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**GELFAND AND WALLMAN-TYPE COMPACTIFICATIONS**

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In this paper we compare the Gelfand and Wallman methods of constructing a compactification for a Tychonoff space  $X$  from a suitable ring of continuous real-valued functions on  $X$ . Every Hausdorff compactification  $T$  of  $X$  is Gelfand constructable; in particular,  $T$  is equivalent, as a compactification of  $X$ , to the structure space of all maximal ideals of the ring of all continuously extendable functions from  $X$  to  $T$ . However, Wallman's method applied to this ring may not yield  $T$ . We thus inquire into some relationships that exist between the Wallman and Gelfand compactification of  $X$  constructed from a suitable ring of functions on  $X$ .

O. Topological preliminaries. All topological spaces in this paper are assumed to be completely regular and Hausdorff. We shall be concerned with methods of constructing compactifications for such spaces.

Let  $X$  be a topological space. The space  $T$  is an extension of  $X$  means there exists a homeomorphism  $h$  from  $X$  into  $T$  such that  $h[X]$  is dense in  $T$ . The function  $h$  is called an embedding. Occasionally the necessary embedding maps will be explicitly mentioned, but usually they will be tacitly assumed. In fact, when  $T$  is given as an extension of  $X$ , we may take  $X$  as a subspace of  $T$ . The space  $T$  is a compactification of  $X$  (denoted  $T \in cX$ ) means that  $T$  is a compact extension of  $X$ . The compactifications  $T$  and  $K$  of a space  $X$  are equivalent as compactifications of  $X$  (denoted  $T = K$ ) means there exists a homeomorphism between  $T$  and  $K$  such that  $h(x) = x$  for each  $x \in X$ .

We shall use the standard notations [4] regarding  $C(X)$ , the ring of continuous real-valued functions. For any  $f \in C(X)$ ,

$$Z(f) = \{x \in X \mid f(x) = 0\}$$

is called the zero-set of  $f$ . If  $\mathcal{A}$  is a subring of  $C(X)$ , we define  $Z[\mathcal{A}] = \{Z(f) \mid f \in \mathcal{A}\}$ ; however,  $Z[C(X)]$  is customarily denoted by  $Z(X)$ . We shall only refer to subrings of  $C(X)$  with unity.

Let  $\mathcal{A}$  be a subring of  $C(X)$ . We shall denote the space of maximal ideals of  $\mathcal{A}$  with the Stone topology [4, 7M], also called the structure space of  $\mathcal{A}$ , by  $H[\mathcal{A}]$ . The space of ultrafilters of  $Z[\mathcal{A}]$  is denoted by  $wZ[\mathcal{A}]$ . This space of ultrafilters is constructed by Wallman's method [1] [2]. We shall be primarily concerned with those subrings  $\mathcal{A}$  of  $C(X)$  for which  $wZ[\mathcal{A}] \in cX$  and how these

subbrings relate to a certain type of "structure space" for  $\mathcal{A}$ .

Let  $\mathcal{L}$  be a collection of subsets of  $X$ . Then  $\mathcal{L}$  is a lattice on  $X$  means

- (1)  $\emptyset, X \in \mathcal{L}$ ;
- (2) if  $A, B \in \mathcal{L}$ , then  $A \cap B \in \mathcal{L}$  and  $A \cup B \in \mathcal{L}$ .

A set in  $\mathcal{L}$  is referred to as an  $\mathcal{L}$ -set.

The lattice  $\mathcal{L}$  on  $X$  is a Wallman base on  $X$  means

- (1)  $\mathcal{L}$  is a base for the closed subsets of  $X$ ;
- (2)  $\mathcal{L}$  is a disjunctive lattice on  $X$  (i.e., if  $A \in \mathcal{L}$  and  $x \in X - A$ , then there exists  $B \in \mathcal{L}$  such that  $x \in B$  and  $A \cap B = \emptyset$ );
- (3)  $\mathcal{L}$  is a normal lattice on  $X$  (i.e., for each  $A, B \in \mathcal{L}$ , if  $A$  and  $B$  are disjoint, then there exists  $C, D \in \mathcal{L}$  such that  $X - A \subset C$ ,  $X - B \subset D$  and  $C \cup D = X$ ).

For any lattice  $\mathcal{L}$  on  $X$ , an  $\mathcal{L}$ -filter is a nonvoid subset  $\mathcal{F}$  of  $\mathcal{L}$  such that

- (1)  $\emptyset \notin \mathcal{F}$ ;
- (2) if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ ;
- (3) if  $A \in \mathcal{F}$ ,  $B \in \mathcal{L}$  and  $A \subset B$ , then  $B \in \mathcal{F}$ .

An  $\mathcal{L}$ -ultrafilter is a maximal (with respect to inclusion)  $\mathcal{L}$ -filter. The set of all  $\mathcal{L}$ -ultrafilters is denoted by  $w\mathcal{L}$ .

Let  $\mathcal{L}$  be a lattice on  $X$ . In order to topologize  $w\mathcal{L}$ , define  $A^* = \{\mathcal{U} \in w\mathcal{L} \mid A \in \mathcal{U}\}$  for each  $A \in \mathcal{L}$ . Then  $\{A^* \mid A \in \mathcal{L}\}$  is a base for the closed sets of some (necessarily unique) topology for  $w\mathcal{L}$ . We shall only consider  $w\mathcal{L}$  with this topology. Now  $w\mathcal{L} \in cX$  if and only if  $\mathcal{L}$  is a Wallman base on  $X$  (with respect to the embedding map  $\varphi: X \rightarrow w\mathcal{L}$  defined by  $\varphi(x) = \{A \in \mathcal{L} \mid x \in A\}$ ). If  $T \in cX$ , then  $T$  is a Wallman-type compactification of  $X$  means there exists a Wallman base  $\mathcal{L}$  on  $X$  such that  $T = w\mathcal{L}$ . It is unknown whether or not every compactification is Wallman-type. If  $T \in cX$ , then  $T$  is a  $z$ -compactification of  $X$  means there exists a Wallman base  $\mathcal{L} \subset Z(X)$  such that  $T = w\mathcal{L}$ .

1. Filter ideals. Let  $X$  be a topological space and  $\mathcal{A}$  a subring of  $C(X)$ .

DEFINITION 1.1. The ideal  $I$  of  $\mathcal{A}$  is a filter ideal of  $\mathcal{A}$  means  $Z[I]$  is a  $Z[\mathcal{A}]$ -filter. The set of all maximal filter ideals is denoted by  $F[\mathcal{A}]$ .

DEFINITION 1.2.  $\mathcal{A}$  is a wallman subring of  $C(X)$  means that  $Z[\mathcal{A}]$  is a Wallman base on  $X$ .

We first give some elementary facts about filter ideals, the proofs of which are straight forward.

PROPOSITION 1.3. *The ideal  $I$  is a filter ideal of  $\mathcal{A}$  if and only if  $Z(f) \neq \emptyset$  for each  $f \in I$ .*

Thus an ideal of  $\mathcal{A}$  need not be a filter ideal. Further, every ideal of  $\mathcal{A}$  is a filter ideal if and only if  $\mathcal{A}$  is inverse closed (if  $f \in \mathcal{A}$  and  $Z(f) = \emptyset$ , then  $1/f \in \mathcal{A}$ ).

PROPOSITION 1.4. *If  $F$  is a  $Z[\mathcal{A}]$ -filter, then*

$$Z^{-}[F] = \{f \in \mathcal{A} \mid Z(f) \in F\}$$

*is a filter ideal of  $\mathcal{A}$ .*

A filter ideal  $I$  of  $\mathcal{A}$  is a  $z$ -filter ideal means if  $f \in \mathcal{A}$  and  $Z(f) \in Z[I]$ , then  $f \in I$ . Then there is a one-to-one correspondence between the  $Z[\mathcal{A}]$ -filters and the  $z$ -filter ideals of  $\mathcal{A}$ . The next two propositions show that there is also a one-to-one correspondence between  $Z[\mathcal{A}]$ -ultrafilters and maximal filter ideals.

PROPOSITION 1.5. *If  $I$  is a maximal filter ideal in  $\mathcal{A}$ , then  $Z[I] \in wZ[\mathcal{A}]$ .*

*Proof.* Now  $Z[I]$  is a  $Z[\mathcal{A}]$ -filter. Suppose  $F$  is a  $Z[\mathcal{A}]$ -filter such that  $Z[I] \subset F$ . Then  $Z^{-}[F]$  is a filter ideal of  $\mathcal{A}$  and  $I \subset Z^{-}[Z[I]] \subset Z^{-}[F]$ . Since  $I$  is a maximal filter ideal, then  $I = Z^{-}[F]$ . Thus  $Z[I] = F$ ; hence,  $Z[I] \in wZ[\mathcal{A}]$ .

PROPOSITION 1.6. *If  $\mathcal{U} \in wZ[\mathcal{A}]$ , then  $Z^{-}[\mathcal{U}]$  is a maximal filter ideal.*

*Proof.* Since  $\mathcal{U} \in wZ[\mathcal{A}]$ , then  $Z^{-}[\mathcal{U}]$  is a filter ideal by 1.4. Suppose  $I$  is an ideal of  $\mathcal{A}$  such that  $Z^{-}[\mathcal{U}] \subset I$ . Then  $\mathcal{U} \subset Z[I]$  where  $Z[I]$  is a  $Z[\mathcal{A}]$ -filter by 1.3. Since  $\mathcal{U}$  is maximal, then  $\mathcal{U} = Z[I]$ . So  $I \subset Z^{-}[Z[I]] = Z^{-}[\mathcal{U}]$ ; thus  $I = Z^{-}[\mathcal{U}]$ . Hence,  $Z^{-}[\mathcal{U}]$  is a maximal filter ideal.

PROPOSITION 1.7. *Every maximal filter ideal of  $\mathcal{A}$  is a prime ideal of  $\mathcal{A}$ .*

*Proof.* Let  $I$  be a maximal filter ideal of  $\mathcal{A}$  and suppose  $I$  is not prime. We select  $f, g \in \mathcal{A}$  such that  $fg \in I$ , but  $f \notin I$  and  $g \notin I$ . So  $I$  is properly contained in the ideals  $I_1 = I + \mathcal{A}f$  and  $I_2 = I + \mathcal{A}g$ . Since  $I_1, I_2$  are not filter ideals, by 1.1 we select  $h_1, h_2 \in I$  and  $k_1, k_2 \in \mathcal{A}$  such that  $Z(h_1 - k_1f) = \emptyset$  and  $Z(h_2 - k_2g) = \emptyset$ . Clearly  $h_1 - k_1f \in I_1$  and  $h_2 - k_2g \in I_2$ . Since  $(Z(h_1) \cap Z(k_1)) \cup (Z(h_1) \cap Z(f)) = \emptyset$  and

$(Z(h_2) \cap Z(k_2)) \cup (Z(h_2) \cap Z(g)) = \emptyset$ , then  $Z(h_1) \cap Z(h_2) \cap Z(fg) = \emptyset$  so,  $Z(h_1^2 + h_2^2 + (fg)^2) = \emptyset$ . But  $h_1^2 + h_2^2 + (fg)^2 \in I$ , contradicting  $I$  is a filter ideal by 1.1. Hence,  $I$  must be a prime ideal of  $\mathcal{A}$ .

The following easily proved characterization of maximal filter ideals we state without proof:

**PROPOSITION 1.8.** *Let  $M$  be a filter ideal of  $\mathcal{A}$ . Then  $M \in F[\mathcal{A}]$  if and only if for every  $f \in \mathcal{A} - M$  there exists  $g \in M$  such that  $Z(f) \cap Z(g) = \emptyset$ .*

**2. Maximal filter ideal spaces.** Let  $X$  be a topological space. Let  $\mathcal{A}$  be a subring of  $C(X)$  (we shall only refer to subrings of  $\mathcal{A}$  with unity). We denote the structure space of  $\mathcal{A}$  by  $H[\mathcal{A}]$  (see [4, 7M]) and the set of maximal filter ideals of  $\mathcal{A}$  by  $F[\mathcal{A}]$ . We seek to define a "structure space" topology for  $F[\mathcal{A}]$  and to examine the relationships between the spaces  $F[\mathcal{A}]$  and  $wZ[\mathcal{A}]$ . In particular, we show  $F[\mathcal{A}] = wZ[\mathcal{A}]$  equivalent as compactifications of  $X$  if and only if  $Z[\mathcal{A}]$  is a Wallman base on  $X$ . Furthermore,  $F[\mathcal{A}]$  is a compactification of  $X$  if and only if  $Z[\mathcal{A}]$  is a Wallman base on  $X$ . Accordingly, we shall refer to  $\mathcal{A}$  as a Wallman ring on  $X$  if  $Z[\mathcal{A}]$  is a Wallman base on  $X$ .

**THEOREM 2.1.** *Let  $X$  be a topological space and  $\mathcal{A}$  a subring of  $C(X)$ . For each  $x \in X$  define  $M_x = \{f \in \mathcal{A} \mid f(x) = 0\}$ . Then*

- (a)  $M_x \in F[\mathcal{A}]$  for each  $x \in X$  if and only if  $Z[\mathcal{A}]$  is a disjunctive lattice on  $X$ ;
- (b) If  $Z[\mathcal{A}]$  is a disjunctive lattice on  $X$ , then the mapping  $x \rightarrow M_x$  is one-to-one if and only if  $\mathcal{A}$  strongly separates points in  $X$  (i.e., if  $x, y \in X$ ,  $x \neq y$ , then there exists  $f \in \mathcal{A}$  such that  $f(x) = 0$  and  $f(y) \neq 0$ ).

*Proof.* (a) Suppose  $M_x \in F[\mathcal{A}]$  for each  $x \in X$ . Let  $A \in Z[\mathcal{A}]$  and  $x \in X - A$ . Select  $f \in \mathcal{A}$  such that  $A = Z(f)$ . Since  $f \in \mathcal{A} - M_x$ , then by 1.8 we may choose  $g \in M_x$  such that  $Z(f) \cap Z(g) = \emptyset$ . Then  $Z(g) \in Z[\mathcal{A}]$ ,  $x \in Z(g)$  and  $Z(g) \cap A = \emptyset$ . Hence,  $Z[\mathcal{A}]$  is a disjunctive lattice on  $X$ . Conversely, suppose  $Z[\mathcal{A}]$  is disjunctive. By 1.3,  $M_x$  is a filter ideal of  $\mathcal{A}$  for each  $x \in X$ . Suppose  $x \in X$ . Let  $I$  be a filter ideal of  $\mathcal{A}$  properly containing  $M_x$  and select  $f \in I - M_x$ . Since  $Z[\mathcal{A}]$  is disjunctive, select  $Z(g) \in Z[\mathcal{A}]$  such that  $x \in Z(g)$  and  $Z(g) \cap Z(f) = \emptyset$ . Then  $g \in M_x$ , so  $g \in I$ , and thus  $f^2 + g^2 \in I$ , contradicting 1.3. Hence,  $M_x \in F[\mathcal{A}]$ .

(b) Since  $Z[\mathcal{A}]$  is a disjunctive lattice on  $X$ , then  $M_x \in F[\mathcal{A}]$  for each  $x \in X$ . Suppose the mapping  $x \rightarrow M_x$  is one-to-one. Let

$x, y \in X$  such that  $x \neq y$ . Then  $M_x \neq M_y$ . So there exists  $f \in \mathcal{A}$  such that  $f(x) \neq 0$  and  $f(y) = 0$ . So  $\mathcal{A}$  strongly separates points in  $X$ . The converse is obvious. This completes the proof.

We now put a structure space topology on  $F[\mathcal{A}]$ . For each  $f \in \mathcal{A}$ , define  $f^* = \{I \in F[\mathcal{A}] \mid f \in I\}$ . Easily  $0^* = F[\mathcal{A}]$  and  $f^* = \emptyset$  whenever  $Z(f) = \emptyset$ . Since every maximal filter ideal is prime, then  $(fg)^* = f^* \cup g^*$ . Hence,  $\{f^* \mid f \in \mathcal{A}\}$  defines a base for some topology (necessarily unique) on  $F[\mathcal{A}]$ . We shall only consider this topology on  $F[\mathcal{A}]$ . Easily  $\{I\} = \bigcap \{f^* \mid f \in I\}$  for each  $I \in F[\mathcal{A}]$ ; hence,  $F[\mathcal{A}]$  is a  $T_1$ -space.

**THEOREM 2.2.**  $F[\mathcal{A}]$  is compact.

*Proof.* Let  $\mathcal{K}$  be a nonvoid collection of nonvoid basic closed subsets of  $F[\mathcal{A}]$  with the finite intersection property. Let  $\mathcal{K}' = \{Z(f) \mid f \in \mathcal{A}, f^* \in \mathcal{K}\}$ . Then  $\mathcal{K}'$  is a nonempty collection of zero sets of  $\mathcal{A}$  with the finite intersection property. So we may select  $\mathcal{U} \in wZ[\mathcal{A}]$  such that  $\mathcal{K}' \subset \mathcal{U}$ . For each  $f \in \mathcal{A}$  where  $f^* \in \mathcal{K}$ , we have  $Z(f) \in \mathcal{K}' \subset \mathcal{U} \Rightarrow f \in Z^-(\mathcal{U}) \in F[\mathcal{A}]$  (by 1.6)  $\rightarrow Z^-(\mathcal{U}) \in f^*$ ; thus,  $Z^-(\mathcal{U}) \in \bigcap \mathcal{K}$ . Hence,  $F[\mathcal{A}]$  is compact.

We now seek conditions under which  $F[\mathcal{A}]$  is a compactification of  $X$  with respect to the mapping  $x \rightarrow M_x$  ( $= \{f \in \mathcal{A} \mid f(x) = 0\}$ ). By 2.1, we must have a subring  $\mathcal{A}$  of  $C(X)$  such that  $\mathcal{A}$  strongly separates points of  $X$  and  $Z[\mathcal{A}]$  is a disjunctive lattice on  $X$ .

**THEOREM 2.3.**  $F[\mathcal{A}]$  is Hausdorff if and only if  $F_1, F_2 \in F[\mathcal{A}]$ ,  $F_1 \neq F_2 \rightarrow$  there exists  $f, g \in \mathcal{A}$  such that  $(fg)^* = F[\mathcal{A}]$ ,  $f \notin F_1$  and  $g \notin F_2$ .

*Proof.* Suppose  $F[\mathcal{A}]$  is Hausdorff. Let  $F_1, F_2 \in F[\mathcal{A}]$ ,  $F_1 \neq F_2$ . Select  $f, g \in \mathcal{A}$  such that  $F_1 \in F[\mathcal{A}] - f^*$ ,  $F_2 \in F[\mathcal{A}] - g^*$  and  $(F[\mathcal{A}] - f^*) \cap (F[\mathcal{A}] - g^*) = \emptyset$ . Then  $f \notin F_1$ ,  $g \notin F_2$  and  $f^* \cup g^* = (fg)^* = F[\mathcal{A}]$ . Suppose the converse hypothesis holds. Let  $F_1, F_2 \in F[\mathcal{A}]$ ,  $F_1 \neq F_2$ . Select  $f, g \in \mathcal{A}$  such that  $f \notin F_1$ ,  $g \notin F_2$  and  $(fg)^* = F[\mathcal{A}]$ . Then  $F_1 \in F[\mathcal{A}] - f^*$ ,  $F_2 \in F[\mathcal{A}] - g^*$  and  $(F[\mathcal{A}] - f^*) \cap (F[\mathcal{A}] - g^*) = \emptyset$ . This completes the proof.

**COROLLARY 2.4.** Suppose  $Z[\mathcal{A}]$  is a base for the closed subsets of  $X$ . Then  $F[\mathcal{A}]$  is Hausdorff if and only if  $F_1, F_2 \in F[\mathcal{A}]$ ,  $F_1 \neq F_2 \rightarrow$  there exists  $f, g \in \mathcal{A}$  such that  $f \notin F_1$ ,  $g \notin F_2$  and  $fg = 0$ .

**THEOREM 2.5.** Let  $\mathcal{A}$  be a subring of  $C(X)$  such that  $Z[\mathcal{A}]$  is

a disjunctive lattice on  $X$ . Let  $\varphi$  denote the mapping  $x \rightarrow M_x$  from  $X$  into  $F[\mathcal{A}]$ . Then

(a)  $\varphi: X \rightarrow F[\mathcal{A}]$  is continuous,

(b)  $\varphi[X]$  is dense in  $F[\mathcal{A}]$ , and

(c)  $\varphi$  is a homeomorphism between  $X$  and  $\varphi[X]$  if and only if  $\mathcal{A}$  strongly separates points from the closed sets in  $X$  (i.e., if  $F$  is a closed subset of  $X$  and  $x \in X - F$ , then there exists  $f \in \mathcal{A}$  such that  $F \subset Z(f)$  and  $f(x) \neq 0$ ).

*Proof.* By 2.1 (a),  $M_x \in F[\mathcal{A}]$  for every  $x \in X$ .

(a) Since  $\varphi^+[f^*] = Z(f)$  for each  $f \in \mathcal{A}$ , it becomes straightforward to show  $\varphi: X \rightarrow F[\mathcal{A}]$  is continuous.

(b) Let  $f \in \mathcal{A}$ . Then  $F[\mathcal{A}] - f^*$  is a basic open set in  $F[\mathcal{A}]$ . Suppose  $(F[\mathcal{A}] - f^*) \cap \varphi[X] = \emptyset$ . Let  $x \in X$ . Then  $\varphi(x) = M_x \notin F[\mathcal{A}] - f^*$ , so  $M_x \in f^*$ . Thus  $f \in M_x$  for every  $x \in X$ ; i.e.,  $f = 0$ . So  $f^* = F[\mathcal{A}]$ . Hence, every nonvoid basic open set of  $F[\mathcal{A}]$  intersects  $\varphi[X]$ ; i.e.,  $\varphi[X]$  is dense in  $F[\mathcal{A}]$ .

(c) First, suppose  $\mathcal{A}$  strongly separates points and closed sets in  $X$ . Then  $Z[\mathcal{A}]$  is a base for the closed sets in  $X$ . Since

$$\varphi^+[f^* \cap \varphi[X]] = Z(f)$$

for each  $f \in \mathcal{A}$ , then  $\varphi$  and  $\varphi^+$  are continuous. By 2.1 (b),  $\varphi$  is one-to-one. Hence,  $\varphi$  is a homeomorphism between  $X$  and  $\varphi[X]$ . Let  $F$  be a closed subset of  $X$ . Then  $\varphi[F]$  is a closed subset of  $\varphi[X]$ . So we may select  $\mathcal{K} \subset \mathcal{A}$  such that

$$\varphi[F] = \bigcap \{f^* \cap \varphi[X] \mid f \in \mathcal{K}\}.$$

Thus  $F = \bigcap \{\varphi^+[f^* \cap \varphi[X]] \mid f \in \mathcal{K}\} = \bigcap \{Z(f) \mid f \in \mathcal{K}\}$ ; so  $Z[\mathcal{A}]$  is a base for the closed subsets of  $X$ . Hence,  $\mathcal{A}$  strongly separates points from closed sets in  $X$ .

Let  $\mathcal{A}$  be a subring of  $C(X)$  which strongly separates points from closed sets in  $X$  and for which  $Z[\mathcal{A}]$  is disjunctive. Then the mapping  $\varphi: X \rightarrow F[\mathcal{A}]$  defined by  $\varphi(x) = M_x$  embeds  $X$  into the compact  $T_1$ -space  $F[\mathcal{A}]$ . Define  $h: X \rightarrow wZ[\mathcal{A}]$  by  $h(x) = \mathcal{U}_x (= \{A \in Z[\mathcal{A}] \mid x \in A\})$ . By [2, Th. 2.7],  $h$  embeds  $X$  into the compact  $T_1$ -space  $wZ[\mathcal{A}]$ . Define  $H: wZ[\mathcal{A}] \rightarrow F[\mathcal{A}]$  by  $H(\mathcal{U}) = Z^+[\mathcal{U}]$  for each  $\mathcal{U} \in wZ[\mathcal{A}]$ .

**THEOREM 2.6.** *The mapping  $H$  is a homeomorphism between  $wZ[\mathcal{A}]$  and  $F[\mathcal{A}]$ .*

*Proof.* By 1.5 and 1.6,  $H$  is a bijection. Now  $\{Z(f)^* \mid f \in \mathcal{A}\}$ , where  $Z(f)^* = \{\mathcal{U} \in wZ[\mathcal{A}] \mid Z(f) \in \mathcal{U}\}$ , is a base for the closed sets

of  $wZ[\mathcal{A}]$  (see [1] or [2]). Since  $H[Z(f)^*] = f^*$  for each  $f \in \mathcal{A}$ , then both  $H$  and  $H^-$  are continuous. Hence,  $H$  is a homeomorphism.

**THEOREM 2.7.**  *$F[\mathcal{A}] \in cX$  if and only if  $\mathcal{A}$  is a Wallman ring.*

*Proof.* By 2.6,  $H$  defines a homeomorphism between  $F[\mathcal{A}]$  and  $wZ[\mathcal{A}]$ . But  $wZ[\mathcal{A}] \in cX$  if and only if  $Z[\mathcal{A}]$  is a Wallman base on  $X$ . Hence,  $F[\mathcal{A}] \in cX$  if and only if  $\mathcal{A}$  is a Wallman ring.

Hence, the structure space  $F[\mathcal{A}]$  of the maximal filter ideals of a subring  $\mathcal{A}$  of  $C(X)$  is a (Hausdorff) compactification if and only if  $\mathcal{A}$  is a Wallman ring. Moreover,  $F[\mathcal{A}]$  is a Wallman-type compactification of  $X$ .

**3. Maximal ideal spaces and maximal filter ideal spaces.** In this section  $\mathcal{A}$  is a subring of  $C(X)$  containing  $\mathcal{R}$ , the constant real-valued functions on  $X$ . For  $x \in X$ , define  $M_x = \{f \in \mathcal{A} \mid f(x) = 0\}$ . The mapping  $f + M_x \rightarrow f(x)$  is a ring isomorphism between  $\mathcal{A}/M_x$  and  $\mathcal{R}$ ; so,  $M_x \in H[\mathcal{A}]$  for each  $x \in X$ . Similarly,  $M_x \in F[\mathcal{A}]$  for each  $x \in X$  (1.3). We topologize  $H[\mathcal{A}]$  by taking the set of all  $f^* = \{M \in H[\mathcal{A}] \mid f \in M\}$ ,  $f \in \mathcal{A}$ , as a base for the closed sets; i.e.,  $H[\mathcal{A}]$  is the structure space of  $\mathcal{A}$  [4, 7M]. Similarly we topologize  $F[\mathcal{A}]$ , where a basic closed set is denoted  $f^* = \{F \in F[\mathcal{A}] \mid f \in F\}$ ,  $f \in \mathcal{A}$ . Define the mapping  $\varphi: X \rightarrow F[\mathcal{A}]$  by  $\varphi(x) = M_x$  and  $\psi: X \rightarrow H[\mathcal{A}]$  by  $\psi(x) = M_x$ . We obtain  $\varphi[Z(f)] = f^* \cap \varphi[X]$  and  $\psi[Z(f)] = f^* \cap \psi[X]$ . Hence,  $H[\mathcal{A}]$  is an extension of  $X$  (via  $\psi$ ),  $F[\mathcal{A}]$  is an extension of  $X$  (via  $\varphi$ ) if and only if  $Z[\mathcal{A}]$  is a base for the closed sets in  $X$ . Now  $F[\mathcal{A}]$  and  $H[\mathcal{A}]$  are both compact  $T_1$ -spaces [see 2.2 and 4, 7M]. From § 2,  $F[\mathcal{A}] \in cX$  if and only if  $\mathcal{A}$  is a Wallman ring on  $X$ . From [4, 7M],  $H[\mathcal{A}] \in cX$  if and only if  $Z[\mathcal{A}]$  is a base for the closed subsets of  $X$  and  $H[\mathcal{A}]$  is Hausdorff.

We remark that even if both  $H[\mathcal{A}]$  and  $F[\mathcal{A}] \in cX$ , they need not yield equivalent compactifications of  $X$ . For example, let  $X = \mathcal{R}$  (reals with the usual topology) and  $\mathcal{R}^*$  be the one-point compactification of  $\mathcal{R}$ . Let  $\mathcal{A}$  be the ring of all functions in  $C(\mathcal{R})$  having continuous extensions to  $\mathcal{R}^*$ . Then  $\mathcal{A}$  is a Wallman ring and  $F[\mathcal{A}] = wZ[\mathcal{A}] = \beta\mathcal{R}$ , but  $H[\mathcal{A}] = \mathcal{R}^*$ . This situation generalizes to arbitrary locally compact Lindelof spaces [1] [5]. However,  $F[C^*(X)] = wZ(X) = \beta X = H[C^*(X)]$ . Thus, we inquire into possible relationships between  $F[\mathcal{A}]$  and  $H[\mathcal{A}]$ .

We first present the following analogue of the Gelfand-Komolgoroff Theorem [4, 7.3] which yields a representation theorem for the maximal filter ideals of  $\mathcal{A}$  when  $wZ[\mathcal{A}] \in cX$ .

**THEOREM 3.1.** *Let  $\mathcal{A}$  be a Wallman ring on the space  $X$  and  $T = wZ[\mathcal{A}]$ . The maximal filter ideals in  $\mathcal{A}$  are then given by  $F^t = \{f \in \mathcal{A} \mid t \in c1_T Z(f)\}$  ( $t \in T$ ).*

*Proof.* Let  $t \in T$ . Easily  $F^t$  is an ideal. From 1.3,  $F^t$  is a filter ideal. We now show  $F^t \in F[\mathcal{A}]$ . Suppose  $F \in F[\mathcal{A}]$  such that  $F^t \subset F$  and  $F^t \neq F$ . Select  $f \in F$  such that  $t \notin c1_T Z(f)$ . Since  $T = wZ[\mathcal{A}]$ , select  $g \in \mathcal{A}$  such that  $t \in c1_T Z(g)$  and  $Z(f) \cap Z(g) = \emptyset$ . But then  $f, g \in F$  and  $Z(f) \cap Z(g) = \emptyset$ , contradicting  $F \in F[\mathcal{A}]$ . So  $F^t$  is maximal. It remains to show that if  $F \in F[\mathcal{A}]$ , then  $F = F^t$  for some  $t \in T$ . Let  $F \in F[\mathcal{A}]$ . Then  $Z[F] \in wZ[\mathcal{A}]$ , so

$$\cap \{c1_T Z(f) \mid f \in F\} = \{t\}$$

for some  $t \in T$  [1], [6]. Hence,  $F = F^t$ . This completes the proof.

The above theorem also yields an explicit one-to-one correspondence between the points of  $T$  and the maximal filter ideals in  $\mathcal{A}$ .

Since  $C(X)$  is inverse closed and  $wZ(X) = \beta X$ , we have the

**COROLLARY 3.2.** (*Gelfand-Komolgoroff theorem*). *For any space  $X$ ,  $H[C(X)] = F[C(X)] = wZ(X) = \beta X$  and the maximal ideals of  $C(X)$  are given by  $M^t = \{f \in C(X) \mid t \in c1_{\beta X} Z(f)\}$ .*

Now, since  $Z(X) = Z[C^*(X)]$ , then  $C^*(X)$  is also a Wallman ring on  $X$  and  $F[C^*(X)] = wZ(X) = \beta X$ . Since  $H[C(X)] = H[C^*(X)]$  [4, 7.11], then  $H[C^*(X)] = F[C^*(X)]$  (i.e., equivalent as compactifications of  $X$ ).

We now inquire into relationships between maximal ideals and maximal filter ideals.

**THEOREM 3.3.** *Suppose  $H[\mathcal{A}] \in cX$ . Then every maximal filter ideal is contained in a unique maximal ideal.*

*Proof.* Let  $F \in F[\mathcal{A}]$ . Suppose  $M, N \in H[\mathcal{A}]$  where  $F \subset M, N$  and  $M \neq N$ . Select  $f, g \in \mathcal{A}$  such that  $fg = 0$ ,  $f \notin M$  and  $g \notin N$  [4, 7M]. But then  $fg = 0 \in F$  so  $f \in F$  or  $g \in F$  (1.7); hence,  $f \in M$  or  $g \in N$ . From this contradiction, we conclude  $M = N$ .

**COROLLARY 3.4.** *Suppose  $H[\mathcal{A}] \in cX$ . If each maximal ideal, which contains a maximal filter ideal, contains a unique maximal filter ideal, then  $F[\mathcal{A}] \in cX$ .*

*Proof.* Since  $H[\mathcal{A}] \in cX$ , then  $Z[\mathcal{A}]$  is a base for the closed subsets of  $X$ . It then suffices to show that  $F[\mathcal{A}]$  is Hausdorff. Let  $F, G \in F[\mathcal{A}]$ ,  $F \neq G$ . There exist unique  $M, N \in H[\mathcal{A}]$  such that  $F \subset M, G \subset N$  (3.3). Since  $M \neq N$  by hypothesis, we select  $f, g \in \mathcal{A}$  such that  $fg = 0, f \notin M$  and  $g \notin N$ . So  $f, g \in \mathcal{A}, fg = 0, f \notin F$  and  $g \notin G$ . By 2.4,  $F[\mathcal{A}]$  is Hausdorff.

Suppose now that  $T \in cX$  and  $\mathcal{A}$  is a subring of  $E(X, T)$  (the ring of all functions on  $X$  continuously extendable to  $T$ ) such that  $\mathcal{A}$  contains  $\mathcal{R}$  (the constant real-valued functions on  $X$ ) and  $Z[\mathcal{A}]$  is a base for the closed subsets of  $X$ . Then  $\psi: X \rightarrow H[\mathcal{A}]$  and  $\varphi: X \rightarrow F[\mathcal{A}]$  embed  $X$  as a dense subspace of the compact  $T_1$ -spaces  $H[\mathcal{A}]$  and  $F[\mathcal{A}]$ , respectively.

For  $f \in E(X, T)$ , denote the continuous extension by  $f^T$ . For  $t \in T$ , define  $M^t = \{f \in \mathcal{A} \mid f^T(t) = 0\}$ . Then  $M^t \in H[\mathcal{A}]$  for each  $t \in T$  since the mapping  $f + M^t \rightarrow f^T(t)$  is a ring isomorphism between  $\mathcal{A}/M^t$  and  $\mathcal{R}$ . Thus the mapping  $\psi: X \rightarrow H[\mathcal{A}]$  defined by  $\psi(x) = M_x$  is extendable from  $X$  to  $T$  by  $\psi(t) = M^t$ . Note that  $M^x = M_x$  for each  $x \in X$ .

LEMMA 3.5.  $\psi^*[f^*] = Z(f^T)$ .

*Proof.*  $t \in Z(f^T)$  if and only if  $f^T(t) = 0$  if and only if  $f \in M^t$  if and only if  $M^t \in f^*$  if and only if  $\psi(t) \in f^*$  if and only if  $t \in \psi^*[f^*]$ .

Hence,  $\psi: T \rightarrow H[\mathcal{A}]$  is continuous. So  $\psi[T]$  is a compact subspace of  $H[\mathcal{A}]$ . We then obtain the

THEOREM 3.6. *If  $H[\mathcal{A}]$  is Hausdorff, then*

- (1)  $H[\mathcal{A}] \in cX$  (via  $\psi: T \rightarrow H[\mathcal{A}]$ );
- (2)  $H[\mathcal{A}] = \psi[T] = \{M^t \mid t \in T\}$ ;
- (3)  $H[\mathcal{A}] \leq T$ ; and
- (4)  $H[\mathcal{A}] = T$  if and only if  $\psi$  is injective if and only if  $\{f^T \mid f \in \mathcal{A}\}$  separates points in  $T$  if and only if  $\{Z(f^T) \mid f \in \mathcal{A}\}$  is a base for the closed subsets of  $T$ .

*Proof.* (1) and (2). Now  $\psi[T] = \text{cl}_{H[\mathcal{A}]} \psi[T]$  since a compact subspace of a Hausdorff space is closed. Also,  $\text{cl}_{H[\mathcal{A}]} \psi[T] = H[\mathcal{A}]$  since  $\psi[X]$  is dense in  $H[\mathcal{A}]$ .

(3). Obvious.

(4). A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

THEOREM 3.7. *Suppose  $T = F[\mathcal{A}]$ . Then  $T = H[\mathcal{A}]$  if and only if each maximal ideal contains a unique maximal filter ideal and  $H[\mathcal{A}]$  is Hausdorff.*

*Proof.* Suppose  $H[\mathcal{A}] = T$ . Let  $M^t \in H[\mathcal{A}]$ . Then  $F^t \subset M^t$ , so every maximal ideal contains a maximal filter ideal (3.6 (2)). Since  $T = H[\mathcal{A}]$ , then  $\psi: T \rightarrow H[\mathcal{A}]$  is injective (3.6 (4)). Hence, if  $F^t, F^s \subset M^p$  where  $t, s, p \in T$  (3.1), then  $t = s = p$ . So each maximal ideal contains a unique maximal filter ideal. The Hausdorff condition is obvious.

Now assume the converse hypothesis and suppose  $H[\mathcal{A}] < T$  (3.6 (3)). Then  $\psi$  is not injective (3.6 (4)). Select  $t, s \in T$  such that  $t \neq s$ , but  $M^t = M^s$ . Since  $T = wZ[A] = F[A]$ , then  $F^t \neq F^s$  (3.1). Clearly  $F^t \subset M^t$  and  $F^s \subset M^s$ . So  $F^t, F^s \subset M^t$  and  $F^t \neq F^s$ , contradicting our assumption that each maximal ideal contains a unique maximal filter ideal. This completes the proof.

**THEOREM 3.8.** *Suppose  $T = H[\mathcal{A}]$ . Then  $T = F[\mathcal{A}]$  if and only if  $\text{cl}_T Z(f) \cap \text{cl}_T Z(g) = \emptyset$  whenever  $Z(f) \cap Z(g) = \emptyset$  and  $f, g \in \mathcal{A}$ .*

*Proof.* Since  $\{f^x | f \in \mathcal{A}\}$  is a base for the closed subsets of  $T$  (3.6 (4)), then so is  $\{\text{cl}_T Z(f) | f \in \mathcal{A}\}$ . By [1, 3.3],  $T = wZ[\mathcal{A}]$  if and only if  $\text{cl}_T Z(f) \cap \text{cl}_T Z(g) = \emptyset$  whenever  $Z(f) \cap Z(g) = \emptyset$  and  $f, g \in \mathcal{A}$ . This completes the proof since  $F[\mathcal{A}] = wZ[\mathcal{A}]$  (2.6).

Hence, if  $T \in cX$  is “constructable” as a maximal ideal space of  $\mathcal{A}$ , where  $\mathcal{A}$  is a subring of  $E(X, T)$  containing  $\mathcal{B}$ , then  $T$  is also constructable as the ultrafilter space from the zero-sets of  $\mathcal{A}$  if and only if disjoint zero-sets of  $\mathcal{A}$  have disjoint closures in  $T$ . Conversely, if  $T$  is “constructable” as the ultrafilter space from the zero-sets of  $\mathcal{A}$ , then  $T$  is constructable as the maximal ideal space of  $\mathcal{A}$  if and only if each maximal ideal contains a unique maximal filter ideal and the maximal ideal space is Hausdorff.

**THEOREM 3.9.** *Suppose  $H[\mathcal{A}] = T$  and  $F[\mathcal{A}] \in cX$ . Then  $T \leq F[\mathcal{A}]$ .*

*Proof.* Let  $F \in F[\mathcal{A}]$ . Since  $T$  is compact and

$$\mathcal{F} = \{\text{cl}_T Z(f) | f \in F\}$$

is a nonvoid set of nonvoid closed subsets of  $T$  with the *fixp*, then  $\cap \mathcal{F} \neq \emptyset$ . Since  $\{\text{cl}_T Z(f) | f \in \mathcal{A}\}$  is a base for the closed subsets of  $T$ , then  $\cap \mathcal{F}$  is a singleton (denote  $F \rightarrow t$ ). Thus, for each  $F \in F[\mathcal{A}]$  there exists a unique  $t \in T$  such that  $F \rightarrow t$ . Define  $h: F[\mathcal{A}] \rightarrow T$  by  $h(F) = t$  where  $F \rightarrow t$ . Then  $h$  is a surjection and  $h(F_x) = x$  for each  $x \in X$ . Since  $h^{-1}[\text{cl}_T Z(f)] = \cap \{g^* | \text{cl}_T Z(f) \subset \text{int}_T Z(g^*), g \in \mathcal{A}\}$  for each  $f \in \mathcal{A}$ , then  $h$  is continuous. Hence,  $T \leq F[\mathcal{A}]$  (via  $h$ ).

**COROLLARY 3.10.** *Suppose  $H[\mathcal{A}] = T$ . Then  $T = F[\mathcal{A}]$  if and*

only if each maximal ideal contains a unique maximal filter ideal.

*Proof.* Suppose each maximal filter ideal contains a unique maximal filter ideal. Then  $F[\mathcal{A}] \in cX$  by 3.4. The mapping  $h: F[\mathcal{A}] \rightarrow T$  defined in the proof of 3.9 is then injective. Hence,  $T = F[\mathcal{A}]$ . The converse follows from 3.7. This completes the proof.

**4. An application to  $E(X, T)$ .** Let  $T \in cX$ . Easily  $Z[E(X, T)]$  is a base for the closed subsets of  $X$ . In 1964 Frink [3] mentioned that  $Z[E(X, T)]$  was a Wallman base on  $X$ . However, Brooks, in a paper published in 1967 [2], mentioned he could not prove this. Subsequently Hager, in a 1969 paper, provided a “constructive” proof. We offer here a proof that  $Z[E(X, T)]$  is a Wallman base on  $X$  based on 2.4 and 2.7. We first observe

**LEMMA 4.1.** *Suppose  $\mathcal{A}$  is a subring of  $C(X)$  such that if  $f \in \mathcal{A}$ , then  $|f| \in \mathcal{A}$ . Let  $I$  be a  $z$ -filter ideal of  $\mathcal{A}$ . Then the following are equivalent:*

- (1)  $I$  is a prime ideal of  $\mathcal{A}$ ;
- (2)  $I$  contains a prime ideal of  $\mathcal{A}$ ;
- (3) if  $f, g \in \mathcal{A}$  and  $fg = 0$ , then  $f \in I$  or  $g \in I$ ; and
- (4) for each  $f \in \mathcal{A}$  there exists  $g \in I$  such that  $f$  does not change sign on  $Z(g)$ .

*Proof.* The techniques of [4, 2.9] apply verbatim.

**THEOREM 4.2.** *Let  $\mathcal{A}$  be subring of  $C(X)$  such that  $Z[\mathcal{A}]$  is a base for the closed subsets of  $X$  and if  $f \in \mathcal{A}$ , then  $|f| \in \mathcal{A}$ . Then  $\mathcal{A}$  is a Wallman ring on  $X$ .*

*Proof.* It suffices to show that  $F[\mathcal{A}]$  is Hausdorff (2.7). To show this we apply 2.4. Let  $F, G \in F[\mathcal{A}]$ ,  $F \neq G$ . Then  $F \cap G$  is a  $z$ -filter ideal of  $\mathcal{A}$  which is not prime. Using 4.1(3), we select  $f, g \in \mathcal{A}$  such that  $fg = 0$ , but  $f \notin F \cap G$  and  $g \notin F \cap G$ . But  $F$  and  $G$  are prime ideals of  $\mathcal{A}$  (1.7); hence, either  $f \in F$  or  $g \in F$ . Suppose  $f \in F$ . Then  $g \notin F$  and  $f \notin G$ . Also, if  $g \in F$ , then  $f \notin F$  and  $g \notin G$ . By 2.4, then,  $F[\mathcal{A}]$  is Hausdorff. Hence,  $\mathcal{A}$  is a Wallman ring on  $X$ .

**COROLLARY 4.3.** *Let  $T \in cX$ . Then  $Z[E(X, T)]$  is a Wallman base for  $X$ .*

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