

Pacific Journal of Mathematics

AN APPROXIMATION THEOREM FOR SUBALGEBRAS OF H_∞

ARNE STRAY

AN APPROXIMATION THEOREM FOR SUBALGEBRAS OF H^∞

ARNE STRAY

Let E be a closed subset of the unitcircle $T = \{z : |z| = 1\}$ and denote by B_E the algebra of all functions which are bounded and continuous on the set $X = \{z : |z| \leq 1 \text{ \& } z \notin E\}$ and analytic in $D = \{z : |z| < 1\}$.

The main result of this paper (Theorem 1) is that there exist an open set V_E containing X such that every $f \in B_E$ can be approximated uniformly on X by functions being analytic in V_E .

The algebra B_E was introduced in [4] by E. A. Heard and J. H. Wells.

In [4] they characterize the interpolationsets for B_E . At the end of their paper they remark that the question of whether D is dense in the maximal ideal space $M(B_E)$ of B_E is open in case E is a proper nonempty subset of T . As a corollary of Theorem 1 we prove that D is dense in $M(B_E)$. (In proving the corollary we of course use the Carleson corona-theorem [1]).

This corollary has also been proved recently by Jaqueline Detraz in [3] where it follows from the very interesting fact that the restriction map from $M(H^\infty)$ (the maximal ideal space of $H^\infty(D)$) to $M(B_E)$ is onto. This is the main theorem of [3, Th. 2]. [3] contains also other results about B_E that we do not prove her. However, Theorem 2 of [3] can also be proved by using the main result of this paper together with the Carleson corona-theorem since Theorem 2 of [3] is equivalent with the fact that D is dense in $M(B_E)$. But the proof of Theorem 2 in [3] is more direct and do not involve the Carleson corona theorem.

Through the whole paper $r_0 > 1$ will be a fixed real number.

Define an open set V_E by $V_E = X \cup \{z : 1 \leq |z| < r_0 \text{ \& } \frac{z}{|z|} \notin V\}$

THEOREM 1. *For every $f \in B_E$ and every $\varepsilon > 0$ there exist a function g analytic in V_E such that $\|f - g\|_X < \varepsilon$.*

LEMMA 1. *Suppose $f \in B_E$ and e is a continuously differensiabile function on T with compact support contained in $\mathbb{C} \setminus E$.*

If $f = u + iv$ and we define $u_1(\theta) = u(\theta)e(\theta)$ ($\theta \in (-\pi, \pi]$), then the function

$$f_1(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u_1(\theta) d\theta = u_1(z) + v_1(z)$$

is in the disc-algebra $A(D)$ consisting of all continuous functions on \bar{D} being analytic in D .

Proof. Let $\theta \in (-\pi, \pi]$. Then the integral

$$I(\theta) = \frac{-1}{2\pi} \int_{\pi}^{\theta} \frac{u_1(\theta+t) - u_1(\theta-t)}{2 \tan \frac{t}{2}} dt$$

exists because it equals the sum

$$I_1(\theta) + I_2(\theta) + I_3(\theta)$$

where

$$I_1(\theta) = \frac{-1}{2\pi} \int_{\pi}^{\theta} \frac{(u(\theta+t) - u(\theta-t))e(\theta)}{2 \tan \frac{t}{2}} dt$$

$$I_2(\theta) = \frac{-1}{2\pi} \int_{\pi}^{\theta} \frac{u(\theta+t)(e(\theta+t) - e(\theta))}{2 \tan \frac{t}{2}} dt$$

and

$$I_3(\theta) = \frac{-1}{2\pi} \int_{\pi}^{\theta} \frac{u(\theta-t)(e(\theta) - e(\theta-t))}{2 \tan \frac{t}{2}} dt.$$

It is well-known that the existence of $I(\theta)$ is equivalent with the existence of $v_1^*(\theta) \stackrel{\text{def}}{=} \lim_{r \rightarrow 1} v_1(re^{i\theta})$ and that $I(\theta) = v_1^*(\theta)$ if $I(\theta)$ or $v_1^*(\theta)$ exists. (See [5] at pages 78 and 79).

Using the results mentioned in [5] we get that $I_1(\theta) = v(\theta) \cdot e(\theta)$. A change of variable in $I_3(\theta)$ shows that $I_3(\theta) = I_2(\theta)$ and $I_2(\theta)$ exists since $e \in C_0^1(T)$ and u is bounded.

Since $v_1^*(\theta)$ exists for all θ , Lemma 1 is proved if we can show that v_1^* is continuous on T . For then f_1 has the continuous boundary values $f_1(\theta) = u_1(\theta) + iv_1^*(\theta)$ and

$$\int_T e^{in\theta} f_1(\theta) d\theta = 0 \quad \text{for } n = 1, 2, \dots$$

By the remarks above it is sufficient to show that $I_2(\theta)$ is continuous on T .

The proof of this depends on the fact that $e \in C_0^1(T)$ and that for a fixed $f \in L^1(T)$ the map $x \mapsto f_x$ (where $f_x(y) = f(yx^{-1})$ when $x, y \in T$) from T to $L^1(T)$ is uniformly continuous. We omit the details.

Proof of Theorem 1. Let $T \setminus E = \bigcup_{k=1}^{\infty} I_k$ where each I_k is an open interval (arc) and $I_k \cap I_j = \emptyset$ if $k \neq j$.

Consider a fixed I_k . Construct open intervals $\{K_{kn}\}_{n=1}^{\infty}$ such that $I_k = \bigcup_{n=1}^{\infty} K_{kn}$. We also require that each $z \in I_k$ is not contained in more than two such intervals and that $K \cap \bar{K}_{kn} \neq \emptyset$ only for finitely many n if K is a compact subset of I_k . We choose non-negative functions $e_{kn} \in C_0^1(T)$ with the support of e_{kn} contained in K_{kn} and such that $\sum_{n=1}^{\infty} e_{kn}(z) = 1$ if $z \in I_k$.

Having carried out this construction for $k = 1, 2, 3, \dots$ we renumerate the double-sequence $\{e_{kn}\}$ to a sequence $\{\alpha_j\}$ by defining $\alpha_1 = e_{11}, \alpha_2 = e_{12}, \alpha_3 = e_{21}, \alpha_4 = e_{31}, \alpha_5 = e_{22}$ and so on.

The sequence $\{K_{kn}\}$ is renumerated in the same manner to a sequence $\{K_j\}$ so that $\text{support } \alpha_j \subset K_j$ for $j = 1, 2, \dots$

For each N we let S_N denote the union of the supports of the functions α_j for $j \geq N$.

By W_j we mean the compact set of all points in $\{z: |z| \leq r_0\}$ except those z such that $|z| > 1$ and such that the line segment from the origin to z intersects K_j .

The construction of the sets K_j guarantees that for each compact subset K of V_E there exists a number N such that $K \subset W_j$ for $j \geq N$.

Let now $f \in B_E$. We can without loss of generality assume $f = u + iv$ where $v(0) = 0$. Define $u_j(\theta) = u(\theta) \cdot \alpha_j(\theta)$ $j = 1, 2, \dots$ $\theta \in (-\pi, \pi]$.

Now let

$$f_j(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta, z) u_j(\theta) d\theta$$

where

$$H(\theta, z) = \frac{e^{i\theta} + z}{e^{i\theta} - z}.$$

The function f_j is analytic outside the support of u_j , but by Lemma 1 we can view f_j as a function continuous on \bar{D} and analytic in D .

But then it is easy to see that we in fact have that f_j is continuous on W_j and analytic in the interior of W_j .

Let $\varepsilon > 0$ be given. Choose polynomials p_j such that

$$\|f_j - p_j\|_{W_j} < \frac{\varepsilon}{2j} \quad \text{for } j = 1, 2, \dots$$

Define now $f_E(z) = 1/2\pi \int_E H(\theta, z) u(\theta) d\theta$ for $z \in \mathbb{C} \setminus E$. f_E is analytic but not necessarily bounded in $\mathbb{C} \setminus E$.

Let K be a compact subset of V_E . Then there exists a number N_1 such that $K \subset W_j$ if $j \geq N_1$. We can choose a number $N_2 \geq N_1$ such that the distance from S_{N_2} to K is positive.

Then we have that

$$(*) : \quad \left\| \sum_M^M f_j \right\|_K \rightarrow 0$$

as $M \geq N \geq N_2$ and $N \rightarrow \infty$ because $\sup |H(\theta, z)| < \infty$ $z \in K$, $\theta \in S_{N_2}$ and the Lebesgue measure of S_N tends to zero as $N \rightarrow \infty$.

Define $P_N(z) = \sum_1^N f_j(z)$ for all z and let $F_N(z) = \sum_1^N f_j(z)$ if $z \in X$.

From (*) and the fact that $\|p_j - f_j\|_{W_j} < \varepsilon/2j$ it follows that P_N is a uniform Cauchy-sequence on compact subsets of V_E .

Thus

$$P(z) = \lim_{N \rightarrow \infty} P_N(z) \quad (z \in V_E)$$

is analytic in V_E . In the same way that the formula (*) was proved we get that on compact subsets K of X we have that

$$\|f_E + F_N - f\|_K \rightarrow 0$$

as $N \rightarrow \infty$.

Let now $z \in X$. Then we have that

$$\begin{aligned} & |f_E(z) + P(z) - f(z)| \\ & \leq |P - P_N(z)| + |P_N(z) - F_N(z)| + |f_E(z) + F_N(z) - f(z)|. \end{aligned}$$

Let now $N \rightarrow \infty$. Then we get that

$$|f_E(z) + P(z) - f(z)| \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

Since $f_E + P$ is analytic in V_E and $z \in X$ was arbitrary the theorem is proved.

COROLLARY. *D is dense in the maximal ideal space $M(B_E)$ of B_E .*

Proof. If the corollary is not true then there exists $m \in M(B_E)$ and functions $f_1, \dots, f_n \in B_E$ such that $m(f_i) = 0$ $i = 1, 2, \dots, n$ and such that $\sum_1^n |f_i| \geq \delta$ in D for some $\delta > 0$. It is not difficult by Theorem 1 to see that we can assume f_1, \dots, f_n to be analytic in V_E .

Then we can construct an open simply connected set $V \subset V_E$ containing X such that $f_1 \dots f_n$ are bounded in V and $\sum_1^n |f_i| \geq \delta/2$ in V . By the Carleson corona theorem we can find bounded analytic functions g_1, \dots, g_n in V such that $f_1 g_1 + \dots + f_n g_n \equiv 1$ in V . Since

g_1, \dots, g_n restricted to X are in B_E we have the contradiction

$$1 = m(1) = \sum_{i=1}^n m(f_i) \cdot m(g_i) = 0 .$$

REFERENCES

1. L. Carleson, Ann. of Math. **76** (1962), 547-559.
2. Jaqueline Detraz, Comptes Rendus Acad. Sci. Paris **269**, 688-691.
3. ———, Comptes Rendus Acad. Sci. Paris **269**, 833-835.
4. E. A. Heard and J. H. Wells, Pacific. J. Math. **28** (1969), 543-553.
5. K. Hoffman, *Banach Spaces of Analytic functions*, Prentice Hall, Englewood Cliffs, N. J., 1962.

Received December 30, 1969.

UNIVERSITY OF OSLO
OSLO NORWAY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Valentin Danilovich Belousov and Palaniappan L. Kannappan, <i>Generalized Bol functional equation</i>	259
Charles Morgan Biles, <i>Gelfand and Wallman-type compactifications</i>	267
Louis Harvey Blake, <i>A generalization of martingales and two consequent convergence theorems</i>	279
Dennis K. Burke, <i>On p-spaces and $w\Delta$-spaces</i>	285
John Ben Butler, Jr., <i>Almost smooth perturbations of self-adjoint operators</i>	297
Michael James Cambern, <i>Isomorphisms of $C_0(Y)$ onto $C(X)$</i>	307
David Edwin Cook, <i>A conditionally compact point set with noncompact closure</i>	313
Timothy Edwin Cramer, <i>Countable Boolean algebras as subalgebras and homomorphs</i>	321
John R. Edwards and Stanley G. Wayment, <i>A v-integral representation for linear operators on spaces of continuous functions with values in topological vector spaces</i>	327
Mary Rodriguez Embry, <i>Similarities involving normal operators on Hilbert space</i>	331
Lynn Harry Erbe, <i>Oscillation theorems for second order linear differential equations</i>	337
William James Firey, <i>Local behaviour of area functions of convex bodies</i>	345
Joe Wayne Fisher, <i>The primary decomposition theory for modules</i>	359
Gerald Seymour Garfinkel, <i>Generic splitting algebras for Pic</i>	369
J. D. Hansard, Jr., <i>Function space topologies</i>	381
Keith A. Hardie, <i>Quasifibration and adjunction</i>	389
G. Hochschild, <i>Coverings of pro-affine algebraic groups</i>	399
Gerald L. Itzkowitz, <i>On nets of contractive maps in uniform spaces</i>	417
Melven Robert Krom and Myren Laurance Krom, <i>Groups with free nonabelian subgroups</i>	425
James Robert Kuttler, <i>Upper and lower bounds for eigenvalues by finite differences</i>	429
Dany Leviatan, <i>A new approach to representation theory for convolution transforms</i>	441
Richard Beech Mansfield, <i>Perfect subsets of definable sets of real numbers</i>	451
Brenda MacGibbon, <i>A necessary and sufficient condition for the embedding of a Lindelof space in a Hausdorff $\mathfrak{H}\sigma$ space</i>	459
David G. Mead and B. D. McLemore, <i>Ritt's question on the Wronskian</i>	467
Edward Yoshio Mikami, <i>Focal points in a control problem</i>	473
Paul G. Miller, <i>Characterizing the distributions of three independent n-dimensional random variables, X_1, X_2, X_3, having analytic characteristic functions by the joint distribution of $(X_1 + X_3, X_2 + X_3)$</i>	487
P. Rosenthal, <i>On the Bergman integral operator for an elliptic partial differential equation with a singular coefficient</i>	493
Douglas B. Smith, <i>On the number of finitely generated O-groups</i>	499
J. W. Spellmann, <i>Concerning the domains of generators of linear semigroups</i>	503
Arne Stray, <i>An approximation theorem for subalgebras of H_∞</i>	511
Arnold Lewis Villone, <i>Self-adjoint differential operators</i>	517