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**COMPLETIONS OF DEDEKIND PRIME RINGS AS SECOND
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The purpose of this paper is to show that if M is a maximal two-sided ideal of a Dedekind prime ring R and P is any maximal right ideal containing M , then the M -adic completion \bar{R} of R can be realized as the second endomorphism ring of $E = E(R/P)$, the R -injective hull of R/P ; that is, as $\text{end}({}_K E)$ where $K = \text{end}(E_R)$. The ring K turns out to be a complete, local, principal ideal domain.

This paper was motivated by a result of Matlis [6] which says that if P is a prime ideal of a commutative Noetherian ring R , then the P -adic completion of the localization of R at P can be realized as the ring of endomorphisms of $E = E(R/P)$, the R -injective hull of R/P .

Since \bar{R} is a full matrix ring over a complete local domain L [4], we are able to approach the problem by considering first the case that R is a complete local domain, then by means of the Morita theorems we pass to the case $\bar{R} = R$, and finally pass to the general case.

1. Introduction. A prime ring R is called a Dedekind prime ring if it is Noetherian, hereditary, and a maximal order in its classical quotient ring Q (see [3]). A ring R is called local if the nonunits of R form an ideal.

If R is a Dedekind prime ring with a nonzero prime ideal M , then M is a maximal two-sided ideal and $\cap M^n = 0$ (see Robson [7]). Let $\bar{R} = \bar{R}_M$ be the completion of R at M in the sense of Goldie [3]. In this situation combining results of Goldie ([3], Theorem 4.5) and Gwynne and Robson ([4], Theorem 2.3) yields the following theorem.

THEOREM 1.1. *Let R be a Dedekind prime ring with a maximal ideal M . Then (i) \bar{R} has a unique maximal two-sided ideal \bar{M} , \bar{M} is the Jacobson radical of \bar{R} , and $R \cap \bar{M}^p = M^p$.*

(ii) *\bar{R} is a full $k \times k$ matrix ring over a domain L which has a unique maximal ideal N , and $L/N = F$ where F is a division ring. Also $R/M^p \simeq \bar{R}/\bar{M}^p$ (each coset of \bar{M}^p has a representative in R).*

(iii) *\bar{R} is a prime principal ideal ring and L is a complete, local, principal ideal domain. The only one-sided ideals of L are the powers of N .*

For the rest of this section let R , M , \bar{R} , \bar{M} , L , and N be as in Theorem 1.1. Let x be the generator of N ; then $N = xL = Lx$ and $N^k = x^kL = Lx^k$.

2. **The Ring L .** This section will be concerned with the construction of the L -injective hull of $(L/N)_L$ and with showing that Theorem 4.4 holds for L .

LEMMA 2.1. *L/N^k can be embedded in L/N^{k+1} as a right L -module via the map $h_k: L/N^k \rightarrow L/N^{k+1}$ defined by $h_k([u + N^k]) = [xu + N^{k+1}]$.*

Proof. h_k is clearly additive and right L -linear. Suppose $h_k([u + N^k]) = [0 + N^{k+1}]$. From the definition of h_k it follows that $xu \in N^{k+1}$ so that $xu = x^{k+1}u$, for some u' in L and $u = x^k u' \in N^k$. Hence $[u + N^k] = [0 + N^k]$ and h_k is a monomorphism. A similar argument shows that h_k is well-defined.

The maps $\{h_k\}$ and the right L -modules $\{(L/N^k)_L\}$ give rise to a directed system. Let E_L be the direct limit of this system. Then E_L can be considered as an ascending union of a family of submodules, $\{(S_j)_L\}$, which is totally ordered by inclusion and where each $(S_j)_L$ is isomorphic to $(L/N^j)_L$.

LEMMA 2.2. *Consider $(L/N^{p+t+1})_L$. Take $a \in N^p/N^{p+t+1}$ and $d \in N^p \setminus N^{p+1}$. The equation $yd = a$ has a solution in $(L/N^{p+t+1})_L$.*

Proof. $a \in N^p/N^{p+t+1}$ so that $a = [x^p v + N^{p+t+1}]$. $d \in N^p \setminus N^{p+1}$ so that $d = x^p u$ where u is a unit in L . In L , $x^p v u^{-1} = w x^p$ since

$$\begin{aligned} N^p &= x^p L = L x^p. \text{ Let } y = [w + N^{p+t+1}]. \text{ } yd \\ &= [w + N^{p+t+1}]d = [wd + N^{p+t+1}] = [w x^p u + N^{p+t+1}] \\ &= [x^p v u^{-1} u + N^{p+t+1}] = [x^p v N^{p+t+1}] = a. \end{aligned}$$

PROPOSITION 2.3. *E_L is isomorphic to the L -injective hull of the simple right L -module $(L/N)_L$.*

Proof. E_L contains a copy of $(L/N)_L$, namely S_1 . Thus it is enough to show that E is an essential injective extension of S_1 . S_1 is essential in E for if $a \in E$, $a \in S_k$ for some integer k . Let t be the first such integer: then $a \in S_t \setminus S_{t-1}$, a is a generator for S_t , and $aL = S_t$. Thus $aL \cap S_1 = S_1$ and S_1 is essential. Since L is a principal ideal domain, it is a hereditary two-sided order in its quotient division ring. In order to prove E_L is injective it is sufficient by a result of Levy ([5], Theorem 3.4) to show that it is L -divisible. Take $a \in E$ and $0 \neq d \in L$. $a \in S_t$ for some t and $d \in N^p \setminus N^{p+1}$ for some p . $yd = a$ has

a solution in S_{p+t+1} , and hence in E , by Lemma 2.2. E is thus an essential injective extension of S_1 and hence is its injective hull.

Let $K = \text{end}_L(E)$ and let K act on E by left multiplication; E then becomes a left K -module. Let $H = \text{end}_K(E)$; in similar manner E then becomes a right H -module. $Ed = E$ (since E is L -divisible) for all nonzero d in L ; thus E is a faithful right L -module. Hence L may be considered as a unital subring of H .

LEMMA 2.4. *The S_k 's are the only proper L -submodules of E_L .*

Proof. Suppose M_L is a submodule of E with generating set $\{m_i\}$. Since $E = \cup S_k$, each m_i is in some S_k . Let k_i be the first k for which $m_i \in S_k$. Then $m_i \in S_{k_i} \setminus S_{k_i-1}$ and $m_i L = S_{k_i}$. $M = \sum m_i L = \sum S_{k_i}$ so that if $\{k_i\}$ is bounded, $M = S_{k_t}$ where $k_t = \max \{k_i\}$, and if $\{k_i\}$ is not bounded, then $M = E_L$.

LEMMA 2.5. *If $a \in S_n$ and if $b \in S_{n-1}$, then there is a $q \in K$ such that $q(b) = a$.*

Proof. Assume that t is the first integer for which $b \in S_{n+t}$. Then $\text{ann}_L(b) = N^{n+t}$ which is contained in N^n which in turn is contained in $\text{ann}_L(a)$. Thus the map $\bar{q}: bL \rightarrow aL$ defined by $\bar{q}(bd) = ad$ is well defined. E_L is L -injective so that \bar{q} can be extended to an endomorphism q of E . $q \in K$.

PROPOSITION 2.6. *Each S_n is a cyclic left K -submodule of ${}_K E$, the composition length of ${}_K(S_n)$ is n , and the S_n 's are the only proper K -submodules of E .*

Proof. If $q \in K$, $q(S_n)$ is an L -submodule of E of composition length less than or equal to n and hence must be contained in S_n by Lemma 2.4; hence each S_n is a left K -submodule. Each ${}_K(S_n)$ is cyclic via Lemma 2.5; in fact, any L generator of S_n will be a K generator of S_n . This implies that ${}_K(S_1)$ is simple and inductively that the composition length of ${}_K(S_n)$ is n . The proof of Lemma 2.4 shows that these are the only K -submodules of E .

LEMMA 2.7. *Let H_i be the annihilator of S_i in H . Then H_i is a two-sided ideal of H , H_{i+1} is properly contained in H_i , and $\cap H_i = 0$.*

Proof. H_i is clearly a right ideal of H . If $h \in H$, then $(S_i)h$ is a K -submodule of E of composition length less than or equal to i . By Proposition 2.6 it must be that $(S_i)h \subset S_i$ so that each S_i is H -invariant. As a result H_i is a left ideal and hence an ideal. The inclusions are

proper, for $H_i \cap L = N^i$ and $N_i \neq N^{i+1}$. Since $E = \bigcup S_i$, anything in $\bigcap H_i$ would annihilate all of E and hence be zero.

PROPOSITION 2.8. $H = L$. That is, L is the second endomorphism ring of E_L .

Proof. Take $f \in H$. By Proposition 2.6 there is a nonzero $y \in S_1$ such that $S_1 = Ky = yL$. Hence there is a $p_1 \in L$ such that $yf = yp_1$. Also, if $z \in S_1$, $z = ky$ for some $k \in K$ and

$$z(f - p_1) = (ky)(f - p_1) = k0 = 0. \text{ Hence } f - p_1 \in \text{ann}_H(S_1) = H_1.$$

Inductively suppose that there is a $p_i \in L$ such that $f - p_i \in H_i$. Now take $y \in S_{i+1} \setminus S_i$. $y(f - p_i) \in S_{i+1}$ so that there is a $d \in L$ such that $y(f - p_i) = yd$. If $z \in S_{i+1}$, $z = ky$ for some $k \in K$. Then $z(f - p_i) = (ky)(f - p_i) = k(y(f - p_i)) = k(yd) = (ky)d = zd$ and hence $f - p_i - d$ is in H_{i+1} . Let $p_{i+1} = p_i + d$; then $f - p_{i+1} \in H_{i+1}$.

The sequence $\{p_i\}$ is Cauchy in L , for $p_n - p_m = (p_n - f) + (f - p_m)$ an element of $H_n + H_m$; but $H_n + H_m = H_n$ if $n \leq m$. Thus $p_n = p_m$ is in $H_n \cap L = N^n$. L is complete; therefore $\{p_i\}$ converges to some element p of L . It only remains to be shown that $p = f$. Take $z \in E$; $z \in S_n$ for some n . $\{p_i\}$ converges to p so that there is a positive integer M such that $p_m - p \in N^n$ for all m greater than M . Take m greater than $M + n$. $zf = zp_m = zp$. z was arbitrary; therefore $f = p$.

3. The Ring K . In this section it will be shown that K is a complete, local, principal ideal domain.

LEMMA 3.1. Let L , E , and K be as in §2. Let J denote the Jacobson radical of K and let $A_n = \text{ann}_K(S_n)$. Then

- (i) K is a local domain.
- (ii) $J = A_1$, $J^n \subset A_n \cap A_n = 0$, and $\bigcap J^n = 0$.
- (iii) K is complete in the topology induced by the A_n 's.

Proof. (i) K is local since it is the endomorphism ring of an indecomposable injective module. To prove that K is a domain it is sufficient to show that every nonzero endomorphism of E_L is an epimorphism. Let $0 \neq k \in K$. If $k(E) \neq E$, $k(E) = S_n$ for some n by Lemma 2.4. $\text{Ann}_L(S_n) = N^n$; take $0 \neq b \in N^n$. Since E is L -divisible, $Eb = E$. As a result $S_n = k(E) = k(Eb) = k(E)b = S_nb = 0$ contradicting the fact that $k \neq 0$.

(ii) The radical of K , J , is the set of all endomorphisms of E_L whose kernel is essential (see [2], page 44). Since $(S_1)_L$ is the unique minimal submodule of E , $\ker(k)$ is essential if and only if $k(S_1) = 0$;

therefore $J = A_1$ and $JS_1 = 0$. Inductively suppose that $J^{n-1}S_{n-1} = 0$. $JS_n \subset S_{n-1}$ since it is contained in the radical of $K(S_n)$, S_{n-1} . Hence $J^n s_n = J^{n-1}(JS_n)$ which is contained in $J^{n-1}S_{n-1}$ which is zero, hence $J^n \subset A_n \cap A_n = 0$ since anything in $\cap A_n$ would annihilate all of the S_n 's and hence all of E . $\cap J^n = 0$ since $J^n \subset A_n$.

(iii) Let $\{f_i\}$ be a Cauchy sequence in K with respect to the topology induced by the decreasing family $\{A_n\}$. Let $x \in E$. $x \in S_p$ for some p . Since $\{f_i\}$ is Cauchy, there is an integer M such that $f_n - f_m \in A_p$ for n, m greater than M . Define $f(x) = f_{M+1}(x)$. It is clear that $f \in K$ and that $f_i \rightarrow f$ by the nature of the construction.

Pick $j \in J \setminus A_2$. There is such a j , for if $y_2 \in S_2 \setminus S_1$ and if $0 \neq y_1 \in S_1$, then there is a $j \in K$ such that $j(y_2) = y_1$ by Lemma 2.5. $j \in J \setminus A_2$. In fact if $s \in S_{n+1} \setminus S_n$, then $j^n s$ is a nonzero element of S_1 . The proof is by induction. If $s \in S_2 \setminus S_1$, then $s = y_2 u$ for u a unit in L . Hence $js = jy_2 u = y_1 u \neq 0$. Inductively suppose that $j^{n-1}s$ is nonzero for all s in $S_n \setminus S_{n-1}$ and take $s \in S_{n+1} \setminus S_n$. $js \in S_n$ by an argument in the previous proof. The claim is that $js \notin S_{n-1}$. If it were, then $j^{n-1}s = 0$ which contradicts the induction hypothesis since $sd \in S_n \setminus S_{n-1}$ for some d in L . Hence $js \notin S_{n-1}$ so again by the induction hypothesis $j^n s = j^{n-1}(js) \neq 0$.

LEMMA 3.2. *Let K, J, j, E , and L be as above.*

- (i) $J = jK$.
- (ii) $J = Kj$.
- (iii) $J^n = j^n K = Kj^n$.

Proof. (i) Let $x \in J$. Let $y_2 \in S_2 \setminus S_1$. $x(y_2) = y \in S_1$ since $x \in J$. Let $j(y_2) = y_1$; y_1 is a nonzero element of S_1 since $j \in J \setminus A_2$. Then there is an element d in L such that $y = y_1 d = j(y_2)d = j(y_2 d)$. By Lemma 2.5 there exists $k_1 \in K$ such that $k_1(y_2) = y_2 d$. If $s \in S_2$, then $s = y_2 c$ for some c in L . $x(s) = x(y_2 c) = X(y_2)c = uc = (jk_1(y_2))c = jk_1(y_2 c) = jk_1(s)$. This says that $x - jk_1 \in A_2$.

Inductively suppose that there exist k_1, \dots, k_{n-1} such that

$$z = x - (jk_1 + j^2 k_2 + \dots + j^{n-1} k_{n-1}) \in A_n. \quad \text{If}$$

$$y_{n+1} \in S_{n+1} \setminus S_n, \text{ then } j^n(y_{n+1}) = y_1$$

a nonzero element of S_1 by the above choice of j . Also $z(y_{n+1}) \in S_1$ since $z \in A_n$. Hence by the argument above there is a $k_n \in K$ such that $z - j^n k_n \in A_{n+1}$. The sequence $\{jk_1 + \dots + j^n k_n\}$ converges to x in the A_n topology by the nature of the construction. Also, since $J^n \subset A_n$ the sequence $\{k_1 + \dots + j^{n-1} k_n\}$ is Cauchy and hence by the completeness of K converges to some element k of K . Also by the

construction $jk = x$. Since x was arbitrary in J , $J = jk$.

(ii) is proven by an argument similar to that of (i).

(iii) $J = jK = Kj$ by (i) and (ii). Inductively suppose that $J^n = j^n K = Kj^n$. Then $J^{n+1} = J^n J = (j^n K)(jK) = j^n (Kj)K = j^n (jK)K = j^{n+1} K$. Similarly $J^{n+1} = Kj^{n+1}$.

PROPOSITION 3.3. K as above.

(i) $J^n = A_n$ for all n .

(ii) J^n are the only one-sided ideals of K .

(iii) K is a complete principal ideal domain.

Proof. (i) $J = A_1$ by Lemma 3.1. Inductively suppose that $A_n = J^n$. $J^{n+1} \subset A_{n+1} \subset A_n = J^n$. $J^n/J^{n+1} = j^n K/j^{n+1} K \simeq K/jK = K/J$ which is simple. Therefore either $A_{n+1} = J^{n+1}$ or $A_{n+1} = J^n$. But by the induction hypothesis $j^n \notin A_{n+1}$ so that $A_{n+1} = J^{n+1}$.

(ii) It is sufficient to show that given $x \in K$, $xK = K$ or that $xK = J^p$ for some p . Take $x \in K$ and suppose that $xK \neq K$, then x is not a unit and hence $x \in J^{p+1}$ for some p . By Lemma 3.1 $x = j^p k$, and k must be a unit; for otherwise $k = jk_1$ for some k_1 in K and $x = j^p jk_1 \in J^{p+1}$. As a result $xK = j^p kK = j^p K = J^p$. Similarly $Kx = J^p$.

(iii) K is a principal ideal domain by Lemma 3.2 and (ii). K is complete by (i) and Lemma 3.1.

4. The Ring R . Let R , M , \bar{R} , and L be as in Theorem 1.1. Then \bar{R} is the full $k \times k$ matrix ring over L . Let e_{ij} , $i, j = 1, 2, \dots, n$ be a complete set of matrix units for \bar{R} . Let M_L be a right L -module and let $M^* = M_1 \oplus \dots \oplus M_n$, a direct sum of n copies of M . Let f_1 be the identity map on M_1 , and let f_i , $i = 2, \dots, n$ be an isomorphism from M_1 to M_i . Then M^* can be made into an \bar{R} -module by defining $f_i(m)e_{ij} = f_j(m)$ and $f_i(m)e_{kj} = 0$ if $i \neq k$. “*” is a category isomorphism from the category of right L -modules to the category of right \bar{R} -modules. There is also a category isomorphism e_{11} from the category of right \bar{R} -modules to the category of right L -modules defined by $(A_{\bar{R}})e_{11} = Ae_{11}$. M and M^*e_{11} are isomorphic for any right L -module M (see [1], or [5] page 137).

PROPOSITION 4.1. \bar{R} is the second endomorphism ring of the \bar{R} -injective hull of the simple right \bar{R} -module.

Proof. Let E be the L -injective hull of the simple right L -module as in §2. Then E^* is the \bar{R} -injective hull of a simple right \bar{R} -module since “*” is a category isomorphism. \bar{R}/\bar{M} is simple Artinian and \bar{M} is the Jacobson radical of \bar{R} so there is only one isomorphism class of simple right- \bar{R} -modules. Let $K = \text{end}_{\bar{R}}(E^*)$ and take $q \in K$.

$q(E^*e_{ii}) = q(E^*_{ii}e_{ii}) = q(E^*e_{ii})e_{ii}$; thus each E^*e_{ii} is K -invariant and ${}_K E^* = {}_K F^*e_{11} \oplus {}_K Ee_{22} \oplus \cdots \oplus {}_K E^*e_{kk}$. Each e_{ij} is a K -isomorphism so that E^* is decomposed as a direct sum of k mutually isomorphic K -modules. Thus each K -endomorphism of E^* can be given by multiplication by a matrix of homomorphisms. The remainder of the proof shows that the entries in this matrix are of the desired forms. Each $q \in K$ restricted to E^*_{ii} is an L -endomorphism of E^*_{ii} . Each L -endomorphism of E^*_{ii} can be extended in one and only one way to an \bar{R} -endomorphism of E^* ; namely, if \bar{q} is an L -endomorphism of E^*_{ii} , then its unique extension q is defined by $q(z) = \sum_{j=1}^k \bar{q}(ze_{ji})e_{ij}$ for $z \in E^*$. Hence $K \simeq \text{end}_L(E^*_{ii})$ via the restriction map. By proposition 2.8 each element of $\text{end}_K(E^*_{ii})$ can be given by right multiplication by an element of $e_{ii}\bar{R}e_{ii}$. If $h: E^*e_{ii} \rightarrow E^*e_{jj}$ is a K -homomorphism, then $h\bar{e}_{ji}$ is a K -endomorphism of E^*e_{ji} where \bar{e}_{ji} denotes right multiplication by e_{ji} . Hence $h\bar{e}_{ji} = \bar{e}_{ii}r\bar{e}_{ii}$ for some $r \in \bar{R}$. If $z \in E^*_{ii}$, then $(z)h = zhe_{jj} = zhe_{ji}e_{ij} = ze_{ii}re_{ii}e_{ij} = ze_{ii}re_{ij}$ so that h is given by right multiplication by an element of $e_{ii}\bar{R}e_{ij}$. As a result every K -endomorphism of E^* is given by right multiplication by an element of \bar{R} .

R can be considered as a subring of \bar{R} ; as a result every \bar{R} -module is automatically an R -module. Also, if \bar{M} is the maximal two-sided ideal of \bar{R} , then $\bar{M}^p \cap R = M^p$ and every coset of \bar{R}/\bar{M}^p has a representative in R (Theorem 1.1).

LEMMA 4.2. *E^* as in the proof of Proposition 4.1, then $(E^*)_R$ is the ascending union of \bar{R} -modules $0 \subset B_1 \subset B_2 \subset \cdots$ where the composition length of B_n is n . These are the only \bar{R} -submodules of E^* . Furthermore, the B_i 's are the only R -submodules of E^* and every R -endomorphism of E^* is an \bar{R} -endomorphism. That is, the structure of E^* as an R -module is identical to its structure as an \bar{R} -module.*

Proof. The first part follows since it was true of E and $*$ is a category isomorphism. Let $B_i = S_i^*$. A category isomorphism preserves the submodule lattice. Note that the composition length of $(B_n)_{\bar{R}}$ is n ; since \bar{M} is the radical of \bar{R} , $B_n\bar{M}^n = 0$. In order to prove that the B_n 's are the only R -submodules of E^* it is sufficient to show that $aR = a\bar{R}$ for all $a \in E^*$. Take $a \in E^*$. Clearly $aR \subset a\bar{R}$. Take $\bar{r} \in \bar{R}$. $a \in B_n$ for some n so that $a\bar{M}^n = 0$. By theorem 1.1 there is an m in \bar{M}^n so that $\bar{r} + m = r \in R$, then $a\bar{r} = a\bar{r} + 0 = a\bar{r} + am = a(\bar{r} + m)ar$. Thus $a\bar{R} \subset aR$ and $a\bar{R} = aR$.

Let q be an R -endomorphism of E^* and take $a \in E^*$ and $\bar{r} \in \bar{R}$. It must be shown that $q(a\bar{r}) = q(a)\bar{r}$. Since $a \in E^*$, $a \in B_n$ for some n . The B_n 's are the only R -submodules of E^* and the composition length

of B_n is n , so that $q(B_n) \subset B_n$ and $q(a) \in B_n$. As above there is an $m \in \bar{M}^n$ such that $\bar{r} + m = r \in R$. $B_n \bar{M}^n = 0$. Then

$$\begin{aligned} q(a\bar{r}) &= q(a\bar{r} + 0) = q(a\bar{r} + am) = q(a(\bar{r} + m)) = q(ar) \\ &= q(a)r = q(a)(\bar{r} + m) = q(a)\bar{r} + q(a)m \\ &= q(a)\bar{r} + 0 = q(a)\bar{r}. \end{aligned}$$

Thus q is an \bar{R} -endomorphism.

LEMMA 4.3 E^* is the R -injective hull of $(B_1)_R$.

Proof. By Lemma 4.2 $(B_1)_R$ is an essential submodule of E^*_R . E^* is an injective \bar{R} -module since $*$ is a category isomorphism; in particular E^* is a divisible \bar{R} -module so that E^* is a divisible R -module. R is a hereditary two-sided order so that E^* is an injective R -module by [5], Theorem 3.4.

THEOREM 4.4. Let R be a Dedekind prime ring with a maximal two-sided ideal M , and let P be a maximal right ideal of R containing M . Then the R -endomorphism ring of the R -injective hull of R/P is a complete principal ideal domain.

Proof. Let R, \bar{R}, L, E_L , and E^* be as above. Then by Lemma 4.3 E^* is the injective hull of a simple right R -module which is annihilated by M . $(B_1)_R \simeq R/P$ since both are simple modules over the simple Artinian ring R/M ; thus $E^* \simeq E(R/P)$. By Lemma 4.2 $\text{end}_R(E^*) = \text{end}_{\bar{R}}(E^*)$ which is isomorphic to $\text{end}_L(E)$ since $*$ is a category isomorphism. Hence the result follows by Proposition 3.3.

THEOREM 4.5. (Main Theorem) Let R be a Dedekind prime ring with a nonzero prime ideal M , and let P be a maximal right ideal containing M with $E(R/P)$ the R -injective hull of R/P . Then \bar{R} , the completion of R at M , is isomorphic to the second endomorphism ring of $E(R/P)$.

Proof. Consider E^* ; as above $E^* \simeq E(R/P)$. By Lemma 4.2 the R and \bar{R} structures of E^* are identical. Thus \bar{R} is second endomorphism ring of $E(R/P)$ by Proposition 4.1.

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E. M. Alfsen and B. Hirsberg, <i>On dominated extensions in linear subspaces of $\mathcal{C}_C(X)$</i>	567
Joby Milo Anthony, <i>Topologies for quotient fields of commutative integral domains</i>	585
V. Balakrishnan, G. Sankaranarayanan and C. Suyambulingom, <i>Ordered cycle lengths in a random permutation</i>	603
Victor Allen Belfi, <i>Nontangential homotopy equivalences</i>	615
Jane Maxwell Day, <i>Compact semigroups with square roots</i>	623
Norman Henry Eggert, Jr., <i>Quasi regular groups of finite commutative nilpotent algebras</i>	631
Paul Erdős and Ernst Gabor Straus, <i>Some number theoretic results</i>	635
George Rudolph Gordh, Jr., <i>Monotone decompositions of irreducible Hausdorff continua</i>	647
Darald Joe Hartfiel, <i>The matrix equation $AXB = X$</i>	659
James Howard Hedlund, <i>Expansive automorphisms of Banach spaces. II</i>	671
I. Martin (Irving) Isaacs, <i>The p-parts of character degrees in p-solvable groups</i>	677
Donald Glen Johnson, <i>Rings of quotients of Φ-algebras</i>	693
Norman Lloyd Johnson, <i>Transition planes constructed from semifield planes</i>	701
Anne Bramble Searle Koehler, <i>Quasi-projective and quasi-injective modules</i>	713
James J. Kuzmanovich, <i>Completions of Dedekind prime rings as second endomorphism rings</i>	721
B. T. Y. Kwee, <i>On generalized translated quasi-Cesàro summability</i>	731
Yves A. Lequain, <i>Differential simplicity and complete integral closure</i>	741
Mordechai Lewin, <i>On nonnegative matrices</i>	753
Kevin Mor McCrimmon, <i>Speciality of quadratic Jordan algebras</i>	761
Hussain Sayid Nur, <i>Singular perturbations of differential equations in abstract spaces</i>	775
D. K. Oates, <i>A non-compact Krein-Milman theorem</i>	781
Lavon Barry Page, <i>Operators that commute with a unilateral shift on an invariant subspace</i>	787
Helga Schirmer, <i>Properties of fixed point sets on dendrites</i>	795
Saharon Shelah, <i>On the number of non-almost isomorphic models of T in a power</i>	811
Robert Moffatt Stephenson Jr., <i>Minimal first countable Hausdorff spaces</i>	819
Masamichi Takesaki, <i>The quotient algebra of a finite von Neumann algebra</i>	827
Benjamin Baxter Wells, Jr., <i>Interpolation in $C(\Omega)$</i>	833