**THE DERIVED SET OF THE SPECTRUM OF A DISTRIBUTION FUNCTION**

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Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be real sequences with $b_n > 0$, $b_n \to 0\,(n \to \infty)$. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials satisfying the recurrence

$$xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x)\,, \quad (n \geq 0)\,,$$
$$P_{-1}(x) = 0\,, \quad P_0(x) = 1\,.$$

Then there is a substantially unique distribution function $\psi$ with respect to which the $P_n(x)$ are orthogonal. This paper verifies a conjecture of D. P. Maki that the set of all limit points of the sequence $\{a_n\}$ is the derived set of the spectrum of $\psi$.

Let $\{P_n(x)\}$ be a sequence of orthonormal polynomials defined by the recurrence formula

(1.1)
$$xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) \quad (n \geq 0)\,,$$
$$P_{-1}(x) = 0,\ P_0(x) = 1,\ a_n \text{ real},\ b_n > 0\,.$$

Then it is well known that there is a bounded, nondecreasing function $\psi$ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)d\psi(x) = \delta_{mn}\,,$$

the spectrum of $\psi$,

$$S(\psi) = \{t \mid \psi(t + \varepsilon) - \psi(t - \varepsilon) > 0 \text{ for all } \varepsilon > 0\}\,,$$

being an infinite set.

If we impose the additional hypothesis

(1.2)
$$\lim_{n \to \infty} b_n = 0\,,$$

then the Hamburger moment problem associated with (1.1) is determined (by Carleman's criterion — see [4, p. 59]) and the distribution function $\psi$ is substantially unique (is uniquely determined up to an arbitrary additive constant at all points of continuity).

In [3], D. P. Maki proved that every (finite) sequential limit point of $\{a_n\}_{n=0}^{\infty}$ is a point in $S(\psi)$ and he conjectured that $\lambda$ is a limit point of $\{a_n\}$ if and only if $\lambda$ is a point of the derived set, $S(\psi)'$. Maki's conjecture is correct as will be proved below.

If we denote the smallest and largest limit points (in the extended real number system) of $S(\psi)$ by $\sigma$ and $\tau$, respectively, then it was proved in [2, Th. 7] that under the hypothesis (1.2),

$$(1.3) \qquad \sigma = \liminf_{n\to\infty} a_n, \quad \tau = \limsup_{n\to\infty} a_n \,.$$

It follows that Maki's conjecture will remain valid if we allow infinite limit points also.

2. In the sequel, we will have reference to the $J$-fraction,

$$(2.1) \qquad \frac{1}{\mid z - a_0} - \frac{b_0^2}{\mid z - a_1} - \frac{b_1^2}{\mid z - a_2} - \frac{b_2^2}{\mid z - a_2} - \cdots .$$

With the hypothesis (1.2), we are dealing with the determinate case so (2.1) converges uniformly on every closed half-plane,

$$\mathrm{Im}\,(z) \geqq \delta > 0 \,,$$

to an analytic function $F$ which is not a rational function. $\psi$ can be obtained from $F$ by the Stieltjes inversion formula (see [5, p. 250]) and this shows that if the analytic continuation of $F$ is regular in a region containing a real open interval $(a, b)$, then $\psi$ is constant on $(a, b)$.

We will denote the $n^{th}$ convergent of (2.1) by $A_n(z)/B_n(z)$ so that $B_n(z)$ is the monic orthogonal polynomial, $B_n(z) = b_0 b_1 \cdots b_{n-1} P_n(z)$.

We also recall that if the Hamburger moment problem is determined, then (see [4, Corollary 2.6])

$$(2.2) \qquad \rho(x) \equiv \left\{ \sum_{n=0}^{\infty} P_n^2(x) \right\}^{-1}$$

vanishes at all points of continuity of $\psi$ and equals the jump of $\psi$ at a point of discontinuity.

3. We now state our main result.

THEOREM. *Let* $\lim_{n\to\infty} b_n = 0$. *Then* $\lambda$ *is a limit point of the sequence* $\{a_n\}_{n=0}^{\infty}$ *if and only if* $\lambda$ *is a limit point of* $S(\psi)$.

*Proof.* In view of (1.3), it is sufficient to consider $\lambda$ finite.

First let $\lambda$ be a finite limit point of $\{a_n\}$. Then Maki has shown that $\lambda \in S(\psi)$ and also that there is a subsequence $\{P_{n_k}\}$ such that

$$(3.1) \qquad \lim_{k\to\infty} \int_{-\infty}^{\infty} (x - \lambda)^2 P_{n_k}^2(x) d\psi(x) = 0 \,.$$

We will now show that $\lambda$ cannot be an isolated point of $S(\psi)$.

Assume $\lambda$ is isolated. Then $\psi$ has a jump at $\lambda$ so let $J_\lambda > 0$ denote this jump. It follows that there is an $\varepsilon > 0$ such that $S(\psi)$ contains no points in either of the half-open intervals, $[\lambda - \varepsilon, \lambda)$ and $(\lambda, \lambda + \varepsilon]$. Thus, writing $f_k = P_{n_k}$, we have by a modification of a technique used by Maki,

$$\int_{-\infty}^{\infty} f_k^2 d\psi = \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi + f_k^2(\lambda)J_\lambda = 1 \ ,$$

$$\int_{-\infty}^{\infty} (x - \lambda)^2 f_k^2 d\psi = \int_{-\infty}^{\lambda-\varepsilon} (x - \lambda)^2 f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} (x - \lambda)^2 f_k^2 d\psi$$

$$\geqq \varepsilon^2 \left\{ \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi \right\}$$

$$= \varepsilon^2 \left[ 1 - f_k^2(\lambda)J_\lambda \right] \geqq 0 \ .$$

Therefore, according to (3.1),

$$\lim_{k \to \infty} P_{n_k}^2(\lambda) = J_\lambda^{-1} > 0$$

but this contradicts the fact that $\rho(\lambda) = J_\lambda$ (see (2.2)). Thus $\lambda \in S(\psi)'$.

Conversely, let $\lambda \in S(\psi)'$ and assume that $\lambda$ is not a limit point of $\{a_n\}$. Then there is a $\delta > 0$ and an index $N_1$ such that $|a_n - \lambda| \geqq 2\delta$ for $n \geqq N_1$, hence

$$|z - a_n| \geqq \delta \text{ for } |z - \lambda| \leqq \delta \ , \qquad\qquad n \geqq N_1 \ .$$

Since by hypothesis, $b_n \to 0$, there is an index $N$ such that

$$\left| \frac{b_n^2}{(z - a_n)(z - a_{n+1})} \right| \leqq \frac{b_n^2}{\delta^2} < \frac{1}{4} \qquad\qquad \text{for } n \geqq N$$

and $z \in D = \{w \mid |w - \lambda| \leqq \delta\}$.

It now follows from a Theorem of Worpitzky (see [5, Th. 10.1]) that the $J$-fraction

$$\frac{b_N^2 \mid}{\mid z - a_{N+1}} - \frac{b_{N+1}^2 \mid}{\mid z - a_{N+2}} - \frac{b_{N+2}^2 \mid}{\mid z - a_{N+3}} - \cdots$$

converges uniformly on $D$ to an analytic function $F_N$ and, from (2.1), we have

$$F(z) = \frac{A_N(z) - A_{N-1}(z)F_N(z)}{B_N(z) - B_{N-1}(z)F_N(z)}$$

for $z \in D$, $z$ not a zero of $B_N - B_{N-1}F_N$.

Since $F_N$ cannot be a rational function, $B_N - B_{N-1}F_N$ can have at most finitely many zeros in $D$. That is, $F$ has at most finitely many singularities in $D$, which in turn implies that $\psi$ has at most

finitely many points of increase in $[\lambda - \delta, \lambda + \delta]$. Thus we again reach a contradiction.

4. REMARKS. 1. W. R. Allaway has shown (private communication) that when $\{a_n\}$ is bounded, then a theorem of Krein [1, pp. 230–231] can be used to prove Maki's conjecture in the case $S(\psi)'$ is finite.

2. Maki's proof shows that (3.1) holds if for some $\{n_k\}$,

(4.1) $\qquad\qquad b_{n_k-1} \to 0, \; b_{n_k} \to 0, \; a_{n_k} \to \lambda \; \text{(finite)} \; (n \to \infty)$

so that (4.1) is sufficient for $\lambda \in S(\psi)'$ (assuming a determined moment problem).

3. Maki showed that if $A$ denotes the self-adjoint operator, $Af = xf$, defined on a dense subset of $L^2(\psi)$, then $\sigma(A) \subset S(\psi)$, where $\sigma(A)$ denotes the spectrum of $A$.

Consideration of the characteristic function for the singleton set $\{\lambda\}$ shows that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is a point of discontinuity of $\psi$. That is, $P\sigma(A) = D(\psi)$, $C\sigma(A) \subset S(\psi)' \backslash D(\psi)$, where $P\sigma(A)$, $C\sigma(A)$ denote the point and continuous spectra of $A$ and $D(\psi)$ is the set of jump points of $\psi$.

On the other hand, if $\lambda \in S(\psi)' \backslash D(\psi)$ ($\lambda$ finite), then there is a measurable function $f_n$ with support in $[\lambda - 1/n, \lambda + 1/n]$ and with $\| f_n \| = 1$. Then

$$\| (A - \lambda) f_n \|^2 \leqq n^{-2} \int_{\lambda-1/n}^{\lambda+1/n} f_n^2 \, d\psi = n^{-2}$$

so that $\lambda \in \sigma(A)$. It follows that $C\sigma(A) = S(\psi)' \backslash D(\psi)$.

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