3.7. *$B - Z$ is nonempty.*

Proof. If $B = Z$, then P acts fixed-point-free on N/Z whence N/Z is nilpotent by a result of Thompson. It follows that G is solvable.

5.3.8. *There exists $b \in B - Z$ such that $C(b)$ is not p-closed.*

Proof. Suppose on the contrary that $C(b)$ is p-closed for all $b \in B - Z$. We shall show that N/Z is a Frobenius group with complement B/Z. Let $\bar{G} = G/Z$ and let $\bar{H}, \bar{x}$ denote, respectively, the subgroup HZ/Z and the element Zx of $\bar{G}$ where $H \leqq G$ and $x \in G$.

Let $\bar{y} \in \bar{B} \cap \bar{B}^{\bar{x}}$, $y \notin Z$, $x \in N$. Then y and $y^{x^{-1}}$ are in B. Therefore P and P^x are in $C(y)$. By assumption, $P = P^X$, so $x \in N(P) \cap N = B$ and therefore $\bar{x} \in \bar{B}$. Therefore $\bar{N}$ is a Frobenius group with abelian complement $\bar{B}$. Consequently, $\bar{N}$ is solvable and it follows that G is solvable.

5.3.9. *For all $b \in B - Z$, $C(b) \cap N = C(B) \cap N$ and this group is abelian.*

Proof. By the preceding step there exists $b_1 \in B - Z$ such that $C(b_1)$ is not p-closed. $\chi \mid C(b_1)$ is reducible because $b_1 \notin Z$ and $\chi \mid C(b_1)$ contains an irreducible constituent χ_1 of degree $p - 1$ or p because $C(b_1)$ is not p-closed. By (2.5), $\chi_1(1) = p$ because p is not a Fermat prime. By (5.3.3), $C(b_1) \cap N$ is abelian. Because $B \leqq C(b_1) \cap N$, $C(b_1) \cap N \leqq C(B) \cap N$ and therefore $C(b_1) \cap N = C(B) \cap N$ and $C(b_1) = C(B)$. Thus $C(B)$ is not p-closed. If $b \in B - Z$, $C(B) \leqq C(b)$ and $C(b)$ cannot be p-closed. Repeating the argument, we have $C(b) \cap N = C(B) \cap N$ as desired.

From (5.3.5), (5.3.6) and (5.3.9), we get

5.3.10. *$|N: C(B) \cap N|$ is a power of 2.*

Let q be an odd prime divisor of $|N|$. Because $C(B)$ is p-nilpotent, there is a q-Sylow subgroup Q of $C(B)$ normalized by P. By (5.3.4),

PQ is not p-closed. Since PQ has odd order, it follows from (2.5) that $\chi | PQ$ contains an irreducible constituent χ_1 of degree p. By (2.6), $\chi_1 | Q$ is a sum of distinct linear characters. Therefore $\chi | Q$ is a sum of $p + 1$ distinct linear characters because $\chi | N(Q)$ is a irreducible by (5.3.5) and Clifford's theorem may be applied. By a result of Brauer ([2], (3F)) $C(Q)/Z$ is a $(2, q)$-group. By unimodularity of X, $|Z| \mid (p + 1)$. Since p is a Mersenne prime Z is a 2-group and therefore $C(Q)$ is a $(2, q)$-group. $C(B) \cap N \leqq C(Q)$ because $C(B) \cap N$ is abelian. Therefore by (5.3.10), 2 and q are the only prime divisors of $|N|$. It follows that N and therefore G are solvable. This completes the proof of (5.3).

5.4. $O_p(G) \nleqq Z(G)$.

Proof. By [10], $|P: O_p(G)| = p$. Assume (5.4) does not hold. As in the proof of (4.2), it can be shown that $|P| = p$, $O_p(G) = \langle 1 \rangle$. Because $PO_{p'}(G) \lhd G$, $PO_{p'}(G)$ is not p-closed and $\chi | PO_{p'}(G)$ is irreducible by (2.5) and (5.1). Let $C(P) = P \times V$. Then $\chi | PVO_{p'}(G)$ is also irreducible. Therefore $PVO_{p'}(G)$ is solvable by either (5.3) or minimality of $|G|$. Because $N(P)/PV$ is cyclic, $N(P)$ is solvable. But by the Frattini argument $G = N(P)PO_{p'}(G)$ and therefore G is solvable, proving (5.4).

Now a final contradiction can be obtained. $\chi | O_p(G)$ must be a sum of $p + 1$ linear characters. If they are all equal, (5.4) is contradicted. If they are not all equal, X is imprimitive contradicting (5.2).

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Pacific Journal of Mathematics

Vol. 34, No. 3 July, 1970

Richard Hindman Bouldin, *The peturbation of the singular spectrum* 569
Hugh D. Brunk and Søren Glud Johansen, *A generalized Radon-Nikodym derivative* 585
Henry Werner Davis, F. J. Murray and J. K. Weber, *Families of L_p-spaces with inductive and projective topologies* 619
Esmond Ernest Devun, *Special semigroups on the two-cell* 639
Murray Eisenberg and James Howard Hedlund, *Expansive automorphisms of Banach spaces* 647
Frances F. Gulick, *Actions of functions in Banach algebras* 657
Douglas Harris, *Regular-closed spaces and proximities* 675
Norman Lloyd Johnson, *Derivable semi-translation planes* 687
Donald E. Knuth, *Permutations, matrices, and generalized Young tableaux* 709
Herbert Frederick Kreimer, Jr., *On the Galois theory of separable algebras* 729
You-Feng Lin and David Alon Rose, *Ascoli's theorem for spaces of multifunctions* 741
David London, *Rearrangement inequalities involving convex functions* 749
Louis Pigno, *A multiplier theorem* 755
Helga Schirmer, *Coincidences and fixed points of multifunctions into trees* 759
Richard A. Scoville, *Some measure algebras on the integers* 769
Ralph Edwin Showalter, *Local regularity of solutions of Sobolev-Galpern partial differential equations* 781
Allan John Sieradski, *Twisted self-homotopy equivalences* 789
John H. Smith, *On S-units almost generated by S-units of subfields* 803
Masamichi Takesaki, *Algebraic equivalence of locally normal representations* 807
Joseph Earl Valentine, *An analogue of Ptolemy's theorem and its converse in hyperbolic geometry* 817
David Lawrence Winter, *Solvability of certain p-solvable linear groups of finite order* 827