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SOME NUMBER THEORETIC RESULTS

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(In memory of our good friend Leo Moser)

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The paper first establishes the order of magnitude of maximal sets, S , of residues (mod p) so that the sums of different numbers of elements are distinct.

In the second part irrationalities of Lambert Series of the form $\sum f(n)/a_1 \cdots a_n$ are obtained where $f(n) = d(n)$, $\sigma(n)$ or $\varphi(n)$ and the a_i are integers, $a_i \geq 2$, which satisfy suitable growth conditions.

This note consists of two rather separate topics. In §1 we generalize a topic from combinatorial number theory to get an order of magnitude for the number of elements in a maximal set of residues (mod p) such that sums of different numbers of elements from this set are distinct. We show that the correct order is $cp^{1/3}$ although we are unable to establish the correct value for the constant c .

Section 2 consists of irrationality results on series of the form $\sum f(n)/a_1 a_2 \cdots a_n$ where $f(n)$ is one of the number theoretic functions $d(n)$, $\sigma(n)$ or $\varphi(n)$ and a_n are integers ≥ 2 . For $f(n) = d(n)$ it suffices that the a_n are monotonic while for $\sigma(n)$ and $\varphi(n)$ we needed additional conditions on their rates of growth.

1. Maximal sets in a cyclic group of prime order for which subsets of different orders have different sums. In an earlier paper [4] one of us has given a partial answer to the question:

What is the maximal number $n = f(x)$ of integers a_1, \dots, a_n so that $0 < a_1 < a_2 < \cdots < a_n \leq x$ and so that

$$a_{i_1} + \cdots + a_{i_s} = a_{j_1} + \cdots + a_{j_t} \text{ for some } 1 \leq i_1 < \cdots < i_s \leq n \\ 1 \leq j_1 < \cdots < j_t \leq n$$

implies $s = t$? it is conjectured that the maximal set is obtained (loosely speaking) by taking the top $2\sqrt{x}$ integers of the interval $(1, x)$. We were indeed able to prove that $f(x) < c\sqrt{x}$ for suitable c (for example $4/\sqrt{3}$) by using the fact that a set of n positive integers has a minimal set of distinct sums of t -tuples ($1 \leq t \leq n$) if it is in arithmetic progression.

It is natural to pose the analogous question for elements of cyclic groups of prime order, as was done at the Number Theory Symposium in Stony Brook [5]. Here again we may conjecture that a maximal set of residues (mod p) is attained by taking a set of consecutive residues, this time not at the upper end but near $p^{2/3}$.

Conjecture 1.1. Let $f(p)$ be the maximal cardinality of a set of residues mod p so that sums of different numbers of residues in this set are different, then $f(p) = (4p)^{1/3} + o(p^{1/3})$ where the maximum is attained, for example, by taking consecutive residues in an interval of length $(4p)^{1/3} + o(p^{1/3})$ containing the residue $[(p/2)^{1/3}]$.

It is easy to see that we can indeed get a set of about $(4p)^{1/3}$ residues by taking the residues in the interval $[(p/2)^{2/3} - (4p)^{1/3}, [(p/2)^{2/3}]]$. Here sums of distinct numbers of elements are distinct integers, and since all sums are $< p$ it follows that they are distinct residues.

The observation which led to the upper bound in [4] is much less obvious (mod p):

Conjecture 1.2. A set $A = \{a_1, a_2, \dots, a_k\}$ of residues (mod p) has a minimal number of distinct sums of subsets of t elements if A is in arithmetic progression.

Conjecture 1.2 would give us a simple upper bound for $f(p)$:

COROLLARY 1.3. *If Conjecture 1.2 holds then*

$$f(p) < (6p)^{1/3} + o(p^{1/3}).$$

Proof. The sums of t elements from the set of residues

$$\{1, 2, \dots, k-1, k\}$$

fill the interval $((\binom{t+1}{2}), tk - \binom{t}{2})$ that is to say there are $tk - t^2 + O(t)$ such sums. Since for different t we get different sums we must have

$$p \geq \sum_{t=1}^k (tk - t^2 + O(t)) = \frac{k^3}{6} + O(k^2)$$

$$\text{and hence } k < (6p)^{1/3} + o(p^{1/3}).$$

Using methods employed by Erdős and Heilbronn [2] we can show that $f(p) = O(p^{1/3})$. We use the following lemma from [2].

LEMMA 1.4. *Let $1 < m \leq l < p/2$ and let $B = \{b_1, \dots, b_l\}$, $A = \{a_1, \dots, a_m\}$ be sets of residues (mod p). Then there exists an $a_i \in A$ such that the number of solutions of $a_i = b_j - b_k$; $b_j, b_k \in B$ is less than $l - m/6$.*

We now can get a lower bound for the number of distinct sums of t elements from a set of residues.

LEMMA 1.5. *Let $A = \{a_1, \dots, a_k\}$ be a set of residues (mod p)*

and let $A_t = \{a_{i_1} + \dots + a_{i_t} \mid 1 \leq i_1 < \dots < i_t \leq k\}$ then for $1 \leq t \leq k/4$ we have

$$(1.6) \quad |A_t| \geq l + \frac{(t-1)m}{6} - \frac{t(t-1)}{6}$$

where

$$l = \left\lfloor \frac{k+1}{2} \right\rfloor, m = \left\lfloor \frac{k}{2} \right\rfloor.$$

Proof. We divide the set A into two disjoint sets

$$A = \{a_1, a_2, \dots, a_l\}, B = \{b_1, b_2, \dots, b_m\}$$

and prove the inequality (1.6) for the subset of A_t consisting of the sums

$$A_t^* = \{a_i + b_{2-\varepsilon_1} + b_{4-\varepsilon_2} + \dots + b_{2t-2-\varepsilon_{t-1}} \mid \varepsilon_j = 0 \text{ or } 1\},$$

where the b_i are a suitable ordering of the elements of B .

The inequality holds for $t = 1$ since

$$A_t^* = \{a_i\} = A \text{ and } |A| = l.$$

Now assume that (1.6) holds for A_t^* with $t \leq (m/2) - 1$. Then the set $A_t^* + b_{2t} \subset A_{t+1}^*$ and according to Lemma 1.3 there exists a $b_j \in \{b_{2t+1}, b_{2t+1}, \dots, b_m\}$, say $b_j = b_{2t+1}$ so that the equation

$$b_{2t+1} - b_{2t} = a_i^* - a_j^*, \quad a_i^*, a_j^* \in A_t^*$$

has no more than $|A_t^*| - \frac{1}{6}(m - 2t)$ solutions. Hence the set

$$((b_{2t+1} - b_{2t}) + (A_t^* + b_{2t})) \cap (A_t^* + b_{2t})$$

contains no more than $|A_t^*| - \frac{1}{6}(m - 2t)$ elements and

$$\begin{aligned} |A_{t+1}^*| &= |(A_t^* + b_{t+1}) \cup (A_t^* + b_t)| \\ &\geq |A_t^*| + \frac{1}{6}(m - 2t) \\ &\geq l + \frac{(t-1)m}{6} - \frac{t(t-1)}{6} + \frac{1}{6}m - \frac{t}{3} \\ &= l + \frac{tm}{6} - \frac{(t+1)t}{6}. \end{aligned}$$

This completes the proof.

THEOREM 1.7. *The maximal number $f(p)$ of a set A of residues (mod p) so that sums of different numbers of distinct elements of A are distinct satisfies*

$$(1.8) \quad (4p)^{1/3} + o(p^{1/3}) < f(p) < (288p)^{1/3} + o(p^{1/3}).$$

Proof. According to Lemma 1.5 there are at least

$$k/2 + k(t-1)/12 - t^2/6 + O(t)$$

distinct sums of t elements (and hence, by symmetry, sums of $k-t$ elements) for $t < [k/4]$ out of a set A with k elements. Thus if A has the desired property we must have

$$\begin{aligned} p &\geq 2 \sum_{t=1}^{k/4} (k/2 + k(t-1)/12 - t^2/6) + O(k^2) \\ &= 2k^3 \left(\frac{1}{384} - \frac{1}{3} \frac{1}{384} \right) + O(k^2) = k^3/288 + O(k^2). \end{aligned}$$

Thus

$$f(p) < (288 p)^{1/3} + o(p^{1/3}).$$

The lower bound for $f(p)$ was established above.

2. On some irrational series. One of us [1] proved that the series $\sum_{n=1}^{\infty} d(n)t^{-n}$ is irrational for every integer t , $|t| > 1$. In this section we generalize this result to series of the form

$$(2.1) \quad \xi = \sum_{n=1}^{\infty} \frac{d(n)}{a_1 a_2 \cdots a_n}$$

where the a_n are positive integers with $2 \leq a_1 \leq a_2 \leq \cdots$. It is clear that we need some restriction, such as monotonicity, on the a_n since the choice $a_n = d(n) + 1$ would lead to $\xi = 1$.

We divide the proof into two cases depending on the rate of increase of a_n . The first case is very similar to [1].

LEMMA 2.2. *The series (2.1) is irrational if there exists a $\delta > 0$ so that the inequality $a_n < (\log n)^{1-\delta}$ holds for infinitely many values of n .*

Proof. Let n be a large integer so that $a_n < (\log n)^{1-\delta}$. Then by the monotonicity of a_i there exists an interval I of length $n/\log n$ in $(1, n)$ so that for all integers $i \in I$ we have $a_i = t$ where t is a fixed integer, $t \leq (\log n)^{\delta/10}$.

Now put $k = [(\log n)^{\delta/10}]$ and let p_1, p_2, \dots be the consecutive primes greater than $(\log n)^2$. Let

$$A = \left(\prod_{1 \leq i \leq k(k+1)/2} p_i \right)^t$$

then

$$(2.3) \quad \begin{aligned} A &< (2(\log n)^2)^{t \cdot k(k+1)/2} < e^{(\log n)^{1-\delta} (\log n)^{\delta/4}} \\ &< e^{(\log n)^{1-\delta/2}}. \end{aligned}$$

By the Chinese remainder theorem the congruences

$$(2.4) \quad \begin{aligned} x &\equiv p_1^{t-1} (\text{mod } p_1^t) \\ x + 1 &\equiv (p_2 p_3)^{t-1} (\text{mod } (p_2 p_3)^t) \\ &\vdots \\ x + k - 1 &\equiv (p_u p_{u+1} \cdots p_{u+k-1})^{t-1} (\text{mod } (p_u p_{u+1} \cdots p_{u+k-1})^t) \end{aligned}$$

where $u = 1 + k(k-1)/2$, have solutions determined (mod A). The interval I contains at least $[n/(A \log n)]$ solutions of (2.4).

Now assume that $\xi = a/b$ and choose $x \in I$ to be a solution of (2.4) so that $(x, x+k) \subset I$. Then

$$(2.5) \quad \begin{aligned} b a_1 \cdots a_{x-1} \xi &= \text{integer} + b \sum_{l=0}^{k-1} \frac{d(x+l)}{t^{l+1}} \\ &+ b \sum_{s=0}^{\infty} \frac{d(x+k+s)}{t^k a_{x+k} \cdots a_{x+k+s}}. \end{aligned}$$

But (2.4) implies that $d(x+l) \equiv 0 \pmod{t^{l+1}}$ for $l = 0, 1, \dots, k-1$. Thus (2.5) implies that

$$(2.6) \quad b a_1 \cdots a_{x-1} \xi = \text{integer} + \frac{b}{t^k} \sum_{s=0}^{\infty} \frac{d(x+k+s)}{a_{x+k} \cdots a_{x+k+s}}.$$

We now wish to show that for suitable choice of x the sum on the right side of (2.6) is less than 1 and hence $b\xi$ cannot be an integer. We first consider the sum

$$(2.7) \quad \begin{aligned} &\frac{b}{t^k} \sum_{s > 10 \log n} \frac{d(x+k+s)}{a_{x+k} \cdots a_{x+k+s}} \\ &< \frac{b}{t^k} \sum_{s > 10 \log n} \frac{x+k+s}{t^{s+1}} < b(x+k) \sum_{s > 10 \log n} \frac{s}{t^s} \\ &< \frac{2bn}{n^2} < \frac{1}{2} \text{ for large } n. \end{aligned}$$

Next we wish to show that it is possible to choose x so that

$$(2.8) \quad d(x+k+s) < 2^{k/4} \text{ for } 0 \leq s < 10 \log n.$$

We first observe that

$$(2.9) \quad (x+k+s, A) = 1 \text{ for all } 0 \leq s < 10 \log n$$

since otherwise

$$(2.10) \quad x+k+s \equiv 0 \pmod{p_j} \text{ for some } 1 \leq j \leq k(k+1)/2$$

and

$$(2.11) \quad x+i \equiv 0 \pmod{p_j} \text{ for some } 0 \leq i < k.$$

But

$$0 < k + s - i < 11 \log n < (\log n)^2 < p_j$$

so that (2.10) and (2.11) are incompatible.

Let $x = x_0, x_0 + A, \dots, x_0 + zA$ be the solutions of (2.4) for which $(x, x + k) \subset I$. From (2.9) we get

$$(2.12) \quad \sum_{y=0}^z d(x_0 + k + s + yA) < 2 \sum_{l=1}^{\sqrt{n}} \left(\frac{n}{Al} + 1 \right) < c \frac{n \log n}{A}.$$

Thus the number of y 's for which $d(x_0 + k + s + yA) > 2^{k/4}$ is less than $c n \log n / (A \cdot 2^{k/4})$, and the number of y 's so that for some $0 \leq s < 10 \log n$ we have $d(x_0 + k + s + yA) > 2^{k/4}$ is less than

$$10c n \log^2 n / (A \cdot 2^{k/4}) < 1/2 n / (A \log n) < z.$$

It is therefore possible to choose $x = x_0 + yA \in I$ so that (2.8) holds. For such a choice we get

$$(2.13) \quad \frac{b}{t^k} \sum_{s=0}^{10 \log n} \frac{d(x + k + s)}{a_{x+k} \cdots a_{x+k+s}} < \frac{b}{t^k} 2^{k/4} \sum_{s=0}^{\infty} \frac{1}{t^s} < b \cdot 2^{-3k/4} < \frac{1}{2}.$$

Combining (2.7) and (2.13) we see that ξ is irrational.

LEMMA 2.14. *If there exists a positive constant c so that $|a_n| > c(\log n)^{3/4}$ for all n then the series (2.1) is irrational.*

Note that in this lemma we need not assume the monotonicity of a_n (or even that they are positive, however for simplicity we give the proof for positive a_n only).

Proof. We use two results. The Dirichlet divisor theorem

$$(2.15) \quad \sum_{n=1}^N d(n) \sim N \log N$$

and the average order of $d(n)$, [3]

$$(2.16) \quad d(n) < (\log n)^{\log 2 + \epsilon} \text{ for almost all } n.$$

From (2.15) we get the following.

LEMMA 2.17. *Given constants $b, c > 0$, then for almost all integers x*

$$(2.18) \quad d(x + y) < b^{-1}(2c)^{-y}(\log x)^{3y/4}; y = 3, 4, \dots$$

Proof. If we choose x large enough so that $\log x > (2bce)^{4/3}$ then the right side of (2.18) is greater than e^y which exceeds $x + y$, and hence $d(x + y)$, whenever $y > 2 \log x$. Thus, if (2.18) fails to hold for sufficiently large x then it must fail to hold for some y with $3 \leq y \leq 2 \log x$.

Now if there are $c_1 N$ integers x below N so that (2.18) fails to hold then we have more than $c_2 N$ integers x with $\sqrt[3]{N} \leq x \leq N - 2 \log N$ and

$$(2.19) \quad \begin{aligned} d(x + y) &> b^{-1}(2c)^{-y}(\log x)^{3y/4} \geq b^{-1}(2c)^{-y}(\tfrac{1}{2} \log N)^{3y/4} \\ &\geq b^{-1}(4c)^{-3}(\log N)^{9/4} = c_3(\log N)^{9/4}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=1}^N d(n) &\geq c_2 N \cdot \frac{1}{2 \log N} c_3 (\log N)^{9/4} \\ &= c_4 N (\log N)^{5/4} \end{aligned}$$

which contradicts (2.15) for large N .

Combining Lemma 2.17 with (2.16) we find that there exists an infinite set S of integers x so that

$$(2.21) \quad d(x + 1) < \frac{b^{-1}c}{2} (\log x)^{3/4}, d(x + 2) < \frac{b^{-1}c^2}{4} (\log x)^{3/4}$$

and (2.18) both hold.

Now assume that $\xi = a/b$ is a rational value of (2.1) and choose $n \in S$. Then

$$(2.22) \quad a_1 \cdots a_n b \xi = \text{integer} + b \sum_{y=1}^{\infty} \frac{d(n + y)}{a_{n+1} \cdots a_{n+y}}$$

where

$$0 < \sum_{y=1}^{\infty} \frac{d(n + y)}{a_{n+1} \cdots a_{n+y}} < \sum_{y=1}^{\infty} \frac{(2c)^{-y}(\log n)^{3y/4}}{(c(\log n)^{3/4})^y} = 1,$$

in contradiction to the fact that the left side of (2.22) is an integer.

Summing up we have

THEOREM 2.23. *The series (2.1) is irrational whenever*

$$2 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

With considerable additional effort one can weaken the monotonicity condition on the a_n to $a_m/a_n \geq c > 0$ for all $m > n$.

We have not been able to prove the following

Conjecture 2.24. The series (2.1) is irrational whenever $a_n \rightarrow \infty$.
If we consider series of the form

$$(2.25) \quad \sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 \cdots a_n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{a_1 \cdots a_n}$$

then we cannot make conjectures analogous to 2.24 since the choice $a_n = \varphi(n) + 1$ or $\sigma(n) + 1$ would make these series converge to 1. It is reasonable to conjecture that the series (2.25) must be irrational if the a_n increase monotonically, however we can prove this only under more restrictive conditions.

THEOREM 2.26. *If $\{a_n\}$ is a monotonic sequence of integers with $a_n \geq n^{11/12}$ for all large n then the series in (2.25) are irrational.*

For the proof we need the following simple lemmas.

LEMMA 2.27. *Let $\{a_n\}$ be a sequence of positive integers with $a_n \geq 2$ and $\{b_n\}$ a sequence of positive integers so that $b_{n+1} = o(a_n a_{n+1})$. If*

$$(2.28) \quad \xi = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$$

is rational then $a_n = O(b_n)$.

Proof. Assume $\xi = a/b$ and choose N so that for all $n > N$ we have $bb_n < a_{n-1}a_n/4$. If there existed an $n > N$ so that $a_n > 2bb_n$ then we would have

$$ba_1 \cdots a_{n-1} \xi = aa_1 \cdots a_{n-1} = \text{integer} + \sum_{k=0}^{\infty} \frac{bb_{n+k}}{a_n \cdots a_{n+k}}$$

but

$$\begin{aligned} 0 &< \sum_{k=0}^{\infty} \frac{bb_{n+k}}{a_n \cdots a_{n+k}} = \frac{bb_n}{a_n} + \sum_{k=1}^{\infty} \frac{bb_{n+k}}{a_{n+k-1} \cdots a_{n+k}} \cdot \frac{1}{a_n \cdots a_{n+k-2}} \\ &< \frac{1}{2} + \frac{1}{4} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^l = 1, \end{aligned}$$

a contradiction. Thus $a_n \leq 2bb_n$ for all large n .

LEMMA 2.29. *If the series (2.28) is rational, say $\xi = a/b$, and $b_{n+1} = o(a_n a_{n+1})$, then there exists a sequence of positive integers $\{c_n\}$ so that for all large n we have*

$$(2.30) \quad bb_n = c_n a_n - c_{n+1}, \quad 0 < c_{n+1} < a_n, \text{ and } c_{n+1} = o(a_n).$$

Conversely, if these conditions hold then the series (2.28) is rational.

Proof. Choose N so that for all $n > N$ we have $bb_n < a_n a_{n+1}/4$.
Now for $n \geq N$ choose c_n, c_{n+1} so that

$$bb_n = c_n a_n - c_{n+1}, \quad c_n > 0$$

$$0 < c_{n+1} < a_n$$

and c'_{n-1}, c'_{n+2}

$$bb_{n+1} = c'_{n+1} a_{n+1} - c'_{n+2}, \quad c'_{n+1} > 0$$

$$0 < c'_{n+2} < a_{n+1}.$$

Then

$$\begin{aligned} ba_1 \cdots a_{n-1} \xi &= aa_1 \cdots a_{n-1} \\ &= \text{integer} + \frac{bb_n}{a_n} + \frac{bb_{n+1}}{a_n a_{n+1}} + \sum_{k=2}^{\infty} \frac{bb_{n+k}}{a_n \cdots a_{n+k}} \\ &= \text{integer} - \frac{c_{n+1}}{a_n} + \frac{c'_{n+1}}{a_n} - \frac{c'_{n+2}}{a_n a_{n+1}} \\ &\quad + \frac{1}{a_n} \sum_{k=2}^{\infty} \frac{bb_{n+k}}{a_{n+1} \cdots a_{n+k}} \\ &= \text{integer} - \frac{c_{n+1}}{a_n} + \frac{c'_{n+1}}{a_n} - \frac{c'_{n+2}}{a_n a_{n+1}} + \frac{\theta}{a_n}, \end{aligned} \quad (2.31)$$

$$0 < \theta < \frac{1}{2}.$$

Thus

$$\frac{1}{a_n} \left(-c_{n+1} + c'_{n+1} - \frac{c'_{n+2}}{a_{n+1}} + \theta \right) = \text{integer}$$

and since $0 < c_{n+1} < a_n$, $0 < c'_{n+1} \leq [a_n/4] + 1$, $0 < c'_{n+2}/a_{n+1} < 1$,

$0 < \theta < \frac{1}{2}$, this is possible only if $c_{n+1} = c'_{n+1}$.

Now choose N so large that $bb_{n+1} < \varepsilon a_n a_{n+1}$ for all $n > N$, then from (2.31) we have

$$\begin{aligned} \text{integer} &= -\frac{c_{n+1}}{a_n} + \sum_{k=1}^{\infty} \frac{bb_{n+k}}{a_n a_{n+1} \cdots a_{n+k}} < -\frac{c_{n+1}}{a_n} + \varepsilon \sum_{k=1}^{\infty} \frac{1}{a_n \cdots a_{n+k-2}} \\ &\leq -\frac{c_{n+1}}{a_n} + 2\varepsilon. \end{aligned}$$

Thus $c_{n+1} < 2\varepsilon a_n$ for all $n > N$.

If condition (2.30) holds for all $n \geq N$ then

$$\begin{aligned} \sum_{n=N}^{\infty} \frac{bb_n}{a_1 \cdots a_n} &= \sum_{n=N}^{\infty} \frac{c_n a_n - c_{n+1}}{a_1 \cdots a_n} \\ &= \frac{c_N}{a_1 \cdots a_{N-1}} - \sum_{n=N}^{\infty} c_{n+1} \left(\frac{1}{a_1 \cdots a_n} - \frac{a_{n+1}}{a_1 \cdots a_{n+1}} \right) \\ &= \frac{c_N}{a_1 \cdots a_{N-1}} \end{aligned}$$

is clearly rational.

Finally we need a fact from sieve theory. We are grateful to R. Miehch for supplying the correct constants.

LEMMA 2.32. *Given an integer a and $\varepsilon > 0$ then for large y the number of integers m satisfying*

$$m \not\equiv 0, m \not\equiv a \pmod{p}$$

for all primes p , with $2 < p < y^{1/5}$ exceeds $y^{1-\varepsilon}$.

Proof of Theorem 2.26. Let $f(n)$ stand for either $\sigma(n)$ or $\varphi(n)$ and assume that

$$\sum_{n=1}^{\infty} \frac{f(n)}{a_1 \cdots a_n} = \frac{a}{b}.$$

Since $a_n > n^{11/12}$ for large n the hypothesis of Lemma 2.29 is satisfied and we get

$$(2.33) \quad bf(n) = c_n a_n - c_{n+1} \text{ for large } n.$$

Since $f(n) = o(n^{1+\varepsilon})$ for all $\varepsilon > 0$ we get

$$(2.34) \quad c_n < n^{1/12+\varepsilon} \text{ for large } n.$$

From Lemma 2.28 we get

$$(2.35) \quad a_n = O(f(n)) = O(n^{1+\varepsilon})$$

and hence the number of integers $n \leq x$ for which

$$\frac{a_{n+1}}{a_n} > 1 + x^{-1/2}$$

is $O(x^{3/4})$, since otherwise we would have

$$a_x = \prod_{n < x} \frac{a_{n+1}}{a_n} > (1 + x^{-1/2})^{x^{3/4}} > x^2$$

for large x , in contradiction to (2.35). From now on we restrict our attention to integers n for which

$$(2.36) \quad \frac{a_{n+1}}{a_n} < 1 + n^{-1/2}.$$

For such integers we get from (2.33) and (2.35) that

$$\begin{aligned}
 \frac{f(n+1)}{f(n)} &= \frac{c_{n+1} a_{n+1}}{c_n a_n} \left(1 - \frac{c_{n+2}}{c_{n+1} a_{n+1}}\right) / \left(1 - \frac{c_{n+1}}{c_n a_n}\right) \\
 (2.37) \quad &= \frac{c_{n+1}}{c_n} (1 + O(n^{-1/2})) (1 + O(n^{-3/4+\varepsilon})) \\
 &= \frac{c_{n+1}}{c_n} + O(n^{-1/2+\varepsilon})
 \end{aligned}$$

Now consider a prime q , $\frac{1}{2} x^{1/11} \leq q \leq x^{1/11}$, then according to Lemma 2.32 there exist more than $y^{1-\varepsilon}$ integers $m \leq y = x^{10/11}$ so that

$$(2.38) \quad m \not\equiv 0, m \not\equiv -2q \pmod{p}$$

for all primes p with $2 < p < y^{1/5}$. We may even assume that m is odd. The number of integers $n = 2qm$ where m satisfies (2.38) exceeds $x^{10/11-\varepsilon} > x^{3/4}$ and hence we can pick such an n that satisfies (2.37) with $x/2 \leq n \leq x$.

Now

$$f(n) = f(2q)f(m)$$

where

$$\frac{f(2q)}{2q} = \begin{cases} \frac{3(q+1)}{2q} & \text{if } f = \sigma \\ \frac{q-1}{2q} & \text{if } f = \varphi \end{cases}$$

in either case

$$(2.39) \quad f(2q) = A/q, \quad A \text{ an integer not divisible by } q.$$

Since m has at most 5 prime factors all exceeding $y^{1/5}$ we have

$$\begin{aligned}
 (1 - y^{-1/5})^5 &< \frac{f(m)}{m} < (1 + y^{-1/5})^5 \\
 (2.40) \quad f(m) &= m(1 + O(y^{-1/5})) = m(1 + O(x^{-2/11})).
 \end{aligned}$$

By the same reasoning we get

$$(2.41) \quad f(n+1) = n(1 + O(x^{-2/11})).$$

Substituting (2.39), (2.40) and (2.41) in (2.37) we get

$$(2.42) \quad \frac{f(n+1)}{f(n)} = \frac{A}{q} (1 + O(x^{-2/11})) = \frac{c_{n+1}}{c_n} + O(x^{-1/2+\varepsilon}).$$

But since $q > x^{1/12}$ and $c_n < x^{1/12}$ we get

$$(2.43) \quad \frac{1}{qc_n} \leq \left| \frac{A}{q} - \frac{c_{n+1}}{c_n} \right| < x^{-2/11+\varepsilon}.$$

Since $qc_n < x^{1/11+1/12} < x^{2/11-\varepsilon}$ this leads to a contradiction.

We could get similar irrationality results if the functions $\sigma(n)$ or $\varphi(n)$ are replaced by $\sigma_k(n)$ ($k \geq 1$) or products of powers of $\sigma_k(n)$ and $\varphi(n)$. In each case we would need the assumption that the a_n are monotonic, increasing faster than a certain fractional power of the numerators.

From Lemma 2.29 it is clear that there is a set of power 2^{\aleph_0} of series (2.25) which are rational even if we restrict the integers c_n to the values 1 or 2 since for $c_n = 1$ we can choose $a_n = \sigma(n) - 1$ or $\sigma(n) - 2$ to get $c_{n+1} = 1$ or 2 respectively and for $c_n = 2$ we choose $a_n = [(\sigma(n)-1)/2]$ to get $c_{n+1} = 1$ if $\sigma(n)$ is odd and $c_{n+1} = 2$ if $\sigma(n)$ is even. For the series with numerators $\varphi(n)$ we would have to use $c_n = 1, 2$ or 3 since all $\varphi(n)$ are even for $n > 2$.

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