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A GENERALIZATION OF MARTINGALES AND TWO CONSEQUENT CONVERGENCE THEOREMS

LOUIS HARVEY BLAKE

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Loeve has observed that a discrete stochastic process can be interpreted as a game and that a martingale can be interpreted as a "fair" game. In this context, the notion of a martingale is enlarged to a game which becomes "fairer with time" and then this concept is utilized to establish two convergence theorems.

Let $(\Omega, \mathfrak{A}, p)$ be a probability space with $\{\mathfrak{A}_n\}_{n \geq 1}$ an increasing family of sub σ -algebras of \mathfrak{A} to which the process $\{X_n\}_{n \geq 1}$ is adapted, (see [3, p. 65]). Henceforth, the process $\{X_n\}_{n \geq 1}$ will be referred to as a game.

DEFINITION. The game $\{X_n\}_{n \geq 1}$ will be said to become *fairer with time* if for every $\varepsilon > 0$.

$$p[|E(X_n | \mathfrak{A}_m) - X_m| > \varepsilon] \rightarrow 0$$

as $n, m \rightarrow \infty$ with $n \geq m$.

It should be noted that any martingale is a game which becomes fairer with time. An easy example of a game which is not a martingale or a sub or a super martingale but does become fairer with time is constructed by considering a game which consists of tossing a die. Here, let

$$\mathfrak{A}_n = \mathfrak{A}, \text{ all } n$$

and

$$X_n(\{i\}) \equiv i + (-1)^n/n .$$

The main results. Let $\{\alpha_n: n \geq 1\}$ be a monotonic sequence decreasing to zero with finite sum. The game $\{X_n\}_{n \geq 1}$ may be decomposed with respect to $\{\alpha_n: n \geq 1\}$ as

$$(1.1) \quad X_n = Y_n - Z_n, \text{ where } \{Y_n\}_{n \geq 1} \text{ and } \{Z_n\}_{n \geq 1}$$

are defined inductively by:

$$(1.2) \quad \begin{aligned} Y_1 &= X_1 \\ &\vdots \\ Y_n &= Y_{n-1} + [X_n - E(X_n | \mathfrak{A}_{n-1})] + \alpha_{n-1} \end{aligned}$$

$$(1.3) \quad Z_n = Z_{n-1} + [X_{n-1} - E(X_n | \mathfrak{A}_{n-1})] + \alpha_{n-1}.$$

We note that $\{Y_n\}_{n \geq 1}$ is adapted to the sequence of σ -algebras $\{\mathfrak{A}_n\}_{n \geq 1}$ and forms a submartingale with respect to it.

We will call the decomposition of the game $\{X_n\}_{n \geq 1}$ according to (1.1) – (1.3) a Doob-like decomposition. (See [3, p. 104–105].)

Also, we define the collection of sets $\{B_{n,m}^\alpha\}$ for $m = 1, 2, \dots$ and $n \geq m$ by

$$B_{n,m}^\alpha \equiv \{w: |E(X_n | \mathfrak{A}^m) - X_m| > \alpha_m\}.$$

THEOREM 1. *Let $\{X_n\}_{n \geq 1}$ be a uniformly integrable game and $\{Y_n\}_{n \geq 1}$, the submartingale associated with its Doob-like decomposition, be uniformly dominated in absolute value by an element of $L_1(\Omega, \mathfrak{A}, p)$. Suppose for every $\delta > 0$ there exists an integer $N(\delta)$, such that*

$$(1.4) \quad P[B_{n,m}^\alpha] < \delta \text{ whenever } n \geq m \geq N(\delta),$$

and

$$(1.5) \quad \sim B_{n,n}^\alpha \subset \sim B_{k,k-1}^\alpha \text{ whenever } n \geq k \geq k-1 \geq m \geq N(\delta).$$

Then, there exists a function X in $L_1(\Omega, \mathfrak{A}, p)$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |X_n - X| dp = 0.$$

Proof. It will be sufficient to show the game $\{X_n\}_{n \geq 1}$ is Cauchy in the L_1 norm. For every pair (n, m) of positive integers write:

$$\int_{\Omega} |X_n - X_m| dp = \int_{B_{n,m}^\alpha} |X_n - X_m| dp + \int_{\sim B_{n,m}^\alpha} |X_n - X_m| dp.$$

Since $p[B_{n,m}^\alpha] \rightarrow 0$ as $n, m \rightarrow \infty$ and since the game $\{X_n\}_{n \geq 1}$ is uniformly integrable (see [1, p. 89]), it is immediate that $\int_{B_{n,m}^\alpha} |X_n - X_m| dp$ can be made arbitrarily small for sufficiently large n and m .

By utilizing the Doob-like decomposition of $\{X_n\}_{n \geq 1}$, we can write

$$\int_{\sim B_{n,m}^\alpha} |X_n - X_m| dp \leq \int_{\sim B_{n,m}^\alpha} |Y_n - Y_m| dp + \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp.$$

Since there exists an integrable function which uniformly dominates the process $\{Y_n\}_{n \geq 1}$ in absolute value, it is immediate that $\{Y_n\}_{n \geq 1}$ is a convergent submartingale. Moreover, the dominated convergence theorem can be used to show that $\int_{\sim B_{n,m}^\alpha} |Y_n - Y_m| dp$ can be made arbitrarily small for sufficiently large n and m .

Thus, it remains to show that $\int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp$ can be made arbitrarily small for sufficiently large n and m and the proof will be complete.

On $\sim B_{n,m}^\alpha$ it follows that

$$X_m \geq E(X_n | \mathfrak{A}_m) - \alpha_m .$$

In particular, on $\sim B_{n,n-1}^\alpha$

$$X_{n-1} \geq E(X_n | \mathfrak{A}_{n-1}) - \alpha_{n-1}$$

and so where

$$(1.6) \quad Z_n - Z_{n-1} = X_{n-1} - E(X_n | \mathfrak{A}_{n-1}) + \alpha_{n-1}$$

we can say

$$(1.7) \quad Z_n - Z_{n-1} \geq 0 \text{ on } \sim B_{n,n-1}^\alpha .$$

Thus, choose any $\delta > 0$ and there exists $N(\delta)$ such that

$$(1.8) \quad \sum_{k=m}^\infty \alpha_k < \delta/2 \text{ for } m \geq N(\delta)$$

and such that (1.5) holds. Hence, with $n \geq m \geq N(\delta)$, (1.5) and (1.7), write

$$(1.9) \quad Z_n - Z_{n-1} \geq 0 \text{ on } \sim B_{n,m}^\alpha .$$

By observing the fact that $B_{n,m}^\alpha \in \mathfrak{A}_m$ for all n and m , we can write that

$$(1.10) \quad \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp = \int_\Omega E\{|Z_n - Z_m| I_{\sim B_{n,m}^\alpha} | \mathfrak{A}_m\} dp .$$

By (1.9), $|Z_n - Z_m| I_{\sim B_{n,m}^\alpha} = \sum_{k=m+1}^n (Z_k - Z_{k-1}) I_{\sim B_{n,m}^\alpha}$; this together with (1.6) lets us continue the equality in (1.10) to

$$\begin{aligned} \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp &= \sum_{k=m+1}^n \left\{ \int_{\sim B_{n,m}^\alpha} E\{(X_{k-1} - E(X_k | \mathfrak{A}_{k-1}) + \alpha_{k-1} | \mathfrak{A}_m)\} dp \right\} \\ &= \int_{\sim B_{n,m}^\alpha} \{(X_m - E(X_n | \mathfrak{A}_m)) + \alpha_m + \dots + \alpha_{n-1}\} dp \\ &\leq \int_{\sim B_{n,m}^\alpha} \left\{ \alpha_m + \sum_{k=m}^{n-1} \alpha_k \right\} dp < \delta . \end{aligned}$$

By not demanding that the submartingale $\{Y_n\}_{n \geq 1}$ associated with the Doob-like decomposition of the game $\{X_n\}_{n \geq 1}$ be uniformly bounded above in absolute value by an element of $L_1(\Omega, \mathfrak{A}, p)$, we get the weaker

THEOREM 2. *Let $\{X_n\}_{n \geq 1}$ be a uniformly integrable game satisfying (1.4) and (1.5). Then, there exists some constant c such that*

$$\lim_{n \rightarrow \infty} \int_a X_n dp = c.$$

Proof. It will be sufficient to show the sequence $\left\{ \int_a X_n dp \right\}_{n \geq 1}$ is Cauchy. With respect to the Doob-like decomposition of $\{X_n\}_{n \geq 1}$, we can write

$$(1.11) \quad \left| \int_a (X_n - X_m) dp \right| \leq \left| \int_{B_{n,m}^\alpha} (X_n - X_m) dp \right| + \left| \int_{\sim B_{n,m}^\alpha} (X_n - X_m) dp \right|.$$

Again, $\left| \int_{B_{n,m}^\alpha} (X_n - X_m) dp \right|$ may be made arbitrarily small for sufficiently large m and n by using the uniform integrability of $\{X_n\}_{n \geq 1}$. In order to deal with the second summand in (1.11), write

$$\left| \int_{\sim B_{n,m}^\alpha} (X_n - X_m) dp \right| \leq \left| \int_{\sim B_{n,m}^\alpha} (Y_n - Y_m) dp \right| + \int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp.$$

But $\int_{\sim B_{n,m}^\alpha} |Z_n - Z_m| dp$ can be made arbitrarily small for sufficiently large m and n exactly as in the proof of Theorem 1. Hence, showing that $\left| \int_{\sim B_{n,m}^\alpha} (Y_n - Y_m) dp \right|$ can be made arbitrarily small for sufficiently large m and n will complete the proof. To this end, we use (1.2) and write

$$E((Y_n - Y_m) | \mathfrak{A}_m) = \alpha_m + \dots + \alpha_{n-1}$$

and get

$$\begin{aligned} \int_{\sim B_{n,m}^\alpha} (Y_n - Y_m) dp &= \int_{\sim B_{n,m}^\alpha} E\{(Y_n - Y_m) | \mathfrak{A}_m\} dp \\ &= \int_{\sim B_{n,m}^\alpha} (\alpha_m + \dots + \alpha_{n-1}) dp \leq \sum_{k=m}^{n-1} \alpha_k. \end{aligned}$$

But since $\sum_{k=m}^{n-1} \alpha_k$ can be made arbitrarily small for sufficiently large m and n , the result follows.

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