Pacific Journal of Mathematics

THE ABEL SUMMABILITY OF CONJUGATE MULTIPLE FOURIER-STIELTJES INTEGRALS

S. P. Philipp, Victor Lenard Shapiro and William Hall Sills

Vol. 36, No. 1 November 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 1, 1971

THE ABEL SUMMABILITY OF CONJUGATE MULTIPLE FOURIER-STIELTJES INTEGRALS

S. P. PHILIPP, V. L. SHAPIRO AND W. H. SILLS

Let $K(x) = \Omega(x/|x|)|x|^{-k}$ where $\Omega(\xi)$, $|\xi| = 1$, is a real valued function which is in Lip α, $0 < \alpha < 1$, on the unit $(k-1)$-sphere S in k-dimensional Euclidean space, E_k, $k \geqq 2$ with the additional property that $\int_S \Omega(\xi)d\sigma(\xi) = 0$ where σ is the natural surface measure for S. ($K(x)$ is usually called a Calderón-Zygmund kernel in Lip α.) Let μ be a Borel measure of finite total variation on E_k and set $\hat{\mu}(y) = (2\pi)^{-k}\int_{E_k} e^{-i(y,w)}d\mu(w)$. Also designate the principal-valued Fourier transform of K by $\hat{K}(y)$ and the principal-valued convolution of K with μ by $\tilde{\mu}(x)$. Define $I_R(x) = (2\pi)^k\int_{E_k} e^{-|y|/R}\hat{K}(y)\hat{\mu}(y)e^{i(y,x)}dy$. Then if k is an even integer or if $k = 3$, the following result is established: $\lim_{R\to\infty} I_R(x) = \tilde{\mu}(x)$ almost everywhere.

In [5] V. L. Shapiro proved that the conjugate Fourier-Stieltjes integral of a finite Borel measure μ in the plane E_2, taken with respect to a Calderón-Zygmund kernel $K(x)$ in Lip α, $1/2 < \alpha < 1$, is almost everywhere Abel summable to the principal-valued convolution $K*\mu$. The purpose of this paper is to extend this result to E_3 and to even-dimensional E_k for $K(x)$ in Lip α, $0 < \alpha < 1$. The first author will obtain the corresponding result for the odd-dimensional cases $k = 2s + 1$, $s \geqq 2$, in a paper to appear, by the use of special functions. Also, the results of the present paper should be compared with Theorem 2 of [6, p. 44].

2. Definitions and notation.

For $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k)$ put $(x, y) = x_1y_1 + \cdots + x_ky_k$, $|x| = (x, x)^{1/2}$ and $B(x, t) = \{y: |x - y| < t\}$. We will work with a fixed Calderón-Zygmund kernel $K(x) = \Omega(x/|x|)/|x|^k$ where $\Omega(\xi)$, $|\xi| = 1$, is a real-valued function defined on the unit $(k-1)$-dimensional sphere S in Euclidean space E_k, $k \geqq 2$, and $\int_S \Omega(\xi)d\sigma(\xi) = 0$, where σ is the natural surface measure for S [2, Chapter 11]. We define $K(x)$ to be in Lip α if $|\Omega(\xi) - \Omega(\eta)| = 0(|\xi - \eta|^\alpha)$ for some α, $0 < \alpha < 1$. The Fourier transform of a Borel measure μ in E_k of finite total variation is denoted as usual by

$$(1) \qquad \hat{\mu}(y) = (2\pi)^{-k}\int_{E_k} e^{-i(y,w)}d\mu(w)$$

and by the principal-valued convolution $\tilde{\mu}(x)$ we mean

$$(2) \qquad \lim_{t\to 0}\int_{E_k - B(x,t)} K(x-y)d\mu(y)$$

which is known to exist and be finite almost everywhere [1, p. 118]. The formal conjugate Fourier-Stieltjes integral of μ is given by

$$(3) \qquad (2\pi)^k \int_{E_k} e^{i(x,y)}\hat{\mu}(y)\hat{K}(y)dy$$

where

$$\hat{K}(y) = (2\pi)^{-k} \lim_{t\to 0; T\to\infty} \int_{B(0,T)-B(0,t)} e^{-i(y,x)}K(x)dx$$

is the principal-valued Fourier transform. We will denote the Abel means of (3) by

$$(4) \qquad I_R(x) = (2\pi)^k \int_{E_k} e^{-|y|/R} e^{i(x,y)}\hat{\mu}(y)\hat{K}(y)dy,\ R > 1\,.$$

With $\lambda = (k-2)/2$, P_n^λ will designate the Gegenbauer polynomials defined by the equation

$$(5) \qquad (1 - 2\rho\cos\theta + \rho^2)^{-\lambda} = \sum_{n=0}^{\infty} \rho^n P_n^\lambda(\cos\theta),\ 0 \leqq \rho < 1\,.$$

These functions allow us to form the Laplace series $\sum_{n=1}^{\infty} Y_n(\xi)$ of surface harmonics attached to $\Omega(\xi)$ on the unit sphere S in E_k by means of the equation

$$(6) \qquad Y_n(\xi) = \frac{\Gamma(\lambda)(n+\lambda)}{2\pi^{\lambda+1}} \int_S P_n^\lambda[(\xi,\eta)]\Omega(\eta)d\sigma(\eta)$$

(see [2, Chapter 11]). Formulas (5) and (6) give the Poisson integral representation

$$(7) \qquad \sum_{n=1}^{\infty} \rho^n Y_n(\xi) = \frac{\Gamma(\lambda+1)}{2\pi^{\lambda+1}} \int_S \frac{(1-\rho^2)\Omega(\eta)d\sigma(\eta)}{(1 - 2\rho(\xi,\eta) + \rho^2)^{\lambda+1}}$$

which is valid for $0 \leqq \rho < 1$. The assumptions on $\Omega(\xi)$ imply that $Y_0(\xi) = 0$.

3. The main theorem.

Our principal theorem is

THEOREM 1. *Let $K(x) = \Omega(x/|x|)/|x|^k$ be a Calderón-Zygmund kernel in* Lip α, $0 < \alpha < 1$. *Let μ be a Borel measure in E_k of finite total variation. Let $k = 3$ or $k = 2s$ where s is a positive integer. Then $\lim_{R\to\infty} I_R(x) = \tilde{\mu}(x)$ almost everywhere.*

Our proof will closely follow the original proof in [5]. We shall use, in addition, generalizations of certain statements in [5] obtained by V. L. Shapiro in [6]. Before outlining the proof, we will need some lemmas.

4. **Basic lemmas.** Throughout the balance of this paper, $\sum_{n=1}^{\infty} Y_n(\xi)$ will designate the Laplace series for $\Omega(\xi)$ on the unit sphere S in E_k. We will denote $\sup\{Y_n(\xi): |\xi| = 1\}$ by $\| Y_n \|_\infty$. The proof of the following lemma is given in [6, p. 69].

LEMMA 1. (i) *For each* $\gamma, 0 < \gamma < \alpha, \sum_{n=1}^{\infty} \| Y_n \|_\infty n^\gamma / n^{(k-1)/2} < \infty$.
(ii) $\hat{K}(y)$ *exists everywhere and if*

$$y \neq 0,\ \hat{K}(y) = \sum_{n=1}^{\infty} (-i)^n Y_n(y/|y|)\Gamma(n/2)/2^k \pi^{k/2} \Gamma((n+k)/2) .$$

Also, $\hat{K}(0) = 0$ *and the series converges absolutely and uniformly.*

Next we set

$$H_n^k(R) = \{\Gamma(n/2)/2^{k/2}\Gamma((n+k)/2)\} \int_0^\infty e^{-t/R} t^{k/2} J_{n+(k/2)-1}(t)dt ,$$

$R > 1; n = 1, 2, \cdots; k = 2, 3, \cdots,$ where $J_{n+\lambda}(t), \lambda = (k-2)/2,$ is a Bessel function of the first kind of order $n + \lambda$. The $H_n^k(R)$ arise naturally in the computation of $I_R(x)$.

LEMMA 2. (i) $0 \leqq H_n^k(R) \leqq 1, \lim_{R \to \infty} H_n^k(R) = 1,$
(ii) $0 \leqq H_n^k(R) \leqq$ Const. $R^k n^{-k/2}$,
(iii) $\sum_{n=1}^{\infty} \| Y_n \|_\infty H_n^k(R) = 0(R^k)$ *as* $R \to \infty$.

The first statement of (i) is proved in [6, Lemma 24, p. 64]. Also, as in formula (25) of [6, p. 56], we may express $H_n^k(R)$ by use of Euler's integral representation for hypergeometric functions as follows:

$$(8) \quad H_n^k(R) = (B(1/2, (n-1)/2))^{-1} \int_0^1 t^{-1/2}(1-t)^{(n-3)/2}(1 + 1/tR^2)^{-(n+k)/2} dt ,$$

where $B(p, q)$ is the usual Beta function. From this follows the second statement of (i). Part (ii) is a consequence of the inequalities $|J_{n+\lambda}(t)| \leqq t^\lambda$ [8, p. 60, Ex. 5] and $\Gamma(n/2)/2^{k/2}\Gamma((n+k)/2) \leqq$ Const. $n^{-k/2}$ [8, p. 58]. Part (iii) is a consequence of (ii) and Lemma 1, (i).

In what follows we will set $\rho = \sqrt{1 + 1/R^2} - 1/R, R > 1$. We note that $0 < \rho < 1$ and $\rho \to 1$ as $R \to \infty$. Our proof of the main theorem is based upon showing that $\sum_{n=1}^{\infty} H_n^k(R) Y_n(\xi)$ behaves somewhat like $\sum_{n=1}^{\infty} \rho^n Y_n(\xi)$. Next we state some lemmas which relate ρ^n to the $H_n^k(R)$. In the case that k is a positive even integer, $H_n^k(R)$ can be computed in closed form. Consider, for example, the formula

$\int_0^\infty e^{-at}J_\nu(t)dt = ((1 + a^2)^{1/2} - a)^\nu/(1 + a^2)^{1/2}, a > 0, \nu > -1$ [7, p. 202]. By differentiating the integral and replacing a by R^{-1} and ν by the appropriate integer, one shows that

$$(9)\quad H_n^2(R) = n^{-1}\int_0^\infty e^{-t/R}tJ_n(t)dt = \rho^n(1 + 1/R^2)^{-1}\{1 + n^{-1}(R\sqrt{1 + 1/R^2})^{-1}\}$$

and that

$$H_n^4(R) = \frac{1}{n(n+2)}\int_0^\infty e^{-t/R}t^2J_{n+1}(t)dt$$

$$(10)\quad = \rho^{n+1}(\sqrt{1 + 1/R^2})^{-3}\left\{1 + \frac{3(n+1)}{n(n+2)}(R\sqrt{1 + 1/R^2})^{-1} + \frac{3}{n(n+2)}(R\sqrt{1 + 1/R^2})^{-2}\right\},$$

and so on. The general formula for $H_n^{2s}(R) = (n(n+2)\cdots(n+2s-2))^{-1}\cdot \int_0^\infty e^{-t/R}t^sJ_{n+s-1}(t)dt$, $s \geqq 1$, is obtained by induction. We formalize this in the next lemma, whose proof we leave to the reader.

LEMMA 3. *For* $s = 1$ *put* $C_0^s(n) = 1, C_1^s(n) = 1/n$. *For* $s \geqq 2$ *let the coefficients* $C_j^s(n), n \geqq 1, 1 \leqq j \leqq s$ *be determined by*

$$(11)\quad \begin{aligned} C_j^s(n) = b(n, s-1)\{&(j + s - 1)C_{j-1}^{s-1}(n+1) \\ &+ (n + s - 1)C_j^{s-1}(n+1) - (j+1)C_{j+1}^{s-1}(n+1)\}\end{aligned}$$

where $b(n, s) = (n+1)(n+3)\cdots(n+2s-1)/n(n+2)\cdots(n+2s)$ *and where we agree to set* $C_0^{s-1}(n+1) = 1$ *and* $C_s^{s-1}(n+1) = C_{s+1}^{s-1}(n+1) = 0$. *Then*

$$(12)\quad H_n^{2s}(R) = \rho^{n+s-1}(\sqrt{1 + 1/R^2})^{-(s+1)}\left\{1 + \sum_{j=1}^s C_j^s(n)(R\sqrt{1 + 1/R^2})^{-j}\right\}.$$

Next let $S(\xi, 1-\rho) = \{\eta: |\eta| = 1, (\xi, \eta) > \cos(1-\rho)\}, |\xi| = 1, 0 < 1 - \rho < 1$, denote the spherical cap centered at ξ of curvilinear radius $1 - \rho$. Fix the North pole of S at ξ and write $\sum_{n=1}^\infty \rho^n Y_n(\xi) - \Omega(\xi)$ in the Poisson integral form

$$\frac{\Gamma(\lambda+1)}{2\pi^{\lambda+1}}\int_S \frac{(1-\rho^2)(\Omega(\eta) - \Omega(\xi))d\sigma(\eta)}{(1 - 2\rho(\xi, \eta) + \rho^2)^{\lambda+1}},$$

$\lambda = (k-2)/2$. Using the standard argument [10, p. 90 and Th. 3.15] we split the integral over the sets $S(\xi, 1-\rho)$, $S - S(\xi, 1-\rho)$ and use the inequality $(1-\rho^2)(1 - 2\rho(\xi, \eta) + \rho^2)^{-(\lambda+1)} \leqq \text{Const.}\,(1-\rho) \times (1 - (\xi, \eta))^{-(\lambda+1)}$, $1/2 \leqq \rho < 1$, in the second integral to obtain, for $\Omega(\xi)$ in Lip α,

(13) $$\left|\sum_{n=1}^{\infty}\rho^n Y_n(\xi) - \Omega(\xi)\right| = 0(1-\rho)^{\alpha})$$

uniformly in ξ as $\rho \to 1$.

LEMMA 4. *Let $C_j^s(n)$, $1 \leqq j \leqq s; n \geqq 1$ be as in* (11). *Let $0 \leqq \rho < 1$. Then $|\sum_{n=1}^{\infty}\rho^n Y_n(\xi)C_j^s(n)| = 0(1)$ uniformly in ρ, ξ.*

To establish the lemma we note that the recursion formula (11) implies that the coefficients $C_j^s(n)$ are ratios of polynomials in n with integer coefficients and that the denominators are products of unrepeated factors of the form $n+p$, p a nonnegative integer. Also, because $b(n, s) = 0(n^{-1})$ and $C_1^1(n) = 1/n$, an obvious induction argument shows that $C_j^s(n) = 0(n^{-j})$ as $n \to \infty$. It follows that each $C_j^s(n)$ can be written as a finite sum of the form $\sum A_p^q/(n+p)^q$, the A_p^q being independent of n. Hence, in order to establish the lemma it is enough to prove that for q a positive integer $\sum_{n=1}^{\infty}\rho^n Y_n(\xi)/(n+p)^q$ is uniformly bounded in ρ, ξ. This follows at once from induction, integration, Lemma 1, and the fact that by Lemma 3, $\rho^{p-1}\sum_{n=1}^{\infty}\rho^n Y_n(\xi)$ is uniformly bounded for $1/2 < \rho < 1$ and ξ in S.

LEMMA 5. *Let $K(x) = \Omega(x/|x|)/|x|^k$ be a Calderón-Zygmund kernel in* Lip α, $0 < \alpha < 1$ *on the unit sphere S in E_k. Let $\xi = x/|x|$ and suppose $k = 2s$ where s is a positive integer. Then*

$$\left|\sum_{n=1}^{\infty} H_n^{2s}(R)Y_n(\xi) - \Omega(\xi)\right| = 0(R^{-\alpha})$$

uniformly in ξ as $R \to \infty$.

To establish the lemma, let $0 \leqq \rho < 1$ and put
$I_1 = |\sum_{n=1}^{\infty}(H_n^{2s}(R) - \rho^{n+s-1}(\sqrt{1+1/R^2})^{-(s+1)})Y_n(\xi)|$,
$I_2 = |\sum_{n=1}^{\infty}(\rho^{n+s-1}(\sqrt{1+1/R^2})^{-(s+1)} - \rho^n)Y_n(\xi)|$,
$I_3 = |\sum_{n=1}^{\infty}\rho^n Y_n(\xi) - \Omega(\xi)|$.
Recall that $\rho = \sqrt{1+1/R^2} - 1/R$. It is easy to see that $0(R^{-\alpha})$ as $R \to \infty$ is equivalent to $0((1-\rho)^{\alpha})$ as $\rho \to 1$. Thus, $I_3 = 0(R^{-\alpha})$ follows from (13). The same bound for I_2 follows from $|\rho^{s-1}(\sqrt{1+1/R^2})^{-(s+1)} - 1| = 0(R^{-1})$ and (13). By formula (12) of Lemma 3 and by Lemma 4, I_1 is dominated by a finite sum of terms of the form

$$\text{Const. } (R\sqrt{1+1/R^2})^{-j}\left|\sum_{n=1}^{\infty}\rho^n Y_n(\xi)C_j^s(n)\right|, 1 \leqq j \leqq s,$$

all of which are $0(R^{-1})$.

Lemma 5 is needed to prove the main theorem in the even-dimensional cases. For the case E_3 we shall have need of

LEMMA 6. *Let* $K(x) = \Omega(x/|x|)/|x|^3$ *be a Calderón-Zygmund kernel in* $\mathrm{Lip}\,\alpha, 0 < \alpha < 1$, *in* E_3. *Let* $\xi = x/|x|$ *and* $0 < \gamma < \alpha$, *then* $|\sum_{n=1}^{\infty} H_n^3(R) Y_n(\xi) - \Omega(\xi)| = 0(R^{-\gamma})$ *uniformly in* ξ *as* $R \to \infty$.

To prove the lemma we put $A_0 = 0$ and $A_n = \sum_{k=1}^{n} \|Y_k\|_\infty$. We sum $(1-\rho)\sum_{n=1}^{N} \|Y_n\|_\infty \rho^n$ by parts to obtain $(1-\rho)^2 \sum_{n=1}^{N-1} A_n \rho^n + (1-\rho)\rho^N A_N$. By Lemma 1, (i) $\sum_{n=1}^{\infty} \|Y_n\|_\infty n^\gamma/n = C < \infty$. Since $A_N \leqq \sum_{n=1}^{N} \|Y_n\|_\infty (N/n)^{1-\gamma} \leqq N^{1-\gamma} C$, we have

$$(1-\rho)\sum_{n=1}^{\infty} \rho^n \|Y_n\|_\infty \leqq (1-\rho)^2 C \sum_{n=1}^{\infty} n^{1-\gamma}\rho^n \leqq (1-\rho)^2 \,\mathrm{Const.}\, (1-\rho)^{-2+\gamma} ,$$

where we have used the inequality $\sum_{n=1}^{\infty} n^\beta \rho^n \leqq \mathrm{Const.}\, 1/(1-\rho)^{1+\beta}$, $\beta > 0, 0 \leqq \rho < 1$. Next we observe from (8) that $H_n^k(R)$ is decreasing as a function of k, in particular, $H_n^4(R) \leqq H_n^3(R) \leqq H_n^2(R)$. By (10) and (9) we have $\rho^{n+1}/(1+1/R^2)^{3/2} \leqq H_n^3(R) \leqq \rho^{n-1}/(1+1/R^2)^{3/2}$. It follows that

$$\begin{aligned} |H_n^3(R) - \rho^n| &\leqq |\rho^{n+1}/(1+1/R^2)^{3/2} - \rho^n| + |\rho^{n-1}/(1+1/R^2)^{3/2} - \rho^n| \\ &\leqq \rho^n \,\mathrm{Const.}\, R^{-1} \leqq \mathrm{Const.}\, (1-\rho)\rho^n . \end{aligned}$$

Therefore,

$$\begin{aligned} \left|\sum_{n=1}^{\infty} H_n^3(R) Y_n(\xi) - \Omega(\xi)\right| &\leqq \left|\sum_{n=1}^{\infty} (H_n^3(R) - \rho^n) Y_n(\xi)\right| + 0(R^{-\alpha}) \\ &\leqq \mathrm{Const.}\, (1-\rho)\sum_{n=1}^{\infty} \|Y_n\|_\infty \rho^n + 0(R^{-\alpha}) \\ &= 0(R^{-\gamma}) + 0(R^{-\alpha}) = 0(R^{-\gamma}) . \end{aligned}$$

5. Proof of the main theorem. Let $(D_{\mathrm{Sym}}\mu)(x)$ denote the symmetric derivative of μ [4, p. 175, Ex. 1]. Let $|E|$ denote the Lebesgue measure of E. If the total variations of the measures $\mu(E) - (D_{\mathrm{Sym}}\mu)(x)|E|$ are denoted by $\int_E |d\mu(y) - (D_{\mathrm{Sym}}\mu)(x)dy|$ then it follows as in the proof of Lebesgue's Theorem [4, Th. 8.8] that

$$\lim_{t\to 0} |B(x,t)|^{-1} \int_{B(x,t)} |d\mu(y) - (D_{\mathrm{Sym}}\mu)(x)dy| = 0 \tag{14}$$

almost everywhere. Thus, in order to prove Theorem 1, it is sufficient to prove that at each point x for which (14) holds,

$$\lim_{R\to\infty} \left\{ I_R(x) - \int_{E_k - B(x,1/R)} K(x-y)d\mu(y) \right\} = 0 . \tag{15}$$

With no loss in generality we will assume that $x = 0$. Set $x = 0$ in (4) and interchange the order of integration using (1). Next introduce spherical coordinates $r^{k-1}drd\sigma(\xi') = dy$ where $\xi' = y/|y|$ and $r = |y|$ and use Lemma 1, (ii) to obtain

$$I_R(0) = \int_{E_k} d\mu(w)\int_0^\infty r^{k-1}e^{-r/R}dr\sum_{n=1}^\infty (-i)^n\Gamma(n/2)/2^k\pi^{k/2}\Gamma((n+k)/2)$$
$$\cdot\int_S e^{-ir|w|(\xi',\eta)}Y_n(\xi')d\sigma(\xi') ,$$

where $\eta = w/|w|$. By [9, p. 368 (2)] (with $\nu = \lambda = (k-2)/2$) the integral over S is $(2\pi)^{\lambda+1}(-i)^n J_{n+\lambda}(r|w|)(r|w|)^{-\lambda}Y_n(\eta)$. Next, interchange summation and the integral in r. Letting Δ_R denote the term in brackets in (15) we obtain

$$\Delta_R = \int_{E_k}\sum_{n=1}^\infty H_n^k(R|w|)Y_n(\xi)|w|^{-k}d\mu(w) - \int_{E_k-B(0,1/R)}\Omega(\xi)|w|^{-k}d\mu(w)$$

where $\xi = -w/|w|$. Next we write $\Delta_R = J_1 + J_2 + J_3$ where

$J_1 = \int_{B(0,1/R)}\sum_{n=1}^\infty Y_n(\xi)H_n^k(R|w|)|w|^{-k}d\mu(w)$,

$J_2 = \int_{E_k-B(0,T)}\left[\sum_{n=1}^\infty Y_n(\xi)H_n^k(R|w|) - \Omega(\xi)\right]|w|^{-k}d\mu(w)$, and

$J_3 = \int_{B(0,T)-B(0,1/R)}\left[\sum_{n=1}^\infty Y_n(\xi)H_n^k(R|w|) - \Omega(\xi)\right]|w|^{-k}d\mu(w)$,

$\xi = -w/|w|$. If $d\mu(w)$ is replaced by dw in J_1 or J_3 the resulting integral is zero. This follows from the uniform convergence of the series and $\int_S \Omega(\xi)d\sigma(\xi) = 0$. By Lemma 2, (ii),

$$|J_1| \leqq \text{Const.}\ |B(0, 1/R)|^{-1}\int_{B(0,1/R)}|d\mu(w) - (D_{\text{Sym}}\mu)(0)dw| = o(1)$$

as $R \to \infty$. In the case $k = 2s$, Lemma 5 gives $|J_2| \leqq \text{Const.}\ T^{-k}$ where T can be taken arbitrarily large. For J_3 we again use Lemma 5 to obtain

$$|J_3| \leqq \text{Const.}\ R^{-\alpha}\int_{B(0,T)-B(0,1/R}|w|^{-(k+\alpha)}|d\mu(w) - (D_{\text{Sym}}\mu)(0)dw| .$$

The proof of the fact that for fixed T, $J_3 = o(1)$ as $R \to \infty$ is similar to that given in [5, p. 14]. In the case $k = 3$, we replace α in the above integrals by γ, where γ is chosen so that $0 < \gamma < \alpha$, and use Lemma 6.

REFERENCES

1. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., **88** (1952), 85-139.

2. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental*

Functions, vol. 2, New York, 1953.
3. B. Muckenhoupt and E. M. Stein, *Classical Expansions and Their Relation to Conjugate Harmonic Functions*, Trans. Amer. Math. Soc. **118** (1965), 17-92.
4. W. Rudin, *Real and Complex Analysis*, New York, 1966.
5. V. L. Shapiro, *The conjugate Fourier-Stieltjes integral in the plane*, Bull. Amer. Math. Soc. **65** (1959), 12-15.
6. ———, *Topics in Fourier and geometric analysis*, Memoirs Amer. Math. Soc., No. 39, 1961.
7. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937.
8. ———, *The Theory of Functions*, Oxford, 1950.
9. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge, 1922.
10. A. Zygmund, *Trigonometric Series*, vol. I, Cambridge, 1959.

Received April 16, 1969. This research was sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nos. AF-AFOSR 694-66 and 69-1689.

UNIVERSITY OF CALIFORNIA, RIVERSIDE

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 36, No. 1 November, 1971