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REAL C^* -ALGEBRAS

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Several variants of the classical Gelfand-Neumark characterization of complex C^* -algebras are here extended to characterize real C^* -algebras up to isometric*-isomorphism and also up to homeomorphic *-isomorphism. The proofs depend on norming the complexification of the real algebra and applying the author's characterization of complex C^* -algebras to the result. L. Ingelstam has obtained similar but weaker results by an entirely different method.

An involution on \mathfrak{A} is a map $(*)$: $\mathfrak{A} \rightarrow \mathfrak{A}$ which is a conjugate linear involutive antiautomorphism. A generalized involution is an involution except that it may be either an automorphism or an antiautomorphism (Generalized involutions have been considered previously by B. Yood [12]. If $\mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1$ is a \mathbf{Z}_2 graded real algebra, then $x^0 + x^1 \rightarrow x^0 - x^1$ is an automorphic generalized involution, and conversely the sets of hermitian and skew hermitian elements in a real algebra with an automorphic generalized involution give a \mathbf{Z}_2 grading.) An algebra \mathfrak{A} with a [generalized] involution is called a [generalized] *-algebra. If \mathfrak{A} is also a Banach algebra and the norm and involution satisfy $\|x^* x\| = \|x\|^2$ for all $x \in \mathfrak{A}$ then \mathfrak{A} is called a [generalized] B^* -algebra.

If \mathcal{H} is a real or complex Hilbert space, then $[\mathcal{H}]$, the Banach algebra of all bounded linear transformations from \mathcal{H} into \mathcal{H} , is a B^* -algebra when the involution is defined as the map assigning to each element its Hilbert space adjoint. A subset of a generalized *-algebra is called self adjoint if it is closed under the involution. A self adjoint subalgebra is called a *-subalgebra. Obviously a norm closed *-subalgebra of $[\mathcal{H}]$ is also a B^* -algebra. A homomorphism φ from an algebra \mathfrak{A} with generalized involution into $[\mathcal{H}]$ is called a *-representation if $\varphi(x^*) = \varphi(x)^*$ for all $x \in \mathfrak{A}$. A Banach generalized *-algebra \mathfrak{A} will be called a C^* -algebra if there is an isometric *-representation of \mathfrak{A} on some Hilbert space. In this case the generalized involution is in fact antiautomorphic. A generalized *-algebra \mathfrak{A} is called hermitian if and only if $-h^2$ has a quasi-inverse in \mathfrak{A} for each hermitian element h in \mathfrak{A} , skew hermitian if and only if j^2 has a quasi-inverse in \mathfrak{A} for each skew hermitian element j in \mathfrak{A} . A *-algebra is called symmetric if and only if $-x^* x$ has a quasi-inverse in \mathfrak{A} for each x in \mathfrak{A} . Complex B^* -algebras are necessarily symmetric and therefore hermitian. However the complex numbers, \mathbf{C} considered as a real Banach algebra with the identity map as

involution are an example of a nonhermitian real B^* -algebra. The existence of an involution or generalized involution is a much weaker condition on a real algebra than on a complex algebra since the identity map is an involution on any commutative real algebra and a generalized involution on any real algebra.

It is well known that any complex B^* -algebra is a C^* -algebra. See [4] for a proof and further references (cf. [2], [11]). The analogous result for real B^* -algebras is false without further restriction. In fact we prove the following theorem which extends results of L. Ingelstam [5, 17.7, 18.6, 18.7, 18.8].

THEOREM 1. *The following are equivalent for a real Banach generalized $*$ -algebra \mathfrak{A} :*

- (1) \mathfrak{A} is a C^* -algebra.
- (2) $\|x\|^2 \leq \|x^*x + y^*y\|$ for all x, y in \mathfrak{A} .
- (3) \mathfrak{A} is a hermitian generalized B^* -algebra.

A complex $*$ -algebra \mathfrak{A} with an identity is a C^* -algebra if and only if $\|z^*\| \|z\| \leq \|z^*z\|$ for all normal elements z in \mathfrak{A} [3, 2.5], and any complex $*$ -algebra \mathfrak{A} is a C^* -algebra if and only if the same inequality holds for all elements x in \mathfrak{A} [11]. It is not known whether these results generalize to real hermitian $*$ -algebra.

We call a generalized $*$ -algebra C^* -equivalent if and only if it is homeomorphically $*$ -isomorphic to some C^* -algebra. Thus a generalized $*$ -algebra is C^* -equivalent if and only if it has a homeomorphic $*$ -representation on some Hilbert space.

THEOREM 2. *The following are equivalent for a real Banach generalized $*$ -algebra \mathfrak{A} .*

- (1) \mathfrak{A} is C^* -equivalent.
- (2) There is a constant C such that $\|z^*\| \|z\| \leq C \|z^*z + w^*w\|$ for all commuting pairs of normal elements z, w in \mathfrak{A} .
- (3) \mathfrak{A} is hermitian and there is a constant C such that $\|z^*\| \|z\| \leq C \|z^*z\|$ for all normal elements z in \mathfrak{A} .
- (4) \mathfrak{A} is hermitian and skew hermitian and there is a constant C such that $\|k\|^2 \leq C \|k^2\|$ for all hermitian and all skew hermitian elements k in \mathfrak{A} .

The real group algebra of \mathbf{Z}_2 with ℓ^1 -norm and an involution given by $(a + b\gamma)^* = a - b\gamma$ where γ is the generator of \mathbf{Z}_2 satisfies condition (4) except that it is not skew hermitian. Also the algebra \mathbf{C} of complex numbers with the identity map as involution satisfies (3) and (4) except that it is not hermitian. The equivalence of (1) and (4) can be regarded as a real and noncommutative version of B.

Yood's result [12, 4.1(4)] or as a real version of his Theorem 2.7 in [13] as extended by a remark in [10]. Notice that condition (2), (3), (4) do not assume the continuity of the involution nor do they put any restriction on nonnormal elements of \mathfrak{A} . In these respects Theorem 2 significantly strengthens Theorem 17.6 of L. Ingelstam in [5].

S. Shirali and J. W. M. Ford have recently shown [10] that a complex Banach algebra with a hermitian real linear involution is symmetric. Their arguments also show that a real hermitian and skew hermitian Banach *-algebra is symmetric. Although the full force of the real version of this result could be avoided in our arguments it is noted in Lemma 1 because of its general interest.

The theorems are all proved by embedding the real algebra in a complex algebra and using a recent result of the author on complex C*-algebras:

THEOREM A ([7]). *A complex Banach algebra \mathfrak{A} with an identity element 1 of norm one is isometrically isomorphic to some complex C*-algebra if and only if \mathfrak{A} is the linear span of*

$$\mathfrak{A}_H = \{h \in \mathfrak{A} : \|\exp(ith)\| \leq 1, \forall t \in \mathbf{R}\}.$$

In this case each element of \mathfrak{A} has a unique decomposition $x = h + ik$ with $h, k \in \mathfrak{A}_H$. Furthermore the map $h + ik \rightarrow h - ik$ is an involution on \mathfrak{A} and any isometric isomorphism of \mathfrak{A} into a C-algebra is a *-isomorphism relative to this involution.*

2. Embedding in a complex C*-algebra. The fundamental tool used in this paper is described in Proposition 1 at the end of this section. For convenience we establish some notation to use throughout the paper.

If \mathfrak{A} is a real algebra, we shall denote the associated complex algebra by \mathfrak{B} . That is, \mathfrak{B} is the set of formal expressions $x + iy$ with x and y in \mathfrak{A} and the obvious algebraic operations. Recall that the spectrum of an element in a real algebra \mathfrak{A} is defined to be its usual spectrum in \mathfrak{B} . Notice that with this convention a real algebra \mathfrak{A} with generalized involution is hermitian if and only if each hermitian element in \mathfrak{A} has real spectrum, is skew hermitian if and only if each skew hermitian element has purely imaginary spectrum, and a *-algebra is symmetric if and only if x^*x has nonnegative spectrum for each element x in \mathfrak{A} [8, 4.1.7 and 4.7.6]. Clearly a complex *-algebra is skew hermitian if and only if it is hermitian. If \mathfrak{A} has a generalized involution, then \mathfrak{B} will be endowed with the generalized involution $(x + iy)^* = x^* - iy^*$.

If \mathfrak{A} is an algebra without an identity then \mathfrak{A}^1 will represent the algebra (under the obvious operation) of all formal expressions $x + t$ with x in \mathfrak{A} and t a scalar. If \mathfrak{A} is normed \mathfrak{A}^1 is given the norm $\|x + t\| = \|x\| + |t|$ unless \mathfrak{A} is assumed to be a generalized B^* -algebra in which case the norm

$$\|x + t\| = \sup\{\|xu + tu\| : u \in \mathfrak{A}, \|u\| = 1\}$$

is used instead. If \mathfrak{A} is a Banach algebra the first norm on \mathfrak{A}^1 is complete, and if \mathfrak{A} is a B^* -algebra so is \mathfrak{A}^1 with the second norm [8, 4.1.13].

It is also convenient to introduce once and for all the following notation for the sets of hermitian, skew hermitian, unitary, normal and positive elements in a generalized $*$ -algebra:

$$\begin{aligned} \mathfrak{A}_H &= \{h \in \mathfrak{A} : h = h^*\}, \quad \mathfrak{A}_J = \{j \in \mathfrak{A} : -j = j^*\}, \\ \mathfrak{A}_U &= \{u \in \mathfrak{A} : uu^* = u^*u = 1\}, \quad \mathfrak{A}_N = \{z \in \mathfrak{A} : z^*z = z^*z\}, \\ \mathfrak{A}_+ &= \{h \in \mathfrak{A}_H : h \text{ has nonnegative real spectrum}\}. \end{aligned}$$

Notice that this is only one of several possible notions of positivity. It will be convenient to use \mathfrak{A}_G to denote $\mathfrak{A}_H \cup \mathfrak{A}_J$ in a (real or complex) generalized $*$ -algebra. Denote the spectrum and spectral radius of an element x in a Banach algebra by $\sigma(x)$ and $\nu(x)$, respectively. Note that $\sigma(x^*) = \{\bar{\lambda} : \lambda \in \sigma(x)\}$ so that $\nu(x) = \nu(x^*)$ for all x in \mathfrak{A} .

LEMMA 1. (*Shirali and Ford [10].*) *A real hermitian and skew hermitian Banach $*$ -algebra is symmetric.*

Proof. Ford's square root lemma [1] is proved for a real Banach $*$ -algebra \mathfrak{A} by applying the original proof to the complexification \mathfrak{C} of a closed maximal commutative $*$ -subalgebra of \mathfrak{A} which contains h , and noting that $u = \lim h_n$ lies in the natural image of \mathfrak{A} in \mathfrak{C} . Lemmas 1 through 5 of [10] now follow for real $*$ -algebras without essential change. The proof is completed by constructing the real commutative $*$ -subalgebra \mathfrak{C} as in [10] and noting that θ is defined on the complexification of \mathfrak{C} .

We note that the proof of Ford's square root lemma holds even for real Banach generalized $*$ -algebras.

LEMMA 2. *Let \mathfrak{A} be a (real or complex) Banach generalized $*$ -algebra. Let there be a constant C such that $\|k\|^2 \leq C \|k^2\|$ for all $k \in \mathfrak{A}_G$. Then*

- (a) $\|k\| \leq C \nu(k)$ for all $k \in \mathfrak{A}_G$.
- (b) *The involution is continuous.*

(c) If \mathfrak{A} is hermitian and lacks an identity then $\|k + t\|^2 \leq 9C^2 \|(k + t)^2\|$ for all $k + t \in (\mathfrak{A}^1)_G$.

(d) Let \mathfrak{A} be hermitian and if the involution is antiautomorphic let \mathfrak{A} be skew hermitian. Then \mathfrak{A}_+ is closed under addition.

Proof. (a) $\|k\| \leq (CC^2 \dots C^{2^n-1})^{2^{-n}} \|k^{2^n}\|^{2^{-n}}$

(b) This follows from Theorem 3.4 in [12].

(c) If \mathfrak{A} is real $(\mathfrak{A}^1)_J = \mathfrak{A}_J$ and if \mathfrak{A} is complex the inequality for elements in $(\mathfrak{A}^1)_J$ follows from the inequality for elements in $(\mathfrak{A}^1)_H$. Thus let $h \in \mathfrak{A}_H$ and $t \in \mathbf{R}$. By replacing h by $-h$ if necessary we can assume that $\nu(h)$ is the greatest real number in $\sigma(h)$. Let the convex hull of $\sigma(h)$ be $[-r, s]$. Then r and $s = \nu(h)$ are nonnegative since \mathfrak{A} lacks an identity, and $\sigma(h + t) \subseteq [-r + t, s + t]$.

Case 1. $t \geq 0$. Then $C\nu(h + t) = C(s + t) \geq \|h\| + |t| = \|h + t\|$.

Case 2. $0 > t \geq r - s/2$. Then $3C\nu(h + t) = 3C(s + t) \geq 3C(s + (r - s/2)) \geq 3C(S/2) \geq C(s - (r - s/2)) \geq C(s + |t|) \geq \|h + t\|$.

Case 3. $r - s/2 > t$. Then $3C\nu(h + t) = 3C(r - t) \geq 3C(r - (2/3)(r - s/2) - 1/3t) \geq C(s - t) \geq \|h + t\|$. Thus in any case $3C\nu(h + t) \geq \|h + t\|$ so that $\|h + t\|^2 \leq 9C^2 \nu(h + t)^2 = 9C^2 \nu(h + t)^2 \leq 9C^2 \|(h + t)^2\|$.

(d) If the involution is antiautomorphic this follows from Lemma 1 and [8, 4.7.10] and in any case is an intermediate step in the proof of Lemma 1. If the involution is automorphic then \mathfrak{A}_H is a *-subalgebra of \mathfrak{A} in which every element satisfies $\|h\|^2 \leq C\|h^2\|$ and has real spectrum. Then \mathfrak{A}_H is semisimple by [12, 3.5] and thus is commutative by [6, Th. 4.8]. Thus $\mathfrak{A}_+ \subseteq \mathfrak{A}_H$ is closed under addition since the spectrum is subadditive in a commutative algebra.

The existence of C such that $\|k\|^2 \leq C\|k^2\|$ for all $k \in \mathfrak{A}_G$ is equivalent to the existence of B or D such that $\|k\| \leq B\nu(k)$ for all $k \in \mathfrak{A}_G$ or $\|z\| \leq D\nu(z)$ for all $z \in \mathfrak{A}_N$, since $\|z\| \leq \|(z + z^*)/2\| + \|(z - z^*)/2\| \leq C(\nu(z) + \nu(z^*)) = 2C\nu(z)$.

PROPOSITION 1. *Let \mathfrak{A} be a real hermitian and skew hermitian Banach generalized *-algebra. Let there be a constant C such that $\|k\|^2 \leq C\|k^2\|$ for each $k \in \mathfrak{A}_G$. Then there is a complex C*-algebra \mathfrak{B} and a homeomorphic *-isomorphism of \mathfrak{A} into \mathfrak{B} .*

Proof. \mathfrak{A}^1 is hermitian and skew hermitian. Thus using Lemma 2(c) we may assume \mathfrak{A} has an identity element. We will define a

norm on \mathfrak{B} which makes it a complex Banach algebra satisfying the hypotheses of Theorem A. The norm $\|\cdot\|_{\mathcal{V}}$ for \mathfrak{B} is defined to be the Minkowski functional of the convex hull of $\mathfrak{B}_{\mathcal{V}}$, or directly :

$$\|x + iy\|_{\mathcal{V}} = \inf \{ \sum_{j=1}^n t_j : x + iy = \sum_{j=1}^n t_j u_j ; t_j \in \mathbf{R}, t_j \geq 0 ; u_j \in \mathfrak{B}_{\mathcal{V}} \} .$$

(This norm has been used previously by Russo and Dye [9]).

In order to prove that this expression is always finite and in fact a complete norm, it is easiest to introduce another norm $\| \cdot \|$ on \mathfrak{B} which is obviously finite and complete and then compare $\|\cdot\|_{\mathcal{V}}$ and $\| \cdot \|$. Let $\|x + iy\| = \|x\| + \|y\|$ for all $x, y \in \mathfrak{A}$. With respect to this norm \mathfrak{B} is a real Banach generalized *-algebra.

By Lemma 2(b) the involution in \mathfrak{A} is continuous. Let C be a constant such that $\|x^*\| \leq C\|x\|$ for all $x \in \mathfrak{A}$. If $x \in \mathfrak{A}$ then $x = h + j$ where $h = (x + x^*)/2 \in \mathfrak{A}_H$ and $j = (x - x^*)/2 \in \mathfrak{A}_J$. Clearly $\|h\|$ and $\|j\|$ are bounded by $(1 + C)\|x\|/2 \leq C\|x\|$.

Let s be a real number greater than $C\|x\|$. Then the power series for $V = \cos^{-1}(h/s)$ and $w = \sinh^{-1}(j/s)$ converge and $h = s[\exp(iv) + \exp(-iv)]/2$, $j = s[\exp(w) + (-\exp(-w))]/2$ with each exponential in $\mathfrak{B}_{\mathcal{V}}$. Similarly iy can be expressed as a positive real linear combination of elements in $\mathfrak{B}_{\mathcal{V}}$. Thus $\|x + iy\|_{\mathcal{V}}$ is always finite and in fact $\|x + iy\|_{\mathcal{V}} \leq 2B(\|x\| + \|y\|) = 2B\|x + iy\|$ for all $x, y \in \mathfrak{A}$.

It is obvious from the definition that $\|\cdot\|_{\mathcal{V}}$ is a norm for a real linear space. However \mathfrak{B} is also a complex normed algebra with respect to $\|\cdot\|_{\mathcal{V}}$ since $\mathfrak{B}_{\mathcal{V}}$ is a multiplicative group closed under multiplication by complex numbers of norm one. Furthermore the involution is an isometry.

Any element $u \in \mathfrak{B}_{\mathcal{V}}$ can be written as $u = h + j + i(k + g)$ with $h, k \in \mathfrak{A}_H$ and $j, g \in \mathfrak{A}_J$. Taking the real part of the equations $u^*u = 1$ and $uu^* = 1$ we get

$$h^2 - j^2 + k^2 - g^2 + hj - jh + ky - gk = 1$$

$$h^2 - j^2 + k^2 - g^2 + jh - hj + gk - kg = 1 .$$

Thus $h^2 - j^2 + k^2 - g^2 = 1$. Since \mathfrak{A} is hermitian and skew hermitian, $h^2, k^2, -j^2$ and $-g^2$ all belong to \mathfrak{A}_+ . Thus by Lemma 2(d) $-j^2 + k^2 - g^2 \in \mathfrak{A}_+$. Therefore $\sigma(h^2) \leq \sigma(1 - (-j^2 + k^2 - g^2)) \leq [0, 1]$ and $\nu(h) \leq 1$. Similarly $\nu(j) \leq 1, \nu(k) \leq 1$ and $\nu(g) \leq 1$. Thus

$$\| \|u\| \| = \| \|h + j\| \| + \| \|k + g\| \| \leq \| \|h\| \| + \| \|j\| \| + \| \|k\| \| + \| \|g\| \| \leq 4C$$

for all $u \in \mathfrak{B}_{\mathcal{V}}$. Thus if $x + iy = \sum_{j=1}^n t_j u_j$ with $t_j \geq 0$ and $u_j \in \mathfrak{B}_{\mathcal{V}}$ then $\| \|x + iy\| \| \leq (\sum_{j=1}^n t_j) \| \|u_j\| \| \leq 4C \sum_{j=1}^n t_j$. Therefore $\| \|x + iy\| \| \leq 4C \|x + iy\|_{\mathcal{V}}$ for all $x + iy$ in \mathfrak{B} .

Since $\|\cdot\|_{\mathcal{V}}$ is equivalent to a complete norm it is a complete

norm. Thus \mathfrak{B} is a complex Banach algebra with an identity element of norm one. Furthermore \mathfrak{B} is the linear span of \mathfrak{B}_H . For each h in \mathfrak{B}_H , $\exp(ith)$ is in \mathfrak{B}_U and hence $\|\exp(ith)\|_U \leq 1$. Therefore $(\mathfrak{B}, \|\cdot\|_U)$ satisfies the hypotheses of Theorem A and is a complex C*-algebra with respect to its involution.

We must still show that the natural map of \mathfrak{A} into \mathfrak{B} is a homeomorphism. This is true since, for all x in \mathfrak{A} , $\|x\|_U \leq 2B\|x\| = 2B\|x\| \leq 8BC\|x\|_U$.

COROLLARY 1. *Any generalized *-algebra satisfying the hypotheses of Proposition 1 has an antiautomorphic involution.*

COROLLARY 2. *Let \mathfrak{A} be a real hermitian and skew hermitian generalized B*-algebra. Then there is a complex C*-algebra and a real isometric *-isomorphism of \mathfrak{A} into \mathfrak{B} .*

Proof. Consider \mathfrak{A} as embedded in $(\mathfrak{B}, \|\cdot\|_U)$ as described in Proposition 1. Using Lemma 2(a), Corollary 1 and the fact that a C*-algebra is a B*-algebra we get

$$\|x\|^2 = \|x^*x\| = \nu(x^*x) = \|x^*x\|_U = \|x\|_U^2 \text{ for all } x \in \mathfrak{A}.$$

Thus the embedding is an isometry.

3. Proofs of Theorems 1 and 2. We need three more lemmas. The first one records the connection between real and complex *-representations.

LEMMA 3. *Let φ be an isometric *-representation of the [real, respectively, complex] B*-algebra \mathfrak{A} on the [real, respectively, complex] Hilbert space \mathcal{L} . Then there is a natural isometric *-representation ψ of the [complex, respectively, real] algebra \mathfrak{B} associated with \mathfrak{A} on the complex, respectively, real Hilbert space \mathcal{K} associated with \mathcal{L} .*

Proof. If \mathcal{L} is real let \mathcal{K} be the set of formal expressions $\xi + i\eta$ where ξ and η belong to \mathcal{L} . The inner product in \mathcal{K} is given by

$$(\xi + i\eta, \zeta + i\mu) = (\xi, \zeta) + i(\eta, \zeta) - i(\xi, \mu) + (\eta, \mu)$$

and thus the norm in \mathcal{K} is given by $\|\xi + i\eta\|^2 = \|\xi\|^2 + \|\eta\|^2$. The complex B*-algebra \mathfrak{B} associated to the real B*-algebra \mathfrak{A} is that defined in the proof of Proposition 1. The typical element of \mathfrak{B} is of the form $x + iy$ with x and y elements of \mathfrak{A} . Define ψ by

$$\psi(x + iy)(\xi + i\eta) = \varphi(x)\xi + i\varphi(x)\eta + i\varphi(y)\xi - \varphi(y)\eta.$$

It is easy to check that this is a $*$ -isomorphism, and that the image is closed in the norm of $[\mathcal{H}]$. Thus the complex $*$ -algebra \mathfrak{A} can be provided with a B^* -norm pulled back through ψ . This norm must agree with the B^* -norm defined in the proof of Proposition 1. Thus ψ is an isometry.

Now consider the case where \mathfrak{A} and \mathcal{A} are complex. The associated real algebra and vector space are obtained by merely restricting scalar multiplication to the real numbers. The inner product and norm in \mathcal{H} are $(\xi, \eta)_x = \operatorname{Re}(\xi, \eta)_{\mathcal{A}}$, $\|\xi\|_x = \|\xi\|_{\mathcal{A}}$. Thus φ considered as a $*$ -representation of a real algebra coincides with ψ .

LEMMA 4. *Let \mathfrak{A} be a Banach generalized $*$ -algebra. Let there be a constant C such that $\|z^*\| \|z\| \leq C \|z^*z + w^*w\|$ for all commuting elements z and w in \mathfrak{A}_N . Then \mathfrak{A} is hermitian and skew hermitian.*

Proof. Any $k \in \mathfrak{A}_G$ lies in some closed maximal commutative $*$ -subalgebra \mathfrak{G} [8, 4.1.3] where it has the same spectrum as in \mathfrak{A} . By Lemma 2(b) there is a constant B such that $\|z\|^2 \leq B \|z^*\| \|z\| \leq BC \|z^*z + w^*w\|$ when z and w lie in \mathfrak{G} . Thus \mathfrak{G} satisfies Theorem 4.2.3 in [8] so that it is hermitian and skew hermitian. Thus \mathfrak{A} is also.

LEMMA 5. *Let \mathfrak{A} be a Banach generalized $*$ -algebra satisfying $\|z^*\| \|z\| \leq C \|z^*z\|$ for all $z \in \mathfrak{A}_N$. Then \mathfrak{A} is skew hermitian.*

Proof. Let B be the bound for the generalized involution guaranteed by Lemma 2(b). Then the involution in \mathfrak{A} is also bounded by B . For an arbitrary skew hermitian element j of \mathfrak{A} , $e^j(e^j)^* = e^j e^{-j} = 1 = (e^j)^*(e^j)$ is \mathfrak{A}^1 . If $z + t$ is in $(\mathfrak{A}^1)_G$, then $z^*z + tz^* + tz = 0$ and $t^2 = 1$. Thus $\|z\|^2 \leq B \|z^*\| \|z\| \leq BC \|z^*z\| \leq BC(1 + B) \|z\|$, so $\|z + t\| \leq BC(1 + B) + 1$. Applying this to e^{nj} for $n \in \mathbf{Z}$ gives $\nu(e^j) = \nu(e^{-j}) = 1$. Therefore the spectrum of e^j lies on the unit circle and the spectrum of j is purely imaginary.

Proof of Theorem 1. (1) \Rightarrow (2): Consider \mathfrak{A} as embedded in $[\mathcal{A}]$ for a suitable Hilbert space \mathcal{A} . Then for x and y in $[\mathcal{A}]$

$$\begin{aligned} \|x\|^2 &= \sup \{ \|x \xi\|^2 \} \leq \sup \{ \|x \xi\|^2 + \|y \xi\|^2 \} \\ &= \sup \{ (x^* x \xi, \xi) + (y^* y \xi, \xi) \} = \sup \{ ((x^* x + y^* y) \xi, \xi) \} \\ &\leq \|x^* x + y^* y\| \end{aligned}$$

where each supremum is over all $\xi \in \mathcal{A}$ with $\|\xi\| \leq 1$.

(2) \Rightarrow (3): Lemma 4.

(3) \Rightarrow (1): Lemma 5, Corollary 2 and Lemma 3.

Note that without changing this proof, condition (2) of Theorem 1 can be weakened to: $\|x\|^2 \leq \|x^*x\|$ for all $x \in \mathfrak{A}$ and there exists a constant C such that $\|z^*\| \|z\| \leq C \|z^*z + w^*w\|$ for all commuting pairs z and w in \mathfrak{A}_N . This is essentially the condition Dc^* in [5, 18.6].

Proof of Theorem 2. (1) \Rightarrow (2): Theorem 1 and Lemma 2b.

(2) \Rightarrow (3): Lemma 4.

(3) \Rightarrow (4): Lemma 5.

(4) \Rightarrow (1): Proposition 1 and Lemma 3.

The following corollary bears the same relationship to Theorem 2 that [5, 18.7] bears to Theorem 1 or [5, 18.6].

COROLLARY 3. *Let \mathfrak{A} be a real normed generalized *-algebra. Let there be a constant C such that $\|x\|^2 \leq C \|x^*x + y^*y\|$ for all x and y in \mathfrak{A} . Then \mathfrak{A} has a homeomorphic *-representation on some Hilbert space.*

Proof. The generalized involution is continuous since $\|x\|^2 \leq C \|x^*x\| \leq C \|x^*\| \|x\|$. Thus the completion of \mathfrak{A} is a generalized *-algebra which satisfies the same inequality and hence satisfies Theorem 2.

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