Pacific Journal of Mathematics

SYMPLECTIC BORDISM, STIEFEL-WHITNEY NUMBERS, AND A NOVIKOV RESOLUTION

DON DAVID PORTER

Vol. 35, No. 1 September 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

SYMPLECTIC BORDISM, STIEFEL-WHITNEY NUMBERS, AND A NOVIKOV RESOLUTION

DON PORTER

Using an Adams type spectral sequence due to Novikov, this paper presents a proof of:

THEOREM A. **If M is a manifold representing a class in the symplectic bordism group Ω_m^{Sp}, $m \neq 8k$, then M bounds an unoriented manifold.**

The method of proof yields some further information; a more precise statement may be found in § 4 below.

The complex Thom spectrum MU defines a (generalized) cohomology theory U^*. The ground ring in this theory, $\Lambda_* = U^*(pt)$ is isomorphic to the complex bordism ring Ω_*^U, where Λ_* has nonpositive grading and Ω_*^U nonnegative. Novikov [8] computed the algebra A^U of operations for the theory U^*, $A^U \cong \Lambda^* \hat{\otimes} S$. Here $\hat{\otimes}$ denotes completed tensor product over Z (cf. [5]), and S is a Hopf algebra over Z generated by the set of operations s_α, one for each partition α of an integer $|\alpha|$. Novikov also constructed a spectral sequence

$$E_2 = \operatorname{Ext} A^U(U^*(X), \Lambda^*) \Rightarrow \pi_*(X)$$

converging to the stable homotopy ring of a ring spectrum X (cf. [1]). We apply this theory to derive information about Ω_*^{Sp}, the homotopy of the symplectic Thom spectrum MSp. In section one the structure of $U^*(MSp)$ is investigated; section two describes a resolution for $U^*(MSp)$; section three computes the necessary part of the E_2 term of the spectral sequence; section four completes the proof of Theorem A.

1. Recall that Λ^* is a polynomial ring over Z on generators $t_i \in \Lambda_{-2i}$. Also $H^*(BSp)$ is a polynomial ring over Z on the symplectic Pontrjagin classes $P_i \in H^{4i}(BSp)$. It follows from the Thom isomorphism and the Atiyah-Hirzebruch spectral sequence that there is an isomorphism of Λ_*-modules

$$F: \Lambda_* \hat{\otimes} H^*(BSp) \rightarrow U^*(MSp)$$

given by

$$F(1 \hat{\otimes} P_i) = (-1)^i s_{\Delta_{2i}}(u) .$$

Here u denotes the Thom class in $U^0(MSp)$ and Δ_n is the partition of n consisting entirely of ones. The proof is similar to [3, p. 49].

In order to study the action of A^U on $U^*(MSp)$, let $E\colon A^U \to U^*(MSp)$ be the map which evaluates operations on the Thom class. We will determine the "top dimension" of $E(s_\alpha)$. There is a natural transformation

$$B\colon U^*(\cdot) \to H_*(MU) \hat{\otimes} H^*(\cdot)$$

defined by the commutativity of the diagram

$$\begin{tikzcd} U^*(X) \arrow[r, "B"] \arrow[dr, "i"'] & H_*(MU) \hat{\otimes} H^*(X) \\ & \operatorname{Hom}(H^*(MU), H^*(X)) \arrow[u, "\cong"'] \end{tikzcd}$$

where i is defined by taking induced maps in integral cohomology. Note that on $U^*(pt) = \Lambda_*$, B is just the Hurewicz map. Consider the Z basis for $H^*(BU)$ consisting of an element c_α for every partition α, where c_α is the α symmetric function of the Chern classes $c_i = c_{\Delta_i}$ [cf. 2]. Similarly consider the Λ_*-basis for $U^*(BU)$ consisting of the Conner-Floyd characteristic classes cf_α [4]. Finally let $H_*(MU)$ be given as the integral polynomial ring on classes $a_i \in H_{2i}(MU)$, and for $\omega = (i_1, \cdots, i_n)$ let $a^\omega = a_{i_1} \cdot \cdots \cdot a_{i_n}$.

PROPOSITION 1. *If $B\colon U^*(BU) \to H_*(MU) \hat{\otimes} H^*(BU)$ is the map defined above, then*

$$B(cf_{\Delta_k}) = \sum a^\omega \hat{\otimes} c_{\Delta_k} \cdot c_\omega ,$$

where the sum is over all partitions ω of length at most k.

Proof. Suppose $g\colon CP(\infty) \to MU(1)$ is a homotopy equivalence representing a class $y \in U^2(CP(\infty))$ which generates $U^*(CP(\infty))$ as a polynomial ring over Λ_*. Similarly let $c \in H^2(CP(\infty))$ be a generator for $H^*(CP(\infty))$. Now if $b_i \in H^{2i}(MU)$ is dual to $a_i \in H_{2i}(MU)$, we have $g^*(b_i) = c^{i+1}$. So $B\colon U^*(CP(\infty)) \to H_*(MU) \hat{\otimes} H^*(CP(\infty))$ is given by

$$B(y) = \sum_{i \geqq 0} a_i \hat{\otimes} c^{i+1} .$$

In the limit $CP(\infty) = BU(1) \to BU$, this is the statement of the proposition for $k = 1$, since $c_{\Delta_1} \cdot c_{(n)} = c_{(n+1)} \equiv (c_{\Delta_1})^{n+1}$ modulo the ideal generated by $c_2, c_3, \cdots$. This ideal restricts to zero in $BU(1)$, so $B(cf_{\Delta_1})$ is as claimed. The proposition now follows by an application of the splitting principle.

Let $f\colon BSp \to BU$ classify the universal symplectic bundle γ over BSp. Then we have immediately:

PROPOSITION 2. *The map* $B\colon U^*(BSp) \to H_*(MU) \hat{\otimes} H^*(BSp)$ *is given by*

$$B(cf_{\Delta_k}(\gamma)) = \sum a^\omega \hat{\otimes} f^*(c_{\Delta_k} \cdot c_\omega) ,$$

where the sum is over all partitions ω *of length at most* k.

Note that $f^*(c_\alpha)$ is given by replacing the odd elementary symmetric fuctions in the α symmetric function with zero, and the $2i$th elementary symmetric function with $(-1)^i P_i$. In particular,

$$f^*(c_{\Delta_{2k+1}}) = 0$$
$$f^*(c_{\Delta_{2k}}) = (-1)^k P_k .$$

Next we consider the following commutative diagram:

$$\begin{array}{ccccc} U^*(MU) & \xrightarrow{E} & U^*(MSp) & \xleftarrow{F} & \Lambda^* \hat{\otimes} H^*(BSp) \\ {\scriptstyle\Phi}\uparrow & & {\scriptstyle\Phi}\uparrow & & \\ U^*(BU) & \xrightarrow{U^*(f)} & U^*(BSp) & \xrightarrow{B} & H_*(MU) \hat{\otimes} H^*(BSp) \end{array}$$

where Φ is the Thom isomorphism. By definition, $s_\alpha = \Phi(cf_\alpha)$, so we have $E(s_\alpha) = (cf_\alpha(\gamma))$. Let K be the subring of $U^*(BU)$ generated by $\{cf_{\Delta_{2i}}\}$, so that $U^*(f)|_K$ is an isomorphism of K with $U^*(BSp)$. Now since B is a monomorphism, it will determine the Hurewicz image of coefficients in Λ_* expressing $cf_\alpha(\gamma)$ in terms of $cf_{\Delta_{2i}}(\gamma)$. But F was chosen so that $\Phi(cf_{\Delta_{2i}}(\gamma)) = s_{\Delta_{2i}}(u) = F(1 \hat{\otimes} (-1)^i P_i)$, thus we have the coefficients in $F^{-1}(E(s_\alpha))$ determined recursively. The first step is given by

PROPOSITION 3. *Let* $\rho\colon \Lambda_* \hat{\otimes} H^*(BSp) \to \Lambda_0 \otimes H^*(BSp)$ *be projection on the top dimension in* Λ_*. *Then*

$$\rho \circ F^{-1} \circ E(s_\alpha) = 1 \otimes f^*(c_\alpha) .$$

Proof. Let $\rho'\colon H_*(MU) \hat{\otimes} H^*(BSp) \to H_0(MU) \otimes H^*(BSp)$ be projection, then by Proposition 2

$$\rho' \circ B(cf_{\Delta_k}(\gamma)) = 1 \otimes f^*(c_{\Delta_k}) .$$

Thus $\rho' \circ B(cf_\alpha(\gamma)) = 1 \otimes f^*(c_\alpha)$. Now the Hurewicz map $\Lambda_0 \to H_0(MU)$ is the identity, so $\rho' \circ B = \rho \circ F^{-1} \circ \Phi$, and the proposition follows. This formula is an explicit expression for the top dimension of $E(s_\alpha)$.

2. From this information on the A^U-module structure of $U^*(MSp)$, we will construct a resolution for $U^*(MSp)$. Let κ_α be the unique

element of the subring K of $U^*(BU)$ such that $U^*(f)(\kappa_\alpha) = U^*(f)(cf_\alpha)$. Let $\mathscr{R}_\alpha = \Phi(\kappa_\alpha)$, so $(s_\alpha - \mathscr{R}_\alpha)$ is an element of the kernel of E. Let Θ_n be the set of those partitions ω of n which cannot be written $\omega = (\alpha, \alpha)$, and let $\Theta = \bigcup_{n>0} \Theta_n$.

THEOREM 1. *The set* $\{(s_\beta - \mathscr{R}_\beta) : \beta \in \Theta\}$ *generates the kernel of* E *as a free* Λ_**-module.*

For the proof of this theorem, we require some data on symmetric functions. Recall the classes $c_\omega \in H^*(BU)$, and define $c^\alpha = c_{\Delta_{i_1}} \cdot \cdots \cdot c_{\Delta_{i_n}}$, if $\alpha = (i_1, \cdots, i_n)$. Introduce a linear ordering, $>$, on the set of partitions of k by taking the longest first and ordering lexicographically among partitions of the same length. For every partition ω of k, we define another partition $T(\omega)$ of k as follows: $T(\omega) = (r_1 + \cdots + r_q, r_2 + \cdots + r_q, \cdots, r_q)$, where q is the largest integer in ω, and r_j is the number of j's in ω. Note that $\beta \notin \Theta$ if and only if $T(\beta) = 2\alpha$. Then the following lemmas are elementary.

LEMMA 1. *There are integers* $m(\alpha, \beta)$ *for every pair of partitions* α, β *of* k *such that* $c^\alpha = \sum m(\alpha, \beta) c_\beta$. *Moreover,* $m(\beta, T(\beta)) = 1$ *and* $m(\alpha, \beta) = 0$ *for* $\beta > T(\alpha)$.

LEMMA 2. *There are integers* $\bar{m}(\beta, \alpha)$ *for every pair of partitions* α, β *of* k *such that* $c_\beta = \sum \bar{m}(\beta, \alpha) c^\alpha$. *Moreover,* $\bar{m}(\beta, T(\beta)) = 1$ *and* $\bar{m}(\beta, T(\gamma)) = 0$ *for* $\gamma > \beta$.

Now suppose for every partition α of $|\alpha|$ there is given an element $x_\alpha \in \Lambda_{2|\alpha|-d}$, so that $\sum x_\alpha s_\alpha$ is an operation of degree d in A^U, written in Novikov's notation [8]. Suppose that $E(\sum x_\alpha s_\alpha) = 0$, and that $x_\alpha = 0$ for $|\alpha| < k$. We write ρ_k for the projection $S \hat{\otimes} \Lambda_* \to S_k \otimes \Lambda_*$ onto elements of degree k in S. Now proceeding by induction on k, for the proof of Theorem 1 it will suffice to show

$$\rho_k(\sum x_\alpha s_\alpha) = \rho_k\Big(\sum_{\beta \in \Theta} y_\beta (s_\beta - \mathscr{R}_\beta)\Big)$$

for some unique coefficients $y_\beta \in \Lambda$.

First consider the case of odd k. For $|\alpha| = k$ odd, we have $\alpha \in \Theta$. From Proposition 2 we have that $\rho' \circ B(cf_{\Delta_k}(\gamma))$ is zero for odd k. Thus $\kappa_\alpha = \sum_{|\gamma|>k} y_\alpha c f_\gamma$, and $\rho_k(\mathscr{R}_\alpha) = 0$, and $\rho_k(\sum x_\alpha s_\alpha) = \rho_k(\sum_{|\alpha|=k} x_\alpha (s_\alpha - \mathscr{R}_\alpha))$. By Proposition 3, $k \geqq 1$, so this also provides the initial case for the induction, $k = 1$.

For k even, since $E(\sum x_\alpha s_\alpha) = 0$ we have

$$\rho \circ F^{-1} \circ E(x_\alpha s_\alpha) = 0 ,$$

so

$$\sum_{|\alpha|=k} x_\alpha \otimes f^*(c_\alpha) = 0 ,$$

and

$$\sum_{|\alpha|=k} (-1)^{|\gamma|} x_\alpha \bar{m}(\alpha, 2\gamma) = 0$$

for every γ with $2|\gamma| = k$. Now by Lemma 2, these equations may be solved uniquely for x_α, $\alpha \notin \Theta$ in terms of x_α, $\alpha \in \Theta$. Thus it suffices to prove that the matrix indexed by $\alpha, \beta \in \Theta_k$ whose (α, β) entry is the coefficient of s_α in $(s_\beta - \mathscr{R}_\beta)$ is invertible. Notice that Proposition 3 implies

$$\rho_{|\beta|}(\mathscr{R}_\beta) = \sum_{2|\gamma|=|\beta|} (-1)^{|\gamma|} \bar{m}(\beta, 2\gamma)\Big(\sum_\eta m(2\gamma, \eta) s_\eta\Big) .$$

Then by Lemmas 1 and 2, if the coefficient of s_η is $\mathscr{R}_\beta$ is nonzero, we have $\eta < \beta$. This completes the proof of Theorem 1.

We now construct the first stage of a resolution; the remaining stages may be obtained by a simple iteration. Let $C_0 = A^U$ and let C_1 be the free A^u-module generated by $\{G_\beta : \beta \in \Theta\}$. Define $d_1 : C_1 \to C_0$ by $d_1(G_\beta) = s_\beta - \mathscr{R}_\beta$. Then the following sequence is exact:

$$0 \longleftarrow U^*(MSp) \xleftarrow{E} C_0 \xleftarrow{d_1} C_1 .$$

There is an isomorphism $\operatorname{Hom} A^U(A^U, \Lambda_*) \cong \Omega^U_*$ defined by evaluation on the Thom class followed by the Atiyah duality isomorphism. The gradings are nonnegative here, so we take Ω^U_* rather than Λ_*. Thus if $g_\beta : C_1 \to \Lambda_*$ is the dual of G_β, we have

$$\Omega^U_* \cong \operatorname{Hom}_{A^U}(C_0, \Lambda_*) \xrightarrow{d_1} \operatorname{Hom}_{A^U}(C_1, \Lambda_*)$$

given by

$$d_1^*(y) = \sum_{\beta \in \Theta} (s_\beta - \mathscr{R}_\beta)(y) g_\beta .$$

3. At this point we may compute

$$E_2^{0,*} = \operatorname{Ext}^{0,*}_{A^U}(U^*(MSp), \Lambda_*) = \ker d_1^* .$$

LEMMA 3. *Let $X \in \Omega^U_{2n}$ be dual to $z \in \Lambda_{-2n}$. Then $d_1^*(X) = 0$ if and only if $(s_\omega - \mathscr{R}_\omega)(z) = 0$ for all $\omega \in \Theta_n$.*

Proof. Suppose there is a $\beta \in \Theta$, $|\beta| \neq n$, such that $(s_\beta - \mathscr{R}_\beta)(z) \neq 0$.

It will suffice to find $\gamma \in \Theta_n$ with $(s_\gamma - \mathscr{R}_\gamma)(z) \neq 0$. Let $(s_\beta - \mathscr{R}_\beta)(z) = y \in \Lambda_{-2k}$, $y \neq 0$, $k \neq 0$. Then there is an α, $|\alpha| = k$, such that $s_\alpha(y) \neq 0 \in \Lambda_0$. By Theorem 1, we may express $s_\alpha(s_\beta - \mathscr{R}_\beta)$ in terms of $\{s_\gamma - \mathscr{R}_\gamma : \gamma \in \Theta\}$, so there is a $\gamma \in \Theta_n$ with $(s_\gamma - \mathscr{R}_\gamma)(z) \neq 0$.

THEOREM 2. *$E_2^{0,*}$ is a polynomial ring over Z with one generator X_i in every dimension $4i \geqq 0$.*

Proof. Since $E_2^{0,*}$ is a subring of Ω_*^U given as the kernel of a map of free abelian groups, it suffices to count dimensions. The theorem now follows from Lemma 3.

It is interesting to note that Lemma 3 together with Proposition 3 gives an explicit criterion for the elements $X_i \in \Omega_{4i}^U$. These elements X_i are polynomial generators for $\Omega_*^{Sp} \otimes Q$.

4. The proof of Theorem A requires two further facts.

PROPOSITION 6. *For $X \in E_2^{0,*}$, the image $[X]_2$ of X in the unoriented bordism ring $\mathfrak{N}_*$ is a fourth power.*

Proof. It will suffice to show that the dual Stiefel—Whitney numbers $\bar{w}_\alpha(X)$ vanish for $\alpha \neq (\gamma, \gamma, \gamma, \gamma)$. Recall [10, p. 256] that the ω symmetric function, $\omega \in \Theta$, is contained in the ideal generated by 2 and the odd elementary symmetric functions. Thus $\rho_{|\omega|}(\mathscr{R}_\omega)$ is divisible by 2, and $s_\omega(z) \equiv 0 \pmod 2$ for $\omega \in \Theta_{2n}$, and z the dual of $X \in \ker d_1^*$ in dimension $4n$. But for such X and ω, $s_\omega(z) = c_\omega(\nu X)$, the normal Chern numbers. These reduce mod 2 to the dual Stiefel—Whitney numbers.

$$c_\omega(\nu X) \equiv \bar{w}_{\omega,\omega}(X) \bmod 2\ ,$$

so for $\omega \in \Theta_{2n}$, $\bar{w}_{\omega,\omega}(X) = 0$. Since $X \in \Omega_*^U$, $[X]_2$ is a square [7], so $\bar{w}_\alpha(X) = 0$ for $\alpha \neq (\omega, \omega)$. The only possible α for which $\bar{w}_\alpha(X) \neq 0$ is thus $\alpha = (\gamma, \gamma, \gamma, \gamma)$.

Novikov shows that $\mathrm{Ext}_{AU}^{s,*}(U^*(Y), \Lambda_*)$ is a torsion group for $s > 0$, for any Y [8]. Thus integral multiples of the X_i are generators for Ω_{4i}^{Sp}. Moreover the E_2 term contains only 2-torsion, as may be seen from [6, 8], so the multipliers are all powers of two. Recall the generators $t_i \in \Omega_{2i}^U$, and let $t^\omega = t_{i_1} \cdot \cdots \cdot t_{i_n}$ for $\omega = (i_1, \cdots, i_n)$.

PROPOSITION 7. *Let X_i be as in Theorem 2, with $X_i = \sum a(\omega) t^\omega$ for integer coefficients $a(\omega)$. Suppose $[X_i]_2 \neq 0$. Then there is an $\omega = (2\alpha, 2\alpha)$ with $a(\omega) \equiv 1 \pmod 2$.*

Proof. By Proposition 6 there are $Y, Y' \in \Omega_*^U$ such that $X_i = Y^2 + 2Y'$, since $[Y^2]_2$ is a fourth power, by [7]. Thus $a(\omega) \equiv 0 \pmod 2$ unless $\omega = (\beta, \beta)$. However if β contains an odd number the symplectic Pontrjagin numbers of t^β are all zero for dimensional reasons. Thus if $a(2\alpha, 2\alpha) \equiv 0 \pmod 2$ for all α, the Stiefel—Whitney numbers of X_i vanish, and $[X_i]_2 = 0$.

THEOREM 3. *Suppose $X \in \Omega_*^{Sp}$ and $[X]_2 \neq 0$. Then X is in the subring of Ω_*^{Sp} generated by those $X_{2i} \in E_2^{0,8i} \subset \Omega_{8i}^U$ on which all differentials in the spectral sequence vanish.*

Proof. Since $|(2\alpha, 2\alpha)| = 4|\alpha|$, it follows from Proposition 7 that $[X_i]_2 \neq 0$ implies i is even. The rest of the statement follows immediately from the existence of the spectral sequence.

Now Theorem A is just a simplification of Theorem 3. It should be noted that the map $\Omega_*^{Sp} \to \mathfrak{N}_*$ factors thru Ω_*^U, so any torsion element of Ω_*^{Sp} bounds in $\mathfrak{N}_*$. Moreover $\Omega_*^{Sp} \otimes Q$ is a polynomial algebra on $X_i \in \Omega_{4i}^{Sp} \otimes Q$, so for $X \in \Omega_n^{Sp}$, $[X]_2 = 0$ unless $n = 4k$. Thus the content of Theorem A is that $[\Omega_{8k+4}^{Sp}]_2 = 0$.

The author has been informed of some recent work of E. E. Floyd which overlaps considerably with the above results. Using very different methods, Floyd gives a more refined upper bound for the image of Ω_*^{Sp} in $\mathfrak{N}_*$.

This work formed part of the author's doctoral thesis at Northwestern University, under the direction of Professor Mark Mahowald. A summary appeared as [9].

REFERENCES

1. J. F. Adams, *Lectures on generalized cohomology, in category theory, homology theory and their applications III*, Springer, Berlin, 1969.
2. ———, *S. P. Novikov's work on complex cobordism*, lecture notes, University of Chicago, 1967.
3. P. E. Conner, and E. E. Floyd, *Differentiable periodic maps*, Springer, Berlin, 1964.
4. ———, *The relation of cobordism to k-theories*, Springer, Berlin, 1966.
5. P. S. Landweber, *Cobordism operations and Hopf algebras*, Trans. Amer. Math. Soc. **129** (1967), 94-110.
6. J. W. Milnor, *On the cobordism ring Ω^* and a complex analogue*, Amer. J. Math. **82** (1960), 505-521.
7. ———, *On the Stiefel—Whitney numbers of complex manifolds and of spin manifolds*, Topology **3** (1965), 223-230.
8. S. P. Novikov, *The methods of algebraic topology from the viewpoint of cobordism*

theory, Izv. Akad. Navk SSSR (Ser. Mat.) **31** (1967), 855-956 (Math. USSR—Izv. **1** (1967), 827-913).
9. D. D. Porter, *Novikov resolutions for symplectic cobordism*, Notices Amer. Math. Soc. **17** (1970), 150-151.
10. R. E. Stong, *Notes on cobordism theory*, *Princeton*, 1968.

Received February 17, 1970.

STATE UNIVERSITY OF NEW YORK AT ALBANY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 35, No. 1 September, 1970