ON GENERALIZED TRANSLATED QUASI-CESÀRO SUMMABILITY

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Let $\alpha > 0, \beta > -1$. The (C_t, α, β) transformation of the sequence $\{s_k\}$ is defined by

$$t_n = \frac{\Gamma(\beta+n+2)\Gamma(\alpha+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(\alpha+\beta+n+k+2)} s_k ,$$

and the (C_t, α, β) transformation of the function $s(x)$ is defined by

$$g(y) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^\infty \frac{x^{\alpha-1}s(x)}{(x+y)^{\alpha+\beta+1}} dx .$$

Some properties of the above two transformations are given in this paper and the relation between the summability methods defined by these transformations is discussed.

1. For any sequence $\{\mu_n\}$ the Hausdorff summability (H, μ_n) is defined by the transformation

$$t_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k}\mu_k)s_k ,$$

where

$$\Delta^0_{\mu_k} = \mu_k ,$$
$$\Delta\mu_k = \mu_k - \mu_{k+1} ,$$
$$\Delta^m_{\mu_k} = \Delta\Delta^{m-1}\mu_k .$$

Transposing the matrix of the (H, μ_n), transformation we get the matrix of the quasi-Hausdorff transformation

$$t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n}\mu_n)s_k ,$$

which will be denoted by (H^*, μ_n). Ramanujan [8] introduced the (S, μ_n) summability, which is defined by the transformation

$$t_n = \sum_{k=0}^{\infty} \binom{k+n}{n} (\Delta^k \mu_n)s_k .$$

Thus the elements of row n of the matrix of the (S, μ_n) transformation are those of the corresponding row of the (H^*, μ_n) transformation moved n places to the left.

It is known [8] that if (H, μ_n) is regular and if $\mu_n \to 0$ as $n \to \infty$, then (S, μ_{n+1}) is regular; conversely, if (S, μ_{n+1}) is regular, then (H, μ_n)

can be made regular by a suitable choice of μ_0.

When

$$\mu_n = \frac{1}{\displaystyle\binom{n+\alpha}{n}} \, ,$$

(H, μ_n) reduces to the Cesàro summability (C, α). Borwein [3] introduced the generalized Cesàro summability (C, α, β) which is (H, μ_n) with

$$(1) \qquad\qquad \mu_n = \frac{\displaystyle\binom{n+\beta}{n}}{\displaystyle\binom{n+\alpha+\beta}{n}} \, .$$

The aim of this paper is to discuss properties of the (S, μ_{n+1}) summability with μ_n given by (1) for $\alpha > 0, \beta > -1$ and of the analogous functional transformation. We shall denote this summability by (C_t, α, β). The case in which $\beta = 0$ has been considered by Kuttner [6] and a summability method similar to (C_t, α, β) has been discussed by me [7].

A straightforward calculation shows that the (C_t, α, β) transformation is given by

$$
\begin{aligned}
t_n = t(n, \alpha, \beta) &= \frac{(\beta+1)(\beta+2)\cdots(\beta+n+1)}{n!} \\
(2) \qquad &\times \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)(k+1)(k+2)\cdots(k+n)}{(\alpha+\beta+1)(\alpha+\beta+2)\cdots(\alpha+\beta+n+1+k)} s_k \\
&= \frac{\Gamma(\beta+n+2)\Gamma(\alpha+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(\alpha+\beta+n+k+2)} s_k \, .
\end{aligned}
$$

It is clear that, if (2) converges for one value of n, then it converges for all n. Further, a necessary and sufficient condition for this to happen is that

$$(3) \qquad\qquad \sum_{k=1}^{\infty} \frac{s_k}{k^{\beta+2}}$$

should converge.

Let $s(x)$ be any function L-integrable in any finite interval of $x \geq 0$ and bounded in some right-hand neighbourhood of the origin. Let $\alpha > 0, \beta > -1$, and let

$$(4) \qquad g(y) = g(y, \alpha, \beta) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1} s(x)}{(x+y)^{\alpha+\beta+1}} dx \, .$$

If $g(y)$ exists for $y > 0$ and if

$$\lim_{y \to \infty} g(y) = s \, ,$$

we say that $s(x)$ is summable (C_t, α, β) to s.

It is clear that a necessary and sufficient condition for the convergence of (4) is that

(5) $$\int_1^\infty \frac{s(x)}{x^{\beta+2}} dx$$

should converge.

2. **The relationship between sequence-to-sequence and function-to-functions transformations.** Given any sequence $\{s_n\}$, let the function $f(x)$ be defined by

$$f(x) = s_n \quad (n \leq x < n + 1; n = 0, 1, 2, \cdots) \, .$$

Then the (C_t, α, β) summability of $\{s_n\}$ is equivalent to the (C_t, α, β) summability of $f(x)$ for $\alpha > 0$, $\beta = 0$ (see [6] Theorem 4). However, the proof breaks down when $\beta > 0$. We can prove that they are equivalent for $-1 < \beta \leq 0$ as follows. Write

$$a(n, k) = \frac{\Gamma(\alpha + k)\Gamma(k + n + 1)}{\Gamma(k + 1)\Gamma(\alpha + \beta + n + k + 2)}$$

$$b(y, k) = \int_k^{k+1} \frac{x^{\alpha-1}}{(x + y)^{\alpha+\beta+1}} dx \, .$$

As in [6], we may suppose that $s_0 = 0$. Then the result would follow if, corresponding to equation (11) of [6], we proved that, if (3) converges, then uniformly for $0 \leq \theta < 1$,

(6) $$\sum_{k=1}^\infty [a(n, k) - b(n + \theta, k)]s_k = o\left(\frac{1}{n^{\beta+1}}\right) \, .$$

Choose an integer Q such that $Q \geq \beta + 3$. From equations analogous to those of the last line and line 6 from bottom of p. 709 of [6], we find that

(7) $$a(n, k) - b(n + \theta, k) = \Sigma p(\theta)\frac{k^{\alpha-q}}{(n + k)^{\alpha+\beta+r}} + O\left(\frac{k^{\alpha-Q}}{(k + n)^{\alpha+\beta+1}}\right) \, ,$$

where $p(\theta)$ is a polynomial in θ (which may be different for each term in the sum), and the sum is taken over those integers q, r which are such that

$$q \geq 1, r \geq 1, \quad q, r \text{ not both } 1, \quad q + r \leq Q \, .$$

Since the convergence of (3) implies that

$$s_k = o(k^{\beta+2}) \, ,$$

and since $\alpha > 0$, $Q \geqq \beta + 3$, we see that the contribution to the expression on the left of (6) of the "0" term in (7) is

$$o\!\left(\frac{1}{n^{\beta+1}}\right).$$

Hence the result would follow if (corresponding to Lemma 2 of [6]) we could prove that the convergence of (3) implied that, for relevant q, r,

$$(8) \qquad \sum_{k=1}^{\infty} \frac{k^{\alpha-q}}{(k+n)^{\alpha+\beta+r}} s_k = o\!\left(\frac{1}{n^{\beta+1}}\right).$$

Now write

$$v_k = \sum_{m=k}^{\infty} \frac{s_m}{m^{\beta+2}}$$

so that $v_k \to 0$ (and this is all we know). The sum on the left of (8) is

$$(9) \quad \begin{aligned} &\sum_{k=1}^{\infty} \frac{k^{\alpha+\beta+2-q}}{(k+n)^{\alpha+\beta+r}}(v_k - v_{k+1}) \\ &= \frac{v_1}{(n+1)^{\alpha+\beta+r}} + \sum_{k=2}^{\infty} v_k\left\{\frac{k^{\alpha+\beta+2-q}}{(k+n)^{\alpha+\beta+r}} - \frac{(k-1)^{\alpha+\beta+2-q}}{(k-1+n)^{\alpha+\beta+r}}\right\}. \end{aligned}$$

The first term on the right of (9) is $o(1/n^{\beta+1})$ (since $r \geqq 1$, $\alpha > 0$). The expression in curly brackets in the second term is

$$O\!\left(\frac{k^{\alpha+\beta+1-q}}{(k+n)^{\alpha+\beta+r}}\right)$$

(and this result is best possible). This gives the required result when $\beta \leqq 0$; but if $\beta > 0$, all that we can deduce in the "worst" cases (which are $q = 1$, $r = 2$ or $q = 2$, $r = 1$) is that the sum (9) is $o(1/n)$.

Of course, the fact that the proof breaks down does not imply that the theorem itself is false. My guess is that the theorem probably is false for $p > 0$; but I have not actually got a counter example.

3. **Theorems.** The following two theorems with $\beta = 0$ are Theorem 1' and Theorem 2' given by Kuttner [6]. The proof of Theorem 1 is similar to that of Theorem 1' in [6], and Theorem 2 follows from Lemma 1 and Lemma 2 of this paper.

THEOREM 1. *Let* $\alpha > 0$, $\beta > -1$ *and* $r \geqq 0$ *and let* $s(x)$ *be summable* (C, r)[1] *to* s *and* (4) *converge. Then* $s(x)$ *is summable* (C_t, α, β) *to* s.

THEOREM 2. *Let* $\alpha > \alpha' > 0$, $\beta > -1$, *and let* $s(x)$ *be summable* (C_t, α, β) *to* s. *Then* $s(x)$ *is summable* (C_t, α', β) *to* s.

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[1] For definition of the (C, r) summability of $s(v)$, see [7].

In § 5, we shall prove

THEOREM 3. *Let* $\alpha > 0, \beta > \beta' > -1$. *Suppose that* $s(x)$ *is summable* (C_t, α, β) *to* s *and the integral*

$$\int_1^\infty \frac{s(x)}{x^{\beta'+2}} dx$$

converges. Then $s(x)$ *is summable* (C_t, α, β') *to* s.

The sequence $\{s_n\}$ is said to be summable A_λ to s if

$$f_\lambda(x) = (1 - x)^{\lambda+1} \sum_{n=0}^\infty \binom{n + \lambda}{n} s_n x^n$$

converges for all x in the interval $0 \leq x < 1$ and tend to a finite limit s as $x \to 1-$. The A_0 method is the ordinary Abel method.

It is known (see [1] and [2]) that $A_\mu \supset A_\lambda$ for $\lambda > \mu > -1$. For other properties of this summability method, see [1] and [6]. We shall prove

THEOREM 4. *Let* $\lambda > -1, \beta > -1$. *Suppose that the sequence* $\{s_n\}$ *is summable* A_λ *to* s *and that* (3) *converges. Then the sequence is summable* $(C_t, \lambda + 1, \beta)$ *to* s.

4. Lemmas.

LEMMA 1. *Let* $\alpha > \alpha' > 0, \beta > -1$. *Suppose that* (5) *converges. Then*

$$y^{\alpha-1} g(y, \alpha', \beta) = \frac{\Gamma(\alpha)}{\Gamma(\alpha')\Gamma(\alpha - \alpha')} \int_0^y t^{\alpha'-1}(y - t)^{\alpha-\alpha'-1} g(t, \alpha, \beta) dt \ .$$

The proof of this lemma is similar to that of Lemma 4 in [6].

LEMMA 2. *Let*

$$t(x) = \int_0^\infty c(x, y) s(y) dy \ .$$

Then in order that

$$s(y) \to s \qquad\qquad (y \to \infty)$$

should imply

$$t(x) \to s \qquad\qquad (x \to \infty)$$

for every bounded $s(y)$, *it is sufficient that*

$$\int_0^\infty |c(x, y)| dy < H \ ,$$

where H is independent of x, that

$$\int_0^Y |c(x, y)| dy \to 0$$

when $x \to \infty$, *for every finite* Y, *and that*

$$\int_0^\infty c(x, y) dy \to 1$$

when $x \to \infty$.

This Theorem 6 in [4].

5. Proof of Theorem 3. Let

$$\phi(x) = \int_x^\infty \frac{s(u)}{u^{\beta+2}} du$$

for $x > 0$. Then $\phi(x)$ is continuous in $(0, \infty)$, and $\phi(x) \to 0$ as $x \to \infty$; hence $\phi(x)$ is bounded in (B, ∞) for any $B > 0$, say

$$|\phi(x)| \leqq M$$

for $x \geqq B$, where M may depend on B if B is small, but may be taken as an absolute constant for large B. It follows that

$$
\begin{aligned}
\left| \int_B^\infty \frac{x^{\alpha-1} s(x)}{(x+t)^{\alpha+\beta+1}} dx \right| &= \left| \int_B^\infty \left(\frac{x}{x+t} \right)^{\alpha+\beta+1} d\phi(x) \right| \\
&= \left| \left(\frac{B}{B+t} \right)^{\alpha+\beta+1} \phi(B) \right. \\
&\qquad \left. + (\alpha+\beta+1)t \int_B^\infty \frac{1}{(x+t)^2} \left(\frac{x}{x+t} \right)^{\alpha+\beta} \phi(x) dx \right| \\
&\leqq |\phi(B)| + (\alpha+\beta+1)tM \int_B^\infty \frac{dx}{(x+t)^2} \\
&\leqq (\alpha+\beta+2)M .
\end{aligned}
$$

(10)

Since $s(x)$ is bounded in some right-hand neighbourhood of the origin, there exists $B_0 > 0$ such that

$$|s(x)| \leqq K$$

for $0 < x < B_0$. By partial integration, we obtain

(11)
$$\left| t^{\beta+1} \int_0^{B_0} \frac{x^{\alpha-1} s(x)}{(x+t)^{\alpha+\beta+1}} dx \right| \leqq \frac{K(\alpha+2\beta+2)}{\alpha(\beta+1)} .$$

By combining (10) and (11) it follows that $g(t, \alpha, \beta)$ is bounded in any finite interval $(0, T)$. Since it tends to s as $t \to \infty$, $g(t, \alpha, \beta)$ is bounded in $(0, \infty)$. Thus, for $y > 0$, the integral

$$I = \frac{\Gamma(\alpha)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} \int_y^\infty t^{-\beta-1}(t - y)^{\beta-\beta'-1}g(t, \alpha, \beta)dt$$

converges. In view of the definition of $g(t, \alpha, \beta)$ it follows that

(12) $$I = \lim_{A\to\infty} I(A)$$

where

$$I(A) = \int_y^A (t - y)^{\beta-\beta'-1}dt \int_0^\infty \frac{x^{\alpha-1}s(x)}{(x + t)^{\alpha+\beta+1}}dx .$$

It follows from (10) by dominated convergence that, for fixed A,

$$\int_y^A (t - y)^{\beta-\beta'-1}dt \int_B^\infty \frac{x^{\alpha-1}s(x)}{(x + t)^{\alpha+\beta+1}}dx \to 0$$

as $B \to \infty$. Hence, by Fubini's theorem

(13) $$I(A) = \int_0^\infty x^{\alpha-1}s(x)dx \int_y^A \frac{(t - y)^{\beta-\beta'-1}}{(x + t)^{\alpha+\beta+1}}dt .$$

We will now show that, for fixed y,

(14) $$\int_0^\infty x^{\alpha-1}s(x)dx \int_A^\infty \frac{(t - y)^{\beta-\beta'-1}}{(x + t)^{\alpha+\beta+1}}dt \to 0$$

as $A \to \infty$. It is clear that for large A the inner integral in (14) is $O(A^{-\alpha-\beta'-1})$ uniformly in $0 \leq x \leq 1$, so that the contribution to (14) of the range $0 < x < 1$ tends to 0 as $A \to \infty$. Now write

$$\psi(x) = \int_x^\infty \frac{s(u)}{u^{\beta'+2}}du;$$

thus we are given that $\psi(x)$ exists and that it tends to 0 as $x \to \infty$. The contribution to (14) of $x > 1$ may now be written

(15) $$-\int_1^\infty x^{\alpha+\beta'+1}d\psi(x) \int_A^\infty \frac{(t - y)^{\beta-\beta'-1}}{(x + t)^{\alpha+\beta+1}}dt .$$

It is easily seen that, for fixed y, A and large x, the inner integral in (15) is $O(x^{-\alpha-\beta'-1})$; thus, integrating by parts, (15) becomes

(16) $$\psi(1) \int_A^\infty \frac{(t - y)^{\beta-\beta'-1}}{(1 + t)^{\alpha+\beta+1}}dt$$

$$+ \int_1^\infty x^{\alpha+\beta'}\psi(x)dx \int_A^\infty \frac{(t - y)^{\beta-\beta'-1}[(\alpha + \beta' + 1)t - (\beta - \beta')x]}{(x + t)^{\alpha+\beta+2}}dt .$$

Now for fixed y and large A, uniformly in $0 \leq x \leq A$, the inner integral in (16) is

$$O\left\{\int_A^\infty t^{-\alpha-\beta'-2}dt\right\} = O(A^{-\alpha-\beta'-1}) .$$

Hence

$$\int_1^A x^{\alpha+\beta'} \psi(x)dx \int_A^\infty \frac{(t-y)^{\beta-\beta'-1}[(\alpha+\beta'+1)t - (\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}}dt$$

$$= \left(\int_1^{A/\log A} + \int_{A/\log A}^A\right)x^{\alpha+\beta'}\psi(x)O(A^{-\alpha-\beta'-1})dx$$

$$= O\left(A^{-\alpha-\beta'-1}\int_1^{A/\log A} x^{\alpha+\beta'}dx\right)$$

$$+ O\left(A^{-\alpha-\beta'-1}\sup_{x\geq(A/\log A)}|\psi(x)|\int_{A/\log A}^A x^{\alpha+\beta'}dx\right) = O(1) .$$

Nothing that for fixed y and large t

$$(t-y)^{\beta-\beta'-1} = t^{\beta-\beta'-1} + O(t^{\beta-\beta'-2}) ,$$

and also that

$$\int_0^\infty \frac{t^{\beta-\beta'-1}[(\alpha+\beta'+1)t - (\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}}dt = 0 ,$$

we see that, for large A uniformly in $x \geq A$, the inner integral in (16) is

$$-\int_0^A \frac{t^{\beta-\beta'-1}[(\alpha+\beta'+1)t - (\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}}dt$$

$$+ O\left\{\int_A^\infty \frac{t^{\beta-\beta'-2}|(\alpha+\beta'+1)t - (\beta-\beta')x|}{(x+t)^{\alpha+\beta+2}}dt\right\}$$

$$= O\left\{x^{-\alpha-\beta-1}\int_0^A t^{\beta-\beta'-1}dt\right\} + O\left\{x^{-\alpha-\beta-1}\int_A^x t^{\beta-\beta'-2}dt\right\} + O\left\{\int_x^\infty t^{-\alpha-\beta'-3}dt\right\}$$

$$= O(x^{-\alpha-\beta-1}A^{\beta-\beta'}) + O(x^{-\alpha-\beta'-2})$$

(except that, in the case $\beta - \beta' = 1$, we must insert an extra term $O(x^{-\alpha-\beta-1}\log x)$). It is now clear that the expression (16) tends to 0 as $A \to \infty$, and this completes the proof of (14). We deduce from (12), (13) and (14) that

$$I = \int_0^\infty x^{\alpha-1}s(x)dx \int_y^\infty \frac{(t-y)^{\beta-\beta'-1}}{(x+t)^{\alpha+\beta+1}}dt$$

$$= \frac{\Gamma(\beta-\beta')\Gamma(\alpha+\beta'+1)}{\Gamma(\alpha+\beta+1)}\int_0^\infty \frac{x^{\alpha-1}s(x)}{(x+y)^{\alpha+\beta'+1}}dx$$

$$= \frac{\Gamma(\beta-\beta')\Gamma(\alpha)\Gamma(\beta'+1)}{\Gamma(\alpha+\beta+1)}y^{-\beta'-1}g(y, \alpha, \beta') .$$

Thus, in view of the definition of I, we have

$$g(y, \alpha, \beta') = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \beta')\Gamma(\beta' + 1)} y^{\beta'+1} \int_y^\infty t^{-\beta-1}(t - y)^{\beta-\beta'-1} g(t, \alpha, \beta) dt .$$

The kernel of this last transformation can easily be verified to satisfy the conditions of Lemma 2, and the theorem now follows.

6. Proof of Theorem 4.   It follows from the convergence of (3) that for $\beta > -1$, $s_\nu = o(\nu^{\beta+2})$.   We can easily prove that the function $t^{n+k}(1 - t)^{\lambda+\beta'+1}$ has a maximum when

$$t = \frac{k + n}{k + n + \lambda + \beta' + 1} .$$

For large $k + n$, this maximum is $O((k + n)^{-\lambda-\beta'-1})$.   Hence, if $\beta' > \beta + 2$, we have, the inversion in the order of integration and summation being justified by absolute convergence,

$$\frac{\Gamma(\beta' + n + 2)}{\Gamma(n + 1)\Gamma(\beta' + 1)} \int_0^1 t^n (1 - t)^{\lambda+\beta'+1} \left\{ \sum_{k=0}^\infty \binom{\lambda + k}{k} s_k t^k \right\} dt$$

$$(17) \quad = \frac{\Gamma(\beta' + n + 2)}{\Gamma(n + 1)\Gamma(\beta' + 1)} \sum_{k=0}^\infty \binom{\lambda + k}{k} s_k \int_0^1 t^{k+n} (1 - t)^{\lambda+\beta'+1} dt$$

$$= \frac{\Gamma(\beta'+n+2)\Gamma(\lambda+\beta'+2)}{\Gamma(n+1)\Gamma(\beta'+1)\Gamma(\lambda+1)} \sum_{k=0}^\infty \frac{\Gamma(\lambda+k+1)\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(\lambda+\beta'+n+k+3)} s_k$$

$$= t(n, \lambda + 1, \beta') .$$

By analytic continuation, (17) holds for $\beta' \geqq \beta$.   Hence

$$t(n, \lambda + 1, \beta) = \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)} \int_0^1 t^n (1 - t)^\beta f_\lambda(t) dt$$

$$= \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)} \int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y} f_\lambda(1 - e^{-y}) dy .$$

By Lemma 2 the result with follow if

(i) $\quad \dfrac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)} \displaystyle\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y} dy < H$

where H is independent of n,

(ii) $\quad \dfrac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)} \displaystyle\int_0^Y (1 - e^{-y})^n e^{-(\beta+1)y} dy \to 0$

when $n \to \infty$, for every finite Y, and

(iii) $\quad \dfrac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)} \displaystyle\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y} dy \to 1 ,$

when $n \to \infty$.   Since

$$\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y} dy = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(\beta+n+2)} \,,$$

(i) and (iii) are satisfied. We have $\Gamma(n + \beta + 2) \sim n^{\beta+1}\Gamma(n+1)$, and the integral in (ii) is, by changing the variable,

$$\int_0^{1-e^{-Y}} t^n (1-t)^\beta dt \,.$$

Hence (ii) is satisfied.

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