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# ON A CERTAIN GENERALIZATION OF $\ell_p$ SPACES

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**An $\mathscr{E}_p$ space is a product of finite-dimensional $c_p$ spaces with a weighted $\ell_p$ norm on the product. The first theorem of this paper yields an isometric embedding of $\mathscr{E}_p$ into an appropriate $c_p$ space. From this theorem, known results about $c_p$ are used to deduce, among other things, the Clarkson inequalities for $\mathscr{E}_p$, $1 < p < \infty$, and hence, the uniform convexity of $\mathscr{E}_p$ for $1 < p < \infty$.**

**The second theorem characterizes the conjugate space of $\mathscr{E}_p$ for $0 < p < 1$. This result is then used to describe some spaces of multipliers. Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{E}_p$ spaces, $1 \leqq p \leqq \infty$, or $\mathscr{E}_0$. The spaces $\mathscr{M}(\mathscr{A}, \mathscr{B})$ of multipliers from $\mathscr{A}$ to $\mathscr{B}$ have previously been identified with certain subspaces of $\mathscr{E}(I)$ and determined precisely in some cases. The third theorem is a complete description of these multiplier spaces: the cases $0 < p < 1$ are included and the spaces $\mathscr{M}(\mathscr{A}, \mathscr{B})$ are determined precisely for all pairs $\mathscr{A}, \mathscr{B}$.**

**1. Definitions.** First, we repeat the definition of $c_p$ (called $C_p$ by Dunford and Schwartz [1], $S_p$ by Gohberg and Krein [2], and $c_p$ by McCarthy [6]). See also [3, D. 37] for the case where $H$ is finite-dimensional.

DEFINITION 1.1. Let $H$ be a Hilbert space and let $X$ be a compact operator on $H$. Then $XX^*$ is positive and compact and hence has a unique positive square root which is also compact. We denote this square root by $|X|$. Now let $\mu_n$ be the, at most countably many, nonzero eigenvalues of $|X|$ enumerated with their multiplicity and arranged in a decreasing sequence as $\mu_1 \geqq \mu_2 \geqq \cdots \geqq 0$. For $0 < p < \infty$, we define

$$||X||_{\phi_p} = \left(\sum_{n=1}^{\infty} \mu_n^p\right)^{1/p}$$

whether finite or infinite; and we define

$$||X||_{\phi_\infty} = \sup\{\mu_n: 1 \leqq n < \infty\} = \mu_1 .$$

Equivalently, [1, p. 1089], $||X||_{\phi_\infty}$ is the operator norm of $X$. Then $c_p$ consists of all compact $X$ with $||X||_{\phi_p}$ finite.

See [1], [2], and [6] for a detailed treatment of $c_p$ spaces and for additional references. Also, [3, Appendix D] contains a number of results in case $H$ is finite-dimensional.

We proceed to define $\mathscr{E}_p$ spaces. These spaces were introduced by R. A. Kunze [5] primarily for the purpose of having analogues of $\ell_p$ spaces in the study of harmonic analysis on compact non-Abelian groups. They have been studied and exploited for this purpose especially by Hewitt and Ross [3].

DEFINITION 1.2. Let $I$ be an index set. For each $\iota \in I$, let $H_\iota$ be a finite-dimensional Hilbert space and let $a_\iota \geqq 1$. We let $\mathscr{E}(I)$ denote the $*$-algebra $\prod_{\iota \in I} \mathscr{B}(H_\iota)$ with all operations defined coordinatewise. Let $E = (E_\iota)_{\iota \in I} \in \mathscr{E}(I)$. For $0 < p < \infty$, we define

$$||E||_p = \left(\sum_{\iota \in I} a_\iota ||E_\iota||_{\phi_p}^p\right)^{1/p} ;$$

we also define

$$||E||_\infty = \sup \{||E_\iota||_{\phi_\infty} : \iota \in I\} .$$

For $0 < p \leqq \infty$, $\mathscr{E}_p(I)$ is defined to be the set of all $E \in \mathscr{E}(I)$ for which $||E||_p$ is finite. In addition, $\mathscr{E}_{00}(I)$ is the set of $E \in \mathscr{E}(I)$ for which $\{\iota \in I : E_\iota \neq 0\}$ is finite; and $\mathscr{E}_0(I)$ is the set of $E \in \mathscr{E}(I)$ for which $\{\iota \in I : ||E_\iota||_{\phi_\infty} \geqq \varepsilon\}$ is finite for all $\varepsilon > 0$. Frequently we write $\mathscr{E}_p$ in place of $\mathscr{E}_p(I)$. We notice that if each $H_\iota$ is one-dimensional, then $\mathscr{E}_p(I)$ is just the $\{a_\iota\}$-weighted $\ell_p$ space which we will call $L_p$; namely, $\{c_\iota\}_{\iota \in I} \in L_p$ if and only if $c_\iota \in K$ for each $\iota \in I$ and $||c||_p = (\sum_{\iota \in I} a_\iota |c_\iota|^p)^{1/p} < \infty$. In addition, if each $a_\iota = 1$, then $\mathscr{E}_p(I)$ is just $\ell_p(I)$. Also, it is convenient to think of $\mathscr{E}_p$ as a product of $c_p$ spaces with a weighted $\ell_p$ norm on the product.

**2. An embedding theorem and some consequences.** In Hewitt and Ross [3], several basic facts about $\mathscr{E}_p$ for $1 \leqq p \leqq \infty$ are proved. There it is shown that Hölder's inequality, Minkowski's inequality and certain generalizations of these hold. The major result of this section is (2.2), a theorem describing a linear isometry of $\mathscr{E}_p$ onto a subspace of an appropriate $c_p$ space. The theorem is then used to derive a number of inequalities for $\mathscr{E}_p$ from results known about $c_p$. We begin with a description of the setting.

Let $I$ be an index set and let $H_\iota$ be a finite-dimensional Hilbert space for each $\iota \in I$. Also, let $a_\iota \geqq 1$ for each $\iota \in I$. For $0 < p \leqq \infty$, $||E||_p$ and $\mathscr{E}_p$ will be as in (1.2). Now from the Hilbert space direct sum $\bigoplus_{\iota \in I} H_\iota$; namely

$$\bigoplus_{\iota \in I} H_\iota = \left\{\{\xi_\iota\} \in \prod_{\iota \in I} H_\iota : \sum_{\iota \in I} ||\xi_\iota||^2 < \infty\right\}$$

with addition and scalar multiplication defined coordinatewise and with an inner product defined by $\langle\{\xi_\iota\}, \{\eta_\iota\}\rangle = \sum_{\iota\in I}\langle\xi_\iota, \eta_\iota\rangle$. It is well known that $\bigoplus_{\iota\in I} H_\iota$ is a Hilbert space under these definitions.

DEFINITION 2.1. Let $0 < p < \infty$ and let $E = (E_\iota)_{\iota\in I} \in \mathscr{E}_p$. Define $T_p(E) = T_E$ where $T_E(\{\xi_\iota\}) = \{a_\iota^{1/p} E_\iota(\xi_\iota)\}$ for all $\{\xi_\iota\} \in \bigoplus_{\iota\in I} H_\iota$. If $p = \infty$ and $E \in \mathscr{E}_\infty$, let $T_\infty(E) = T_E$ where $T_E(\{\xi_\iota\}) = \{E_\iota(\xi_\iota)\}$.

If $p = \infty$, it is known that $T_E \in \mathscr{B}(\bigoplus_{\iota\in I} H_\iota)$ and $\|T_E\| = \|E\|_\infty$. In general we have the following theorem.

THEOREM 2.2. *Let $0 < p < \infty$ and let $T_p$ be defined as above. Then $T_p$ is a linear, $*$-preserving isometry of $\mathscr{E}_p(I)$ onto the subspace $e_p = \{T \in c_p(\bigoplus_{\iota\in I} H_\iota)$: $H_\iota$ is invariant under $T$ for all $\iota \in I\}$ of $c_p(\bigoplus_{\iota\in I} H_\iota)$.*

*Proof.* First, let $\xi = \{\xi_\iota\} \in \bigoplus_{\iota\in I} H_\iota$ so that $T_E(\{\xi_\iota\}) = \{a_\iota^{1/p} E_\iota \xi_\iota\}$ for $E = (E_\iota)_{\iota\in I} \in \mathscr{E}_p$. Then using [1, p. 1093, 9 (a)] to obtain the second inequality below, we have

$$\begin{aligned}
\|T_E(\{\xi_\iota\})\|^2 &= \sum_{\iota\in I} \|a_\iota^{1/p} E_\iota(\xi_\iota)\|^2 \\
&\leqq \sum_{\iota\in I} a_\iota^{2/p} \|E_\iota\|_{\phi_\infty}^2 \|\xi_\iota\|^2 \\
&\leqq \sum_{\iota\in I} a_\iota^{2/p} \|E_\iota\|_{\phi_p}^2 \|\xi_\iota\|^2 \\
&= \sum_{\iota\in I} (a_\iota \|E_\iota\|_{\phi_p}^p)^{2/p} \|\xi_\iota\|^2 \\
&\leqq \sum_{\iota\in I} \|E\|_p^2 \|\xi_\iota\|^2 \\
&= \|E\|_p^2 \|\xi\|^2 .
\end{aligned}$$

Therefore, $T_E(\{\xi_\iota\}) \in \bigoplus_{\iota\in I} H_\iota$ and $\|T_E(\{\xi_\iota\})\| \leqq \|E\|_p \|\xi\|$. Also, $T_E$ is clearly linear. Hence, $T_E \in \mathscr{B}(\bigoplus_{\iota\in I} H_\iota)$ and $\|T_E\| \leqq \|E\|_p$. It is easy to check that $T_p$ is linear and $*$-preserving.

We must now see that $T_E$ is compact for $E \in \mathscr{E}_p$. Since $E \to T_E$ is continuous and $\mathscr{E}_{00}$ is dense in $\mathscr{E}_p$, we need only note that $T_E$ is compact for $E \in \mathscr{E}_{00}$. This is obvious since $T_E$ has finite-dimensional range for $E \in \mathscr{E}_{00}$.

To see that $T_p$ is an isometry, we make the following observation. Suppose $\{\phi_\lambda^j : j = 1, 2, \cdots, d_\lambda\}$ is an orthonormal basis for $H_\lambda$ of dimension $d_\lambda$ for each $\lambda \in I$. For each $\lambda \in I$ and $j = 1, 2, \cdots, d_\lambda$, let $\phi^{\lambda,j} = (\phi_\iota^{\lambda,j})_{\iota\in I} \in \bigoplus_{\iota\in I} H_\iota$ be defined by

$$\phi_\iota^{\lambda,j} = \begin{cases} \phi_\lambda^j & \text{if } \iota = \lambda \\ 0 & \text{if } \iota \neq \lambda . \end{cases}$$

Then it is easy to see that $\{\phi^{\lambda,j} : \lambda \in I, j = 1, 2, \cdots, d_\lambda\}$ is an orthonormal basis for $\bigoplus_{\iota\in I} H_\iota$. Now, let $E \in \mathscr{E}_p$ and let $\{\beta_\lambda^{(j)} : j = 1, 2, \cdots, d_\lambda\}$ be the eigenvalues of $|E_\lambda|$ for each $\lambda \in I$. For each $\lambda \in I$, we choose $\{\phi_\lambda^j: j = 1, 2, \cdots, d_\lambda\}$ to be an orthonormal basis for $H_\lambda$ consisting of eigenvectors corresponding to the eigenvalues $(\beta_\lambda^{(j)})^2$ of $E_\lambda E_\lambda^*$; that is, $E_\lambda E_\lambda^* \phi_\lambda^j = (\beta_\lambda^{(j)})^2 \phi_\lambda^j$. Letting $\phi^{\lambda,j}$ be as above, we have that $T_E T_E^* \phi^{\lambda,j} = T_E T_{E^*} \phi^{\lambda,j} = \{\eta_\iota\}_{\iota \in I}$ where $\eta_\iota = a_\lambda^{2/p} E_\lambda E_\lambda^* \phi_\lambda^j = a_\lambda^{2/p} (\beta_\lambda^{(j)})^2 \phi_\lambda^j$, if $\iota = \lambda$ and $\eta_\iota = 0$ for $\iota \neq \lambda$. That is, $T_E T_E^* \phi^{\lambda,j} = (a_\lambda^{1/p} \beta_\lambda^{(j)})^2 \phi^{\lambda,j}$; or $\{\phi^{\lambda,j}: \lambda \in I, j = 1, \cdots, d_\lambda\}$ is an orthonormal basis for $\bigoplus_{\iota \in I} H_\iota$ consisting of eigenvectors corresponding to the eigenvalues $(a_\lambda^{1/p} \beta_\lambda^{(j)})^2$ of $T_E T_E^*$. Hence, by definition, we have

$$\begin{aligned} \| T_E \|_p^p &= \sum_{\substack{\lambda \in I \\ j=1,2,\cdots,d_\lambda}} (a_\lambda^{1/p} \beta_\lambda^{(j)})^p \\ &= \sum_{\lambda \in I} a_\lambda \sum_{j=1}^{d_\lambda} (\beta_\lambda^{(j)})^p \\ &= \sum_{\lambda \in I} a_\lambda \| E_\lambda \|_{\phi_p}^p = \| E \|_p^p . \end{aligned}$$

Thus, $T_p$ is an isometry.

Finally, we show that $T_p$ maps $\mathscr{E}_p$ onto $e_p(\bigoplus_{\iota \in I} H_\iota)$. Consider $S$ in $e_p(\bigoplus_{\iota \in I} H_\iota)$. For each $\iota \in I$, we let $E_\iota = a_\iota^{-1/p} S|_{H_\iota}$. Since $H_\iota$ is invariant under $S$, $E_\iota \in \mathscr{B}(H_\iota)$ for each $\iota \in I$. Also, we notice that $H_\iota$ is invariant under $S^*$ for each $\iota \in I$. Hence, for $\xi_\iota, \eta_\iota \in H_\iota$, we have

$$\begin{aligned} \langle E_\iota \xi_\iota, \eta_\iota \rangle &= \langle a_\iota^{-1/p} S|_{H_\iota} \xi_\iota, \eta_\iota \rangle \\ &= a_\iota^{-1/p} \langle \xi_\iota, S^*|_{H_\iota} \eta_\iota \rangle \\ &= \langle \xi_\iota, a_\iota^{-1/p} S^*|_{H_\iota} \eta_\iota \rangle \end{aligned}$$

and so $E_\iota^* = a_\iota^{-1/p} S^*|_{H_\iota}$ for each $\iota \in I$. Now we essentially repeat an earlier argument. Namely, let $\{\beta_\lambda^{(j)}: j = 1, 2, \cdots, d_\lambda\}$ be eigenvalues of $|E_\lambda|$ for each $\lambda \in I$ and let $\{\phi_\lambda^j: j = 1, \cdots, d_\lambda\}$ be an orthonormal basis for $H_\lambda$ consisting of eigenvectors corresponding to the eigenvalues $\{(\beta_\lambda^{(j)})^2: j = 1, \cdots, d_\lambda\}$ of $E_\lambda E_\lambda^*$. Then, as above, $SS^* \phi^{\lambda,j} = a_\lambda^{2/p} (\beta_\lambda^{(j)})^2 \phi^{\lambda,j}$ so that $\|S\|_p^p = \|E\|_p^p$ where $E = (E_\iota)_{\iota \in I}$, and hence $E \in \mathscr{E}_p$. Clearly, $S(\xi) = T_E(\xi)$ for all $\xi \in H_\iota, \iota \in I$; thus, by linearity, $S(\xi) = T_E(\xi)$ for all $\xi \in \bigoplus_{\iota \in I} H_\iota$ with $\xi_\iota \neq 0$ for only finitely many $\iota \in I$. By the density of the latter set in $\bigoplus_{\iota \in I} H_\iota$, $S(\xi) = T_E(\xi)$ for all $\xi \in \bigoplus_{\iota \in I} H_\iota$. Hence $T_p(E) = S$ and so $T_p$ maps onto $e_p(\bigoplus_{\iota \in I} H_\iota)$.

We state several corollaries which follow immediately from results for $c_p$ spaces found in [1, XI, § 9], [2, III, § 7] and [6]. Also, compare [3, § 28].

COROLLARY 2.3. *Let* $0 < p \leqq q \leqq \infty$. *Then* $\mathscr{E}_p(I) \subset \mathscr{E}_q(I)$ *and* $\|E\|_q \leqq \|E\|_p$.

COROLLARY 2.4. *Suppose* $0 < p \leqq 1$; *let* $E, F \in \mathscr{E}_p(I)$. *Then*

$$\|E + F\|_p^p \leqq \|E\|_p^p + \|F\|_p^p .$$

*Thus,* $\mathscr{E}_p(I)$ *is a metric space with metric* $\rho$ *where* $\rho(A, B) = ||A - B||_p^p$.

Inequalities (i) and (ii) in the following are due to McCarthy [6, Th. 2.7] for $c_p$ spaces.

COROLLARY 2.5. (*Clarkson's inequalities*). *Let* $E, F \in \mathscr{E}(I)$. *Then, for* $1/p + 1/p' = 1$, *we have*

(i) $2^{p-1}(||E||_p^p + ||F||_p^p) \leqq ||E+F||_p^p + ||E-F||_p^p \leqq 2(||E||_p^p + ||F||_p^p)$ $0 < p \leqq 2$,

(ii) $||E + F||_p^{p'} + ||E - F||_p^{p'} \leqq 2(||E||_p^p + ||F||_p^p)^{p'/p}$ $1 < p \leqq 2$,

(iii) $2(||E||_p^p + ||F||_p^p) \leqq ||E+F||_p^p + ||E-F||_p^p \leqq 2^{p-1}(||E||_p^p + ||F||_p^p)$ $2 \leqq p < \infty$,

(iv) $2(||E||_p^p + ||F||_p^p)^{p'/p} \leqq ||E + F||_p^{p'} + ||E - F||_p^{p'}$ $2 \leqq p < \infty$.

COROLLARY 2.6. *For* $1 < p < \infty$, $\mathscr{E}_p(I)$ *is uniformly convex.* (*Recall that a normed linear space* $X$ *is said to be uniformly convex if* $\delta(\varepsilon) = \inf\{1 - 1/2|x + y|: |x| = |y| = 1, |x - y| = \varepsilon\}$ *is strictly positive in some range* $0 < \varepsilon < \varepsilon_0$.)

*Proof.* Use the Clarkson inequalities (2.5) (ii) and the right hand half of (2.5) (iii) to obtain

$$||E + F||_p^{p'} \leqq 2^{p'} - ||E - F||_p^{p'} \text{ for } 1 < p \leqq 2$$

and

$$||E + F||_p^p \leqq 2^p - ||E - F||_p^p \text{ for } 2 \leqq p < \infty$$

when $||E||_p = ||F||_p = 1$. If, in addition, $||E - F||_p = \varepsilon$, we have

$$1 - \frac{1}{2}||E + F||_p \geqq 1 - \frac{1}{2}(2^{p'} - \varepsilon^{p'})^{1/p'} \text{ for } 1 < p \leqq 2\,,$$

and

$$1 - \frac{1}{2}||E + F||_p \geqq 1 - \frac{1}{2}(2^p - \varepsilon^p)^{1/p} \text{ for } 2 \leqq p < \infty\,.$$

The uniform convexity of $\mathscr{E}_p$ for $1 < p < \infty$ is now clear.

COROLLARY 2.7. (*Radon-Riesz theorem*). *Let* $1 < p < \infty$. *Let* $(E^{(n)})$ *be a sequence in* $\mathscr{E}_p(I)$ *and* $E \in \mathscr{E}_p(I)$ *such that* $E^{(n)} \to E$ *weakly and* $||E^{(n)}||_p \to ||E||_p$. *Then* $||E^{(n)} - E||_p \to 0$.

*Proof.* $\mathscr{E}_p(I)$ is locally uniformly convex; see [4, 15.17 (a)]. Hence, apply [4, 15.17 (a)].

**3. The conjugate space of $\mathscr{E}_p$ for $0 < p < 1$.** Theorem (3.4) below is a characterization of the conjugate space of $\mathscr{E}_p$ for $0 < p < 1$. The conjugate spaces of $\mathscr{E}_p$ for $1 \leqq p < \infty$ are described in [3, § 28]. We first state and prove some easy results which will be used in the proof of (3.4).

LEMMA 3.1. *Let $H$ be a finite-dimensional Hilbert space and let $0 < p, q \leqq \infty$. For each $A \in \mathscr{B}(H)$, there exists $B \in \mathscr{B}(H)$ such that $||B||_{\phi_p} = 1$ and $||A||_{\phi_\infty} = ||AB||_{\phi_q} = \operatorname{tr}(AB)$.*

*Proof.* (Compare [3, D.54].) Let $a$ be the eigenvalue of $|A|$ such that $a = ||A||_{\phi_\infty}$. By [3, D.30] there is an operator $V$ in $\mathscr{U}(H)$ such that $AV = |A|$. Let $\{\zeta_1, \zeta_2, \cdots, \zeta_n\}$ be a basis for $H$ such that $|A|\zeta_1 = a\zeta_1$. Let $P$ be the operator on $H$ such that $P\zeta_1 = \zeta_1$ and $P\zeta_j = 0$ for $j > 1$. Finally, let $B = VP$. By [1, p. 1090, 4 (c)], we have $||B||_{\phi_p} = ||P||_{\phi_p} = 1$. Since $AB = AVP = |A|P$, we have $AB = aP$, and hence

$$||AB||_{\phi_q} = ||aP||_{\phi_q} = a = ||A||_{\phi_\infty} ,$$

and

$$\operatorname{tr}(AB) = \operatorname{tr}(aP) = a = ||A||_{\phi_\infty} .$$

LEMMA 3.2. *Let $H$ be a finite-dimensional Hilbert space, and let $A \in \mathscr{B}(H)$. Then*

(i) *For $0 < p \leqq q \leqq \infty$, we have*

$$||A||_{\phi_\infty} = \sup\{||AB||_{\phi_q}: B \in \mathscr{B}(H) \text{ and } ||B||_{\phi_p} \leqq 1\} ,$$

*and*

(ii) *for $0 < p \leqq 1$, we have*

$$||A||_{\phi_\infty} = \sup\{|\operatorname{tr}(AB)|: B \in \mathscr{B}(H) \text{ and } ||B||_{\phi_p} \leqq 1\} .$$

*Also, the supremum is attained in* (i) *and* (ii).

*Proof.* Let $\alpha = \sup\{||AB||_{\phi_q}: ||B||_{\phi_p} \leqq 1\}$ for $0 < p \leqq q \leqq \infty$. Then by [1, p. 1093, 9 (d) and 9 (a)],

$$||AB||_{\phi_q} \leqq ||A||_{\phi_\infty} ||B||_{\phi_q} \leqq ||A||_{\phi_\infty} ||B||_{\phi_p} \leqq ||A||_{\phi_\infty} ,$$

so that $\alpha \leqq ||A||_{\phi_\infty}$.

For $0 < p \leqq 1$, let $\beta = \sup\{|\operatorname{tr}(AB)|: ||B||_{\phi_p} \leqq 1\}$. By [3, D.46], we have

$$|\operatorname{tr}(AB)| \leqq ||AB||_{\phi_1} \leqq ||A||_{\phi_\infty} ||B||_{\phi_1} \leqq ||A||_{\phi_\infty} ||B||_{\phi_p} \leqq ||A||_{\phi_\infty}$$

so that $\beta \leqq ||A||_{\phi_\infty}$.

The opposite inequalities and the fact that the supremum is attained in (i) and (ii) follow from (3.1).

LEMMA 3.3. *Let $0 < p < 1$, $E \in \mathscr{E}_p(I)$ and $F \in \mathscr{E}_\infty(I)$. Then $EF$ and $FE$ are in $\mathscr{E}_p(I)$,*
( i ) $\|EF\|_p \leqq \|E\|_p \|F\|_\infty$, *and*
(ii) $\|FE\|_p \leqq \|F\|_\infty \|E\|_p$.

*Proof.* Use [1, p. 1093, 9 (d)] to write

$$\begin{aligned}\|EF\|_p^p &= \sum_{\iota \in I} a_\iota \|E_\iota F_\iota\|_{\phi_p}^p \leqq \sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p}^p \|F_\iota\|_{\phi_\infty}^p \\ &\leqq \|F\|_\infty^p \sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p}^p = \|F\|_\infty^p \|E\|_p^p .\end{aligned}$$

Assertion (ii) follows similarly.

THEOREM 3.4. *Let $0 < p < 1$, and let $F \in \mathscr{E}(I)$. If there exists a real number $c > 0$ such that $\|F_\iota\|_{\phi_\infty} \leqq c a_\iota^{(1/p)-1}$ for all $\iota \in I$, then $T_F$, defined on $\mathscr{E}_p(I)$ by $T_F(E) = \langle E, F\rangle = \sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota F_\iota^*)$, is a continuous linear functional on $\mathscr{E}_p(I)$. Conversely, if $T$ is a continuous linear functional on $\mathscr{E}_p(I)$, then $T = T_F$ for some $F \in \mathscr{E}(I)$ such that $\|F_\iota\|_{\phi_\infty} \leqq c a^{(1/p)-1}$ for some $c > 0$ and all $\iota \in I$.*

*Proof.* First, suppose there exists $c > 0$ such that $\|F_\iota\|_{\phi_\infty} \leqq c a_\iota^{(1/p)-1}$ for all $\iota \in I$. Then, for $E \in \mathscr{E}_p(I)$, the number $T_F(E) = \sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota F_\iota^*)$ is well-defined (the series converges absolutely) since by (3.2) and an observation below, we have

$$\begin{aligned}|T_F(E)| &= \left|\sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota F_\iota^*)\right| \\ &\leqq \sum_{\iota \in I} a_\iota |\operatorname{tr}(E_\iota F_\iota^*)| \\ &\leqq \sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p} \|F_\iota\|_{\phi_\infty} \\ &\leqq \sum_{\iota \in I} c a_\iota^{1/p} \|E_\iota\|_{\phi_p} \\ &= c \sum_{\iota \in I} (a_\iota \|E_\iota\|_{\phi_p}^p)^{1/p} \\ &\leqq c\left[\sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p}^p\right]^{1/p} = c\|E\|_p .\end{aligned} \tag{1}$$

The last inequality follows since $1 < 1/p$ so that $\|b\|_{1/p} \leqq \|b\|_1$ for $b \in \ell_1$, and in particular for $b = \{b_\iota\}$ where $b_\iota = a_\iota \|E_\iota\|_{\phi_p}^p$.

The linearity of $T_F$ follows immediately from the linearity of tr [3, D.16]. The inequality (1) also shows that $T_F$ is continuous at 0, hence on $\mathscr{E}_p(I)$. (Recall that $\mathscr{E}_p(I)$ is a metric spaces with $\rho(A, B) = \|A - B\|_p^p$.) Thus, $T_F$ is a continuous linear functional on $\mathscr{E}_p(I)$.

Conversely, let $T$ be a continuous linear functional on $\mathscr{E}_p(I)$. Let $\mathscr{A}_\iota = \{E \in \mathscr{E}_p(I) : E_\lambda = 0 \text{ for } \lambda \neq \iota\}$. Then $\mathscr{A}_\iota$ is isomorphic with $\mathscr{B}(H_\iota)$. Restricting $T$ to $\mathscr{A}_1$, we use elementary algebra to see that there exists $F_\iota \in \mathscr{B}(H_\iota)$ such that $T(E) = a_\iota \operatorname{tr}(E_\iota F_\iota^*)$, for all $E \in \mathscr{A}_\iota$. The linearity of $T$ shows that

$$T(E) = \sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota F_\iota^*)$$

for all $E \in \mathscr{E}_{00}(I)$. Let $F = (F_\iota)_{\iota \in I}$, so that $T = T_F$ on $\mathscr{E}_{00}(I)$.

Now suppose that for every real number $c > 0$, there exists $\iota \in I$ such that $\|F_\iota\|_{\phi_\infty} > c a_\iota^{(1/p)-1}$. In particular, for $n \in \{1, 2, \cdots\}$, let $\iota_n \in I$ be such that $\iota_n \neq \iota_m$ for $m \neq n$ and $\|F_{\iota_n}\|_{\phi_\infty} > n^k a_{\iota_n}^{(1/p)-1}$, where $k$ is a real number greater than zero and such that $2/(1+k) < p$.

For each $n \in \{1, 2, \cdots\}$, let $B_{\iota_n} \in \mathscr{B}(H_{\iota_n})$ be such that $\|B_{\iota_n}\|_{\phi_p} = 1$ and $\|F_{\iota_n}\|_{\phi_\infty} = \operatorname{tr}(F_{\iota_n} B_{\iota_n})$ as in (3.1). Let $b_n = (a_{\iota_n} n^2)^{-1/p}$ for each $n$, and define $E = (E_\iota)_{\iota \in I}$, where $E_\iota = b_n B_{\iota_n}^*$ if $\iota = \iota_n$ for some $n$, and $E_\iota = 0$ otherwise. Then

$$\begin{aligned}
\|E\|_p^p &= \sum_{\iota \in I} a_\iota \|E_\iota\|_{\phi_p}^p = \sum_{n=1}^{\infty} a_{\iota_n} \|b_n B_{\iota_n}^*\|_{\phi_p}^p \\
&= \sum_{n=1}^{\infty} a_{\iota_n} b_n^p \|B_{\iota_n}\|_{\phi_p}^p = \sum_{n=1}^{\infty} a_{\iota_n} (a_{\iota_n} n^2)^{-1} \\
&= \sum_{n=1}^{\infty} n^{-2} < \infty
\end{aligned}$$

so that $E \in \mathscr{E}_p(I)$.

For each positive integer $N$, define $E^{(N)} = (E_\iota^{(N)})_{\iota \in I}$, where $E_\iota^{(N)} = E_\iota$ if $\iota = \iota_n$ with $n \leqq N$, and $E_\iota^{(N)} = 0$ otherwise. Then $E^{(N)} \in \mathscr{E}_{00}(I)$ and $\|E^{(N)}\|_p^p \leqq \|E\|_p^p$ for each $N$. However,

$$\begin{aligned}
T(E^{(N)}) = T_F(E^{(N)}) &= \sum_{\iota \in I} a_\iota \operatorname{tr}(E_\iota^{(N)} F_\iota^*) \\
&= \sum_{n=1}^{N} a_{\iota_n} \operatorname{tr}(E_{\iota_n} F_{\iota_n}^*) \\
&= \sum_{n=1}^{N} a_{\iota_n} \operatorname{tr}(b_n B_{\iota_n}^* F_{\iota_n}^*) \\
&= \sum_{n=1}^{N} a_{\iota_n} b_n \operatorname{tr}((F_{\iota_n} B_{\iota_n})^*) \\
&= \sum_{n=1}^{N} a_{\iota_n} b_n \overline{\operatorname{tr}(F_{\iota_n} B_{\iota_n})} \\
&= \sum_{n=1}^{N} a_{\iota_n} (a_{\iota_n} n^2)^{-1/p} \|F_{\iota_n}\|_{\phi_\infty} \\
&> \sum_{n=1}^{N} a_{\iota_n} (a_{\iota_n} n^2)^{-1/p} n^k a_{\iota_n}^{(1/p)-1} \\
&= \sum_{n=1}^{N} n^{k-2/p} > \sum_{n=1}^{N} 1/n .
\end{aligned}$$

A simple argument now shows that $T$ is discontinuous, a contradiction. Therefore, there exists $c > 0$ so that $\|F_\iota\|_{\phi_\infty} \leqq c a_\iota^{(1/p)-1}$ for all $\iota \in I$. Thus, $T_F$ and $T$ are continuous linear functionals on $\mathscr{E}_p(I)$ which agree on $\mathscr{E}_{00}(I)$, a dense subspace of $\mathscr{E}_p(I)$, so that $T = T_F$ on $\mathscr{E}_p(I)$.

Several easy corollaries follow and will be stated without proof. The notation is as in (3.4).

COROLLARY 3.5. *If* $0 < p < 1$ *and if* $\sup_{\iota \in I} a_\iota < \infty$, *then* $\mathscr{E}_p^* = \{T_F\colon F \in \mathscr{E}_\infty\}$.

COROLLARY 3.6. *Let* $0 < p < 1$ *and let* $L_p$ *be a weighted* $\ell_p$ *space; say* $\|b\|_p = (\sum_{\iota \in I} a_\iota |b_\iota|^p)^{1/p}$ *for* $\{b_\iota\} \in L_p$. *For* $b = \{b_\iota\} \in L_p$ *and* $c = \{c_\iota\}$, *let* $T_c(b) = \sum_{\iota \in I} a_\iota b_\iota \bar{c}_\iota$. *Then*

$$L_p^* = \{T_c\colon |c_\iota| \leqq k a_\iota^{(1/p)-1} \text{ for some } k > 0 \text{ and all } \iota \in I\}.$$

COROLLARY 3.7. *If* $0 < p < 1$, *then* $\ell_p^* = \{T_c\colon c \in \ell_\infty\}$.

**4. Some multiplier theorems.** Theorem (4.2) is a collection of results concerning $(\mathscr{E}_p, \mathscr{E}_q)$-multipliers. We use the following definition: Let $\mathscr{A}$ and $\mathscr{B}$ be subsets of $\mathscr{E}(I)$. We say that $E$ in $\mathscr{E}(I)$ is an $(\mathscr{A}, \mathscr{B})$-multiplier if $EA \in \mathscr{B}$ for all $A \in \mathscr{A}$. The set of all $(\mathscr{A}, \mathscr{B})$-multipliers is denoted by $\mathscr{M}(\mathscr{A}, \mathscr{B})$.

Clearly, multipliers may be discussed in a context much wider than that of $\mathscr{E}_p$ spaces. For example, it is known that $\ell_r = \mathscr{M}(\ell_q, \ell_p)$ for $0 < p < q < \infty$ with $1/r = 1/p - 1/q$. Also, it is shown in McCarthy [6, Ths. 2.3 and 5.1] that $\mathscr{M}(c_q, c_p) = c_r$ for $p$, $q$ and $r$ as above.

In Hewitt and Ross [3, 35.4] $\mathscr{M}(\mathscr{A}, \mathscr{B})$ is described for any pair $(\mathscr{A}, \mathscr{B})$ chosen from the spaces $\mathscr{E}_p$, $\mathscr{E}_q$, $\mathscr{E}_0$, $\mathscr{E}_\infty$ with $1 \leqq p < q < \infty$ with the following exceptions: if $\sup_{\iota \in I} a_\iota = \infty$, it is shown only that $\mathscr{M}(\mathscr{A}, \mathscr{B}) \supsetneqq \mathscr{E}_\infty$, where $\mathscr{A} = \mathscr{E}_p$ and $\mathscr{B} = \mathscr{E}_q$ or $\mathscr{B} = \mathscr{E}_0$ with $1 \leqq p < q < \infty$. Our theorem which follows extends the results of [3, 35.4] to all $p$ and $q$ with $0 < p < q < \infty$. Also, it identifies $\mathscr{M}(\mathscr{A}, \mathscr{B})$ precisely in the exceptions mentioned above when $\sup_{\iota \in I} a_\iota = \infty$. The major tool used in the identification of $\mathscr{M}(\mathscr{A}, \mathscr{B})$ in the cases where $\sup_{\iota \in I} a_\iota = \infty$ is (3.4), our characterization of $\mathscr{E}_p^*$ for $0 < p < 1$.

Before stating our theorem we note that the following lemma may easily be verified using [6, Th. 2.3] and the generalized Hölder inequality.

LEMMA 4.1. *Let* $0 < p, q, r < \infty$ *with* $1/p + 1/q = 1/r$. *If* $E \in \mathscr{E}_p(I)$, $F \in \mathscr{E}_q(I)$, *then* $EF \in \mathscr{E}_r(I)$ *and* $\|EF\|_r \leqq \|E\|_p \|F\|_q$.

THEOREM 4.2. *Let* $0 < p < q < \infty$ *and let* $r$ *be so that* $1/r = 1/p - 1/q$. *For each space* $\mathscr{A}$ *listed to the left of the matrix below and each space* $\mathscr{B}$ *listed above the matrix, the corresponding entry of the matrix is exactly* $\mathscr{M}(\mathscr{A}, \mathscr{B})$.

| | $\mathscr{E}_p$ | $\mathscr{E}_q$ | $\mathscr{E}_0$ | $\mathscr{E}_\infty$ |
|---|---|---|---|---|
| $\mathscr{E}_\infty$ | $\mathscr{E}_p$ | $\mathscr{E}_q$ | $\mathscr{E}_0$ | $\mathscr{E}_\infty$ |
| $\mathscr{E}_0$ | $\mathscr{E}_p$ | $\mathscr{E}_q$ | $\mathscr{E}_\infty$ | $\mathscr{E}_\infty$ |
| $\mathscr{E}_q$ | $\mathscr{E}_r$ | $\mathscr{E}_\infty$ | $\mathscr{E}_s^*$ $s=\dfrac{q}{1+q}$ | $\mathscr{E}_s^*$ $s=\dfrac{q}{1+q}$ |
| $\mathscr{E}_p$ | $\mathscr{E}_\infty$ | $\mathscr{E}_s^*$ $s=\dfrac{pq}{q-p+pq}$ | $\mathscr{E}_s^*$ $s=\dfrac{p}{1+p}$ | $\mathscr{E}_s^*$ $s=\dfrac{p}{1+p}$ |

The proof of the above theorem will be broken into several parts.

Part I. For $0 < p \leqq \infty$, $\mathscr{M}(\mathscr{E}_p, \mathscr{E}_p) = \mathscr{E}_\infty$.

*Proof.* In case $1 \leqq p \leqq \infty$, we use the proof of [3, 35.4, Part II] with $d_{\sigma_n}$ replaced by $a_{\sigma_n}$ throughout.

Now let $0 < p < 1$. The fact that $\mathscr{E}_\infty \subset \mathscr{M}(\mathscr{E}_p, \mathscr{E}_p)$ follows from (3.3). The proof of the opposite inclusion is similar to the proof of [3, 35.4, Part II]. Namely, suppose $E \notin \mathscr{E}_\infty(I)$. Then there is a sequence $\{\iota_n\}_{n=1}^\infty$ of distinct elements in $I$ such that $||E_{\iota_n}||_{\phi_\infty} > n$ for each $n$. By (3.1), there exists $B_{\iota_n}$ in $\mathscr{B}(H_{\iota_n})$ such that $||E_{\iota_n}B_{\iota_n}||_{\phi_p} > n$ and $||B_{\iota_n}||_{\phi_p} = 1$. For $n \in \{1, 2, \cdots\}$, let $\alpha_n = (a_{\iota_n}n^{1+p})^{-1/p}$. Define $A \in \mathscr{E}(I)$ as follows: $A_{\iota_n} = \alpha_n B_{\iota_n}$ for $n \in \{1, 2, \cdots\}$ and $A_\iota = 0$ for all other $\iota$'s in $I$. Since

$$||A||_p^p = \sum_{n=1}^\infty a_{\iota_n}||\alpha_n B_{\iota_n}||_{\phi_p}^p$$
$$= \sum_{n=1}^\infty n^{-(1+p)} < \infty ,$$

we have that $A \in \mathscr{E}_p(I)$. On the other hand, $EA$ does not belong to $\mathscr{E}_p(I)$ because

$$||EA||_p^p = \sum_{n=1}^\infty a_{\iota_n}||\alpha_n E_{\iota_n}B_{\iota_n}||_{\phi_p}^p \geqq \sum_{n=1}^\infty a_{\iota_n}(a_{\iota_n}n^{1+p})^{-1}n^p$$
$$= \sum_{n=1}^\infty 1/n = \infty .$$

Thus, $E \notin \mathscr{M}(\mathscr{E}_p, \mathscr{E}_p)$ and so $\mathscr{M}(\mathscr{E}_p, \mathscr{E}_p) \subset \mathscr{E}_\infty(I)$. Hence, entries (1, 4), (3, 2), and (4, 1) are verified.

Part II. For $0 < p < \infty$, we have that $\mathscr{E}_p = \mathscr{M}(\mathscr{E}_0, \mathscr{E}_p) = \mathscr{M}(\mathscr{E}_\infty, \mathscr{E}_p)$. This will verify entries (1, 1), (1, 2), (2, 1), and (2, 2).

*Proof.* Using (3.3) we see that, for $0 < p < 1$, $\mathscr{E}_p \subset \mathscr{M}(\mathscr{E}_\infty, \mathscr{E}_p) \subset \mathscr{M}(\mathscr{E}_0, \mathscr{E}_p)$. The rest of the assertion is proved in [3, 35.4, Part VII] if we replace $d_\sigma$ by $a_\sigma$ throughout.

Part III. Let $0 < p < q < \infty$ and let $s = pq/(q - p + pq)$. Then $\mathscr{E}_s^* = \mathscr{M}(\mathscr{E}_p, \mathscr{E}_q)$.

*Proof.* Consider $T_F \in \mathscr{E}_s^*$ with $s$ as above. Then $0 < s < 1$ so that by (3.4), there exists a real number $c > 0$ such that $||F_\iota||_{\phi_\infty} \leqq ca_\iota^{(1/s)-1}$. Let $E \in \mathscr{E}_p$. The following is seen to be true by using $||\ ||_{\ell_q} \leqq ||\ ||_{\ell_p}$ for $0 < p < q < \infty$ and the results (3.3), [3, D.52.i.], and (2.3).

$$\begin{aligned} ||FE||_q &= \left[\sum_{\iota \in I} (a_\iota^{1/q} ||F_\iota E_\iota||_{\phi_q})^q\right]^{1/q} \\ &\leqq \left[\sum_{\iota \in I} (a_\iota^{1/q} ||F_\iota E_\iota||_{\phi_q})^p\right]^{1/p} \\ &\leqq \left[\sum_{\iota \in I} a_\iota^{p/q} ||F_\iota||_{\phi_\infty}^p ||E_\iota||_{\phi_q}^p\right]^{1/p} \\ &\leqq \left[\sum_{\iota \in I} a_\iota^{p/q} c^p a_\iota^{(p/s)-p} ||E_\iota||_{\phi_p}^p\right]^{1/p} \\ &= c\left[\sum_{\iota \in I} a_\iota ||E_\iota||_{\phi_p}^p\right]^{1/p} \\ &= c||E||_p\,. \end{aligned}$$

Thus, $FE \in \mathscr{E}_q$ so that $F \in \mathscr{M}(\mathscr{E}_p, \mathscr{E}_q)$. Hence, $\mathscr{E}_s^* \subset \mathscr{M}(\mathscr{E}_p, \mathscr{E}_q)$.

On the other hand, suppose $T_F \notin \mathscr{E}_s^*$. Again, by (3.4), we have that for every $c > 0$, there exists $\iota \in I$ such that $||F_\iota||_{\phi_\infty} > ca_\iota^{(1/s)-1}$. Or, in particular, for each $n \in \{1, 2, \cdots\}$, let $\iota_n$ be such that $\iota_n \neq \iota_m$ for $n \neq m$ and

$$||F_{\iota_n}||_{\phi_\infty} > n^k a_{\iota_n}^{(1/s)-1}$$

where $k$ is a real number satisfying $k \geqq 2/p - 1/q$; that is, $1 \geqq q(2/p - k)$. For each $n \in \{1, 2, \cdots\}$, let $B_{\iota_n} \in \mathscr{B}(H_{\iota_n})$ be such that $||B_{\iota_n}||_{\phi_p} = 1$ and $||F_{\iota_n} B_{\iota_n}||_{\phi_q} = ||F_{\iota_n}||_{\phi_\infty}$ as in (3.1). Let $b_n = (a_{\iota_n} n^2)^{-1/p}$ and define $E_\iota = b_n B_{\iota_n}$ if $\iota = \iota_n$ and $E_\iota = 0$ otherwise. Let $E = (E_\iota)_{\iota \in I}$. Then

$$||E||_p^p = \sum_{n=1}^{\infty} a_{\iota_n} ||b_n B_{\iota_n}||_{\phi_p}^p$$

$$= \sum_{n=1}^{\infty} a_{\iota_n}(a_{\iota_n}n^2)^{-1} = \sum_{n=1}^{\infty} 1/n^2 < \infty ,$$

so that $E \in \mathscr{E}_p$. However,

$$\begin{aligned} ||FE||_q^q &= \sum_{n=1}^{\infty} a_{\iota_n}||F_{\iota_n} b_n B_{\iota_n}||_{\phi_q}^q \\ &= \sum_{n=1}^{\infty} a_{\iota_n}(a_{\iota_n}n^2)^{-q/p}||F_{\iota_n}||_{\phi_\infty}^q \\ &\geqq \sum_{n=1}^{\infty} a_{\iota_n}(a_{\iota_n}n^2)^{-q/p} n^{qk} a_{\iota_n}^{q/s-q} \\ &= \sum_{n=1}^{\infty} n^{q(k-2/p)} \geqq \sum_{n=1}^{\infty} 1/n = \infty . \end{aligned}$$

Thus, $FE \notin \mathscr{E}_q$ so that $F \notin \mathscr{M}(\mathscr{E}_p, \mathscr{E}_q)$. We have, therefore, that $\mathscr{M}(\mathscr{E}_p, \mathscr{E}_q) \subset \mathscr{E}_s^*$ and (4, 2) is verified.

Part IV. We verify entries (3, 3), (3, 4), (4, 3) and (4, 4) by showing that for $0 < p < \infty$,

$$\mathscr{E}_s^* = \mathscr{M}(\mathscr{E}_p, \mathscr{E}_0) = \mathscr{M}(\mathscr{E}_p, \mathscr{E}_\infty) \text{ where } s = \frac{p}{1+p} .$$

*Proof.* Let $T_F \in \mathscr{E}_s^*$. We will first show that $F \in \mathscr{M}(\mathscr{E}_p, \mathscr{E}_0)$. By (3.4), there exists a constant $c > 0$ such that $||F_\iota||_{\phi_\infty} \leqq ca_\iota^{1/s-1} = ca_\iota^{1/p}$ for all $\iota \in I$. Let $E \in \mathscr{E}_p$ so that $\sum_{\iota \in I} a_\iota ||E_\iota||_{\phi_p}^p < \infty$. Then, for $\varepsilon > 0$, $a_\iota ||E_\iota||_\phi^p \leqq (\varepsilon/c)^p$ for all except finitely many $\iota \in I$. Thus,

$$\begin{aligned} ||F_\iota E_\iota||_{\phi_\infty} &\leqq ||F_\iota||_{\phi_\infty}||E_\iota||_{\phi_\infty} \leqq ||F_\iota||_{\phi_\infty}||E_\iota||_{\phi_p} \\ &\leqq ca_\iota^{1/p}||E_\iota||_{\phi_p} \leqq c\cdot\frac{\varepsilon}{c} = \varepsilon \end{aligned}$$

for all except finitely many $\iota \in I$. Hence, $FE \in \mathscr{E}_0$ so that $F \in \mathscr{M}(\mathscr{E}_p, \mathscr{E}_0)$. Clearly $\mathscr{M}(\mathscr{E}_p, \mathscr{E}_0) \subset \mathscr{M}(\mathscr{E}_p, \mathscr{E}_\infty)$ so that it remains only to show that $\mathscr{M}(\mathscr{E}_p, \mathscr{E}_\infty) \subset \mathscr{E}_s^*$.

Suppose $T_F \notin \mathscr{E}_s^*$. Then by (3.4), for each $n \in \{1, 2, \cdots\}$, we can choose distinct $\iota_n \in I$ with the property that $||F_{\iota_n}||_{\phi_\infty} > n^{2/p+1}a_{\iota_n}^{-1/p}$. As in (3.1), for each $\iota \in I$, let $B_\iota \in \mathscr{B}(H_\iota)$ be such that $||B_\iota||_{\phi_p} = 1$ and $||F_\iota||_{\phi_\infty} = ||F_\iota B_\iota||_{\phi_\infty}$. For each $n \in \{1, 2, \cdots\}$ let $b_n = (a_{\iota_n}n^2)^{-1/p}$ and let $E = (E_\iota)_{\iota \in I}$ where $E_\iota = b_n B_{\iota_n}$ if $\iota = \iota_n$ and $E_\iota = 0$ otherwise. An in Part III, it is clear that $E \in \mathscr{E}_p$. However, $||F_{\iota_n}E_{\iota_n}||_{\phi_\infty} = ||F_{\iota_n}b_n B_{\iota_n}||_{\phi_\infty} = b_n||F_{\iota_n}||_{\phi_\infty} > n$ for $n \in \{1, 2, \cdots\}$. Thus, $||FE||_\infty$ is not finite so that $FE \notin \mathscr{E}_\infty$. Hence, $F \notin \mathscr{M}(\mathscr{E}_p, \mathscr{E}_\infty)$ and so $\mathscr{M}(\mathscr{E}_p, \mathscr{E}_\infty) \subset \mathscr{E}_s^*$.

Part V. If $0 < p < q < \infty$ and $1/p - 1/q = 1/r$, then $\mathscr{M}(\mathscr{E}_q, \mathscr{E}_p) = \mathscr{E}_r$.

*Proof.* This result is proved for $1 \leqq p < q < \infty$ in [3, 35.4, Part VI]. That proof does not carry over to our wider range for $p, q$ and $r$, however.

The inclusion $\mathscr{E}_r \subset \mathscr{M}(\mathscr{E}_q, \mathscr{E}_p)$ follows immediately from (4.1). To see the opposite inclusion, suppose that $E = (E_\iota)_{\iota \in I}$ is in $\mathscr{E}(I)$ but not in $\mathscr{E}_r$. We will show that $E \notin \mathscr{M}(\mathscr{E}_q, \mathscr{E}_p)$.

Let $\gamma_\iota = a_\iota^{1/r} || E_\iota ||_{\phi_r}$. Since $E \notin \mathscr{E}_r$, $\{\gamma_\iota\}$ does not belong to $\ell_r(I)$. However, since $\ell_r = \mathscr{M}(\ell_q, \ell_p)$, there exists $\{\beta_\iota\} \in \ell_q$ such that $\{\gamma_\iota \beta_\iota\} \notin \ell_p$. We may, and will, choose $\beta_\iota$ so that $\beta_\iota \geqq 0$ for all $\iota \in I$. Using [6, Th. 2.3] choose $F_\iota$ so that $|| E_\iota F_\iota ||_{\phi_p} = || E_\iota ||_{\phi_r} || F_\iota ||_{\phi_q}$ for each $\iota \in I$ and such that $E_\iota \neq 0$ if and only if $F_\iota \neq 0$. [For example, let $F_\iota = | E_\iota |^{r/q}$. That the above equality holds in this case may be seen directly using conditions for equality in Hölder's inequality for $\ell_p$.]

For our convenience below let $\Phi = \{\iota \in I : \gamma_\iota \neq 0\}$. Note also that $\Phi = \{\iota \in I : E_\iota \neq 0\}$. For $\iota \in \Phi$, let $c_\iota = \beta_\iota a_\iota^{-1/q} || F_\iota ||_{\phi_q}^{-1}$, otherwise $c_\iota = 0$. For all $\iota \in I$, let $F'_\iota = c_\iota F_\iota$ and let $F' = (F'_\iota)_{\iota \in I}$. Then

$$|| F' ||_q^q = \sum_{\iota \in I} a_\iota || F'_\iota ||_{\phi_q}^q = \sum_{\iota \in \Phi} a_\iota \beta_\iota^q a_\iota^{-1} || F_\iota ||_{\phi_q}^{-q} || F_\iota ||_{\phi_q}^q = \sum_{\iota \in \Phi} \beta_\iota^q \leqq \sum_{\iota \in I} \beta_\iota^q < \infty$$

since $\{\beta_\iota\} \in \ell_q$. Thus, $F' \in \mathscr{E}_q$. However,

$$\begin{aligned} || EF' ||_p^p &= \sum_{\iota \in I} a_\iota || E_\iota F'_\iota ||_{\phi_p}^p \\ &= \sum_{\iota \in I} a_\iota c_\iota^p || E_\iota F_\iota ||_{\phi_p}^p \\ &= \sum_{\iota \in \Phi} a_\iota \beta_\iota^p a_\iota^{-p/q} || F_\iota ||_{\phi_q}^{-p} || E_\iota ||_{\phi_r}^p || F_\iota ||_{\phi_q}^p \\ &= \sum_{\iota \in \Phi} a_\iota^{1-p/q} \beta_\iota^p || E_\iota ||_{\phi_r}^p \\ &= \sum_{\iota \in \Phi} (a_\iota^{1/r} || E_\iota ||_{\phi_r})^p \beta_\iota^p \\ &= \sum_{\iota \in \Phi} (\gamma_\iota \beta_\iota)^p = \sum_{\iota \in I} (\gamma_\iota \beta_\iota)^p = \infty \end{aligned}$$

since $\{\gamma_\iota \beta_\iota\} \notin \ell_p$. Hence $E \notin \mathscr{M}(\mathscr{E}_q, \mathscr{E}_p)$ and (3, 1) is verified.

Part VI. $\mathscr{M}(\mathscr{E}_0, \mathscr{E}_0) = \mathscr{M}(\mathscr{E}_0, \mathscr{E}_\infty) = \mathscr{E}_\infty$.

*Proof.* The proof in [3, 35.4, Part III] can be adapted to our somewhat more general setting. However, an easy direct proof will be given.

Since $\mathscr{E}_0$ is an ideal of $\mathscr{E}_\infty$, we have $\mathscr{E}_\infty \subset \mathscr{M}(\mathscr{E}_0, \mathscr{E}_0)$. Also, clearly, $\mathscr{M}(\mathscr{E}_0, \mathscr{E}_0) \subset \mathscr{M}(\mathscr{E}_0, \mathscr{E}_\infty)$. Thus we need to show only that $\mathscr{M}(\mathscr{E}_0, \mathscr{E}_\infty) \subset \mathscr{E}_\infty$. Consider any $E$ in $\mathscr{E}(I)$ that is not in $\mathscr{E}_\infty$. Then for each $n \in \{1, 2, \cdots\}$, let $\iota_n$ be such that $\iota_n \neq \iota_m$ for $n \neq m$ and $|| E_{\iota_n} ||_{\phi_\infty} > n^2$. Let $F = (F_\iota)_{\iota \in I}$ where $F_\iota = (1/n) I_{d_{\iota_n}}$ for $\iota = \iota_n$ and $F_\iota =$

0 otherwise. Then we have $F \in \mathscr{E}_0$ and $EF \notin \mathscr{E}_\infty$, so that $E \notin \mathscr{M}(\mathscr{E}_0, \mathscr{E}_\infty)$. Hence, entries (2, 3) and (2, 4) are verified.

Part VII. It remains only to verify (1, 3) by showing that $\mathscr{M}(\mathscr{E}_\infty, \mathscr{E}_0) = \mathscr{E}_0$.

*Proof.* The proof is easy. Namely, $\mathscr{E}_0 \subset \mathscr{M}(\mathscr{E}_\infty, \mathscr{E}_0)$ since $\mathscr{E}_0$ is an ideal in $\mathscr{E}_\infty$. Finally, suppose $E \notin \mathscr{E}_0$. If $F_\iota = I_{d_\iota}$, then $F \in \mathscr{E}_\infty$ but $EF \notin \mathscr{E}_0$ so that $E \notin \mathscr{M}(\mathscr{E}_\infty, \mathscr{E}_0)$.

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